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Cubic Bipolar Fuzzy SA-(subalgebras) ideals with Thresholds of SA-algebra

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Abstract— *The aim of this paper is to present a new notion named cubic bipolar fuzzy **SA**-subalgebras with thresholds of SA-algebra and the notion of cubic bipolar fuzzy SA-ideals with thresholds SA-algebra and provides the necessary examples,theorems and the types of intersection and union with their conditions and examples and we introduce the homomorphism of cubic bipolar fuzzy SA-subalgebras (ideals) with thresholds on SA-algebra with their theorems.*

Keywords— fuzzy SA-subalgebras, fuzzy SA-ideals, cubic bipolar fuzzy SA-subalgebras (ideals)with thresholds of SA-algebra.

1. INTRODUCTION

In1965, the notion of fuzzy sets was introduced by Zadeh **[15]**. Since then this notion has been applied to other algebraic structures. Is´eki and authors introduced two classes of abstract algebras: BCK-algebras and BCI-algebras **[7-11]**. It is known that the class of BCK-algebras is a proper subclass of the class BCI-algebras. In 1985, Bhattacharya and Mukherjee offered the notion of fuzzy relations and fuzzy groups **[2]**. In 1998, Hong and Jun introduced the notion of anti fuzzy ideals in BCK-algebras **[6]**. In 2000, Lee studied bipolar-valued fuzzy sets and their operations **[14]**. In 2001, Jun introduced interval-valued fuzzy ideals in BCI-algebras **[12]**.. In 2010, Jun and authors introduced the notion of cubic sets **[13]**. In 2015, Mustafa and Hammed introduced the SA-algebras, SA-subalgebras , SA-ideals of SA-algebras and fuzzy SA-idesls with degree (λ,κ) of SA-algebra of SA-algebra **[3]**. In 2021, Hammed and Raheem introduced anti-fuzzy SA-idesls with degree (λ,κ) of SA-algebra of SA-algebra and interval-valued fuzzy SA-ideals with degree (λ,κ) of SA-algebra **[4,5]**. In 2024, Hammed and A.S.Abed introduce bipolar fuzzy SA-ideals of SA-algebra and bipolar valued fuzzy SA-subalgebras and fuzzy SA-ideals of SA-algebra [1].

2. Preliminaries

Definition 2.1.[3].

Let $(X;+,-,0)$ be an algebra with two operations $(+)$ and $(-)$ and constant (0). X is named an $\boldsymbol{SA}$**-algebra** if it satisfies the following identities: for any $x,y,z \in X$

(SA_1) $x-x=0,$

(SA_2) $x-0=x,$

(SA_3) $(x-y)-z=x-(z+y),$

(SA_4) $(x+y)-(x+z)=y-z.$

In , we can define a binary relation (≤) by : $x \le y$ if and only if

$x+y=0$ and $x-y=0$, for some $x,y \in X$.

Lemma 2.2.[3].
Let $(X;+,-,0)$ be an SA-algebra. Then for any $x,y \in X$,

1) $x+y=x-(-y),$

2) $x - y = x + (-y)$,

3) $x \in X \Rightarrow -x \in X$,

4) $x + 0 = x$,

5) $x - y = -y + x$,

6) $x + y = y + x$.

Definition 2.3.[3].

Let $(X; +, -, 0)$ be an SA-algebra and let S be a nonempty set of X. S is called **a SA-subalgebra of X** if

$x + y \in S$ *and* $x - y \in S$, whenever $x, y \in S$.

Definition 2.4.[3].

A nonempty subset I of an SA-algebra $(X; +, -, 0)$ is called **an SA-ideal of X** if it satisfies: for $x, y, z \in X$,

1) $0 \in I$,

2) $(x + z) \in I$ *and* $(y - z) \in I$ *imply* $(x + y) \in I$.

Proposition 2.5.[3].

Every SA-ideal of SA-algebra is a SA-subalgebra of X and the converse is not true.

Lemma 2.6.[3].

An SA-ideal I of SA-algebra $(X; +, -, 0)$ has the following property:

For any $x \in X$, for all $y \in I, x \leq y$ implies that $x \in I$.

Proposition 2.7.[3].

1) Let $\{S_i | i \in \Lambda\}$ be a nonempty family of SA-subalgebras of SA-algebra $(X; +, -, 0)$, then $\cap_{i\in\Lambda} S_i$ is SA-subalgebras of X.
2) Let $\{S_i | i \in \Lambda\}$ be a nonempty family of SA-ideals of SA-algebra $(X; +, -, 0)$, then $\cap_{i\in\Lambda} S_i$ is SA-ideal of X.

Definition 2.8.[3].

Let $(X; +, -, 0)$ and $(Y; +', -', 0')$ be SA-algebras, the mapping

$f: (X; +, -, 0) \to (Y; +', -', 0')$ is called a homomorphism if it satisfies: for all $x, y \in X$,

1) $f(x + y) = f(x) +' f(y)$,
2) $f(x - y) = f(x) -' f(y)$.

Definition 2.9.[3].

Let $f: (X; +, -, 0) \to (Y; +', -', 0')$ be a mapping nonempty SA-algebras X and Y respectively. If μ is a fuzzy subset of X, then the fuzzy subset β of Y defined by:

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$$f(\mu)(y) = \begin{cases} sup\{\mu(x): x \in f^{-1}(y)\} & if \quad f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

is said to be the image of μ under f. Similarly if β is a fuzzy subset of Y , then the fuzzy subset $\mu = (\beta \text{ o } f)$ of X, i.e

the fuzzy subset defined by μ (x) = β (f (x)) for all $x \in X$) is called the pre-image of β under f .

Definition 2.10.[12].

A fuzzy subset μ of a set X **has sup property** if for any subset T of X, there exist $t_0 \in T$ such that

$\mu(t_0) = sup\{\mu(t)| t \in T\}$.

Definition 2.11.[4].

Let $(X;\ +,-,0)$ be an SA-algebra, a fuzzy subset μ of X is called **an anti-fuzzy SA- subalgebra of X** if for all $x, y \in X$,

1) $\mu\,(x+y) \leq \ max\{\mu\,(x), \mu\,(y)\}$,

2) $\mu\,(x-y) \leq \ max\{\mu\,(x), \mu\,(y)\}$.

Proposition 2.12.[4].

Let μ be an anti-fuzzy subset of an SA-algebra $(X;\ +,-,0)$.

1) If μ is an anti-fuzzy SA-subalgebra of , then it satisfies for any t∈[0, 1], $L(\mu, t) \neq \emptyset$ implies $L(\mu, t)$ is a SA-subalgebra of X.

2**)** If $L(\mu, t)$ is a SA-subalgebra of X, for all t∈[0, 1], $L(\mu, t) \neq \emptyset$, then μ is an anti-fuzzy SA-subalgebra of X.

3) Let $\{\mu_i : i \in \Lambda\}$ be a famliy of an anti-fuzzy SA-subalgebras of X, then $\bigcap_{i \in \Lambda} \mu_i$

is an anti-fuzzy SA-subalgebra of X, where $\mu_i \subseteq \mu_{i+1}$ for all $i \in \Lambda$.

4) Let $\{\mu_i : i \in \Lambda\}$ be a famliy of an anti-fuzzy SA-subalgebras of X, then $\bigcup_{i \in \Lambda} \mu_i$

is an anti-fuzzy SA-subalgebra of X.

Definition 2.13.[4].

Let $(X;\ +,-,0)$ be an SA-algebra, a fuzzy subset μ of X is called **an anti-fuzzy SA-ideal of X** if it satisfies the following conditions, for all $x, y \in X$,

1) $\mu\,(0) \leq \mu\,(x)$,

2) $\mu(x+y) \leq max\{\mu(x+z), \mu(y-z)\}$.

Proposition 2.14.[4].

Let μ be an anti-fuzzy subset of an SA-algebra $(X;\ +,-,0)$.

1) If μ is an anti-fuzzy SA-ideal of , then it satisfies for any t∈[0, 1],

$L(\mu, t) \neq \emptyset$ implies $L(\mu, t)$ is a SA-ideal of X.

2) If $L(\mu,t)$ is an SA- ideal of X, for all t∈[0, 1], $L(\mu,t) \neq \emptyset$, then μ is an anti-fuzzy SA-ideal of X .

3) Let $\{\mu_i: i \in \Lambda\}$ be a famliy of an anti-fuzzy SA-ideals of X, then $\bigcap_{i\in\Lambda} \mu_i$ is an anti-fuzzy SA-ideal of X, where $\mu_i \subseteq \mu_{i+1}$

for all $i \in \Lambda$.

4) Let $\{\mu_i: i \in \Lambda\}$ be a famliy of an anti-fuzzy SA-ideals of X, then $\bigcup_{i\in\Lambda} \mu_i$ is an anti-fuzzy SA-ideal of X.

Definition 2.15.[4].

Let $f: (X; +, -, 0) \rightarrow (Y; +', -', 0')$be a mapping nonempty SA-algebras X and Y respectively. If μ is anti-fuzzy subset of X, then the anti-fuzzy subset β of Y defined by:

$$f(\mu)(y) = \begin{cases} inf\{\mu(x): x \in f^{-1}(y)\} & if \;\; f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ 1 & otherwise \end{cases}$$

is said to be the image of μ under f. Similarly if β is anti-fuzzy subset of , then the fuzzy subset $\mu = (\beta \circ f)$ of X (i.e the anti-fuzzy subset defined by $\mu(x) = \beta(f(x))$,

for all $x \in X$) is named the pre-image of β under f.

Theorem 2.16.[4].
1) An onto homomorphic pre-image of an anti-fuzzy SA-subalgebra is also an anti-fuzzy SA-subalgebra.
2) An onto homomorphic pre-image of an anti-fuzzy SA-ideal is also an anti- fuzzy SA-ideal.

Definition 2.17,[12].
An anti- fuzzy subset μ of a set X **has inf property** if for any subset T of X, there exist t_0 ∈T such that $\mu(t_0) = inf\{\mu(t) | t \in T\}$.

Theorem 2.18.[4].
Let $f: (X; +, -, 0) \rightarrow (Y; +', -', 0')$ be a homomorphism between SA-algebras X and Y respectively.
1) For every anti-fuzzy SA-subalgebra μ of X and with inf property, $f(\mu)$ is anti-fuzzy SA-subalgebra of Y.
2) For every anti-fuzzy SA-ideal μ of X and with inf property, $f(\mu)$ is anti-fuzzy SA-ideal of Y.

Definition 2.19.[5].

An **interval-valued fuzzy subset** $\widetilde{\mu}_A$on SA-algebra $(X; +, -, 0)$ is defined as $\widetilde{\mu}_A = \{< x, [\mu_A^-(x), \mu_A^+(x)] >| x \in X\}$. Where $\mu_A^-(x) \leq \mu_A^+(x)$, for all $x \in X$. Then the fuzzy subsets $\mu_A^-$and $\mu_A^+$are called a **lower fuzzy subset and an upper fuzzy subset** of $\tilde{\mu}_A$ respectively.

Let $\widetilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$, $\widetilde{\mu}_A$:$X \rightarrow$ D[0, 1], then $A = \{< x, \widetilde{\mu}_A(x) >| x \in X\}$.

Definition 2.20.[1].

Let $(X; +, -, 0)$ be an SA-algebra, a fuzzy subset μ of X is called **a negative anti-fuzzy SA-subalgebra of X** if

$\mu: X \rightarrow [-1,0]$ and for all $x,y \in X$

1) $\mu(x + y) \leq max\{\mu(x), \mu(y)\}$
2) $\mu(x - y) \leq max\{\mu(x), \mu(y)\}$

Definition 2.21.[1]

Let $(X; +, -, 0)$ be an SA-algebra a fuzzy subset μ of X is called **a negative anti-fuzzy SA-ideal** of X if $\mu: X \longrightarrow [-1,0]$

and for all $x, y, z \in X$:

1) $\mu(0) \leq \mu(x)$
2) $\mu(x+y) \leq max\{\mu(x+z), \mu(y-z)\}$.

3. Cubic bipolar Fuzzy SA-subalgebras of SA-algebra with Thresholds $(\tilde{\alpha}, \tilde{\beta})$, (ω, ϑ) of SA-algebra.

Definition 3.1

Let $(X\,;+,-,0)$ be an SA-algebra, a cubic fuzzy subset $\Omega =< \tilde{\mu}_\Omega(x), \lambda_\Omega(x) >$ of X where $\tilde{\alpha} = [\alpha_1, \alpha_2]$, $\tilde{\beta} = [\beta_1, \beta_2]$ such that $\alpha_1, \beta_1 \subset [-1,0]$, $\alpha_2, \beta_2 \subset [0,1]$, $\omega, \vartheta \in [0,1]$,

$\tilde{\alpha} < \tilde{\beta}$ and $\omega < \vartheta$. If for all $x, y \in X$. $\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)} =< \tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_\Omega{}^{(\omega,\vartheta)}(x) >$ is called **a cubic bipolar fuzzy SA-subalgebra with thresholds $(\tilde{\alpha}, \tilde{\beta})$, (ω,ϑ) of X**, if for all $x, y \in X$:

$$\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}(x+y) \succcurlyeq rmin\left\{\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}(x), \tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}(y)\right\},$$

$$\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}(x-y) \succcurlyeq rmin\left\{\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}(x), \tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}(y)\right\},$$

$$\lambda_\Omega{}^{(\omega,\vartheta)}(x+y) \leq max\left\{\lambda_\Omega{}^{(\omega,\vartheta)}(x), \lambda_\Omega{}^{(\omega,\vartheta)}(y)\right\} \text{ and}$$

$$\lambda_\Omega{}^{(\omega,\vartheta)}(x-y) \leq max\{\lambda_\Omega{}^{(\omega,\vartheta)}(x), \lambda_\Omega{}^{(\omega,\vartheta)}(y)\}.$$

i.e.,

1) $\min\{(\tilde{\mu}_\Omega)^N(x+y), \alpha_1\} \leq max\{(\tilde{\mu}_\Omega)^N(x), (\tilde{\mu}_\Omega)^N(y), \beta_1\}$,

$\max\{(\tilde{\mu}_\Omega)^P(x+y), \alpha_2\} \geq min\{(\tilde{\mu}_\Omega)^P(x), (\tilde{\mu}_\Omega)^P(y), \beta_2\}$,

$\min\{\lambda_\Omega(x+y), \omega\} \leq max\{\lambda_\Omega(x), \lambda_\Omega(y), \vartheta\}$.

2) $\min\{(\tilde{\mu}_\Omega)^N(x-y), \alpha_1\} \leq max\{(\tilde{\mu}_\Omega)^N(x), (\tilde{\mu}_\Omega)^N(y), \beta_1\}$,

$\max\{(\tilde{\mu}_\Omega)^P(x-y), \alpha_2\} \geq min\{(\tilde{\mu}_\Omega)^P(x), (\tilde{\mu}_\Omega)^P(y), \beta_2\}$,

$\min\{\lambda_\Omega(x-y), \omega\} \leq max\{\lambda_\Omega(x), \lambda_\Omega(y), \vartheta\}$.

i.e.,

1) $in\ \{(\mu_\Omega^-)^N(x+y), \alpha_1\} \leq max\{(\mu_\Omega^-)^N(x), (\mu_\Omega^-)^N(y), \beta_1\}$,

$\max\{(\mu_\Omega^-)^P(x+y), \alpha_2\} \geq min\{(\mu_\Omega^-)^P(x), (\mu_\Omega^-)^P(y), \beta_2\}$,

$\min\{(\mu_\Omega^+)^N(x+y), \alpha_1\} \leq max\{(\mu_\Omega^+)^N(x), (\mu_\Omega^+)^N(y), \beta_1\}$,

$\max\{(\mu_\Omega^+)^P(x+y), \alpha_2\} \geq min\{(\mu_\Omega^+)^P(x), (\mu_\Omega^+)^P(y), \beta_2\}$,

$\min\{\lambda_\Omega(x+y), \omega\} \leq max\{\lambda_\Omega(x), \lambda_\Omega(y), \vartheta\}$.

2) $\min\{(\mu_\Omega^-)^N(x-y), \alpha_1\} \leq max\{(\mu_\Omega^-)^N(x), (\mu_\Omega^-)^N(y), \beta_1\}$,

$\max\{(\mu_\Omega^-)^P(x-y), \alpha_2\} \geq min\{(\mu_\Omega^-)^P(x), (\mu_\Omega^-)^P(y), \beta_2\}$,

$\min\{(\mu_\Omega^+)^N(x-y),\alpha_1\} \le max\{(\mu_\Omega^+)^N(x),(\mu_\Omega^+)^N(y),\beta_1\}$,

$\max\{(\mu_\Omega^+)^P(x-y),\alpha_2\} \ge min\{(\mu_\Omega^+)^P(x),(\mu_\Omega^+)^P(y),\beta_2\}$,

$\min\{\lambda_\Omega(x-y),\omega\} \le max\{\lambda_\Omega(x),\lambda_\Omega(y),\vartheta\}$.

Example 3.2.

Let X = {0, 1, 2, 3} in which the operations $(+,-)$ be defined by the following tables:

Table 1: **a cubic bipolar fuzzy SA-subalgebra with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X**

+	**0**	**1**	**2**	**3**
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

−	**0**	**1**	**2**	**3**
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

then$(X;\ +,-,0)$ is an SA-algebra. Define a cubic fuzzy subset

$\Omega = <\tilde{\mu}_\Omega, \lambda_\Omega>$ of X of fuzzy subset $\mu_\Omega^-: X\to$ [-1,0] and $\mu_\Omega^+: X\to$ [0,1] by:

$$\tilde{\mu}_\Omega(x) = \begin{cases} [[-0.7,-0.4],[0.3,0.9]] & \text{if } x = \{0,2\} \\ [[-0.6,-0.3],[0.2,0.8]] & \text{otherwise} \end{cases}, \quad \lambda_\Omega = \begin{cases} 0.1 & if x = \{0,2\} \\ 0.6 & \text{otherwise} \end{cases},$$

$\tilde{\alpha} = [[-0.5,-0.3],[0.4,0.7]]$, $\tilde{\beta} = [[-0.4,-0.2],[0.5,0.8]]$, ω=0.4 and $\vartheta = 0.5$

Proposition 3. 3.

Let $\Omega_{(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})} =< \tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_\Omega{}^{(\omega,\vartheta)}(x) >$ be a cubic bipolar fuzzy SA-subalgebra with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of an SA-algebra $(X;+,-,0)$, then $\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}(0) \succcurlyeq \tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}(x)$ and $\lambda_\Omega{}^{(\omega,\vartheta)}(0) \le \lambda_\Omega{}^{(\omega,\vartheta)}(x)$, for all $x \in X$.

Proof:

For all $x \in X$, we have

$min\,\{(\mu_\Omega^-)^N(0),\alpha_1\} \le max\{(\mu_\Omega^-)^N(x),\alpha_1\}$,

$\max\{(\mu_\Omega^-)^P(0),\alpha_2\} \ge min\{(\mu_\Omega^-)^P(x),\alpha_2\}$,

$\min\{(\mu_\Omega^+)^N(0),\alpha_1\} \le max\{(\mu_\Omega^+)^N(x),\alpha_1\}$,

$\max\{(\mu_\Omega^+)^P(0),\alpha_2\} \ge min\{(\mu_\Omega^+)^P(x),\alpha_2\}$ and

$\min\{\lambda_\Omega(0),\omega\} \le max\{\lambda_\Omega(x),\vartheta\}$. Hence $\tilde{\mu}_\Omega(0) \succcurlyeq \tilde{\mu}_\Omega(x)$ and $\lambda_\Omega(0) \le \lambda_\Omega(x)$, for all $\in X$.

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Definition 3.4.

Let $(X;+,-,0)$ be an SA-algebra. A cubic bipolar fuzzy subset $\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)} =< \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\Omega}{}^{(\omega,\vartheta)}(x) >$of X, for $\tilde{t} = [t_1, t_2]$ such that $t_1 \in [-1,0)$, $t_2 \in [0,1]$ and $s \in [0,1]$, where $\tilde{\alpha} = [\alpha_1, \alpha_2]$, $\tilde{\beta} = [\beta_1, \beta_2]$, such that $\alpha_1, \beta_1 \subset [-1,0]$, $\alpha_2, \beta_2 \subset [0,1]$, $\omega, \vartheta \in [0,1]$, $\tilde{\alpha} < \tilde{\beta}$ and $\omega < \vartheta$.The set:

$$\tilde{U}(\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})} | [t_1, t_2]) = \{x \in X |\ \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x) \succcurlyeq [t_1, t_2]\}$$

$$= \{ x \in X \,|[(\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})})^N(x), (\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})})^P(x)\ ] \succcurlyeq [t_1, t_2]\}$$

$$= \begin{cases} \max\{(\mu_{\Omega}^{-})^N(x), \beta_1\} \le t_1 \text{ and } \min\{(\mu_{\Omega}^{-})^P(x), \beta_2\} \ge t_2, \\ \max\{(\mu_{\Omega}^{+})^N(x), \beta_1\} \le t_1 \text{ and } \min\{(\mu_{\Omega}^{+})^P(x), \beta_2\} \ge t_2. \end{cases}$$

is called **upper $[t_1, t_2]$-Level of $\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}$** and

$$L(\lambda_{\Omega}{}^{(\omega,\vartheta)} | s) = \{x \in X | \lambda_{\Omega}{}^{(\omega,\vartheta)}(x) \le s\} = \{x \in X | \max\{\lambda_{\Omega}(x), \vartheta\} \le s\}$$

is called **Lower s-Level of $\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}$**. And

$$\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}(\tilde{t}, s) = \Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}([t_1, t_2], s) = (\tilde{U}(\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}) | [t_1, t_2]) \cap (\mathrm{L}(\lambda_{\Omega}{}^{(\omega,\vartheta)} | \mathrm{s}))$$

$$= \tilde{U}\left(\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}; \tilde{t}, s\right) = \{x \in X |\ \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(\mathrm{x}) \succcurlyeq [\mathrm{t}_1, \mathrm{t}_2] \text{ and } \lambda_{\Omega}{}^{(\omega,\vartheta)}(x) \le \mathrm{s}\}.$$

$$= \begin{cases} \max\{(\mu_{\Omega}^{-})^N(x), \beta_1\} \le t_1 \text{ and } \min\{(\mu_{\Omega}^{-})^P(x), \beta_2\} \ge t_2, \\ \max\{(\mu_{\Omega}^{+})^N(x), \beta_1\} \le t_1 \text{ and } \min\{(\mu_{\Omega}^{+})^P(x), \beta_2\} \ge t_2. \end{cases}$$

and $\max\{\lambda_{\Omega}(x), \vartheta\} \le s\}$. is **a level subset of X**.

Theorem 3.5.

Let $(X;+,-,0)$ be an SA-algebra. A cubic fuzzy subset $\Omega = <\tilde{\mu}_{\Omega}, \lambda_{\Omega}>$ of X. If $\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)} =< \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\Omega}{}^{(\omega,\vartheta)}(x) >$ is a cubic bipolar fuzzy SA-subalgebra with thresholds $(\tilde{\alpha}, \tilde{\beta})$, (ω,ϑ) of X, then for some $\tilde{t} = [t_1, t_2]$ such that $t_1 \in [-1,0)$, $t_2 \in [0,1]$ and $s \in [0, 1]$, the set $\tilde{U}(\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}; \tilde{t}, s)$ is a SA-subalgebra of X.

Proof:

Assume that $\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)} =< \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\Omega}{}^{(\omega,\vartheta)}(x) >$is a cubic bipolar fuzzy

SA-subalgebra with thresholds $(\tilde{\alpha}, \tilde{\beta})$, (ω,ϑ) of X and let $\tilde{t} = [t_1, t_2]$ such that

$t_1 \in [-1,0)$, $t_2 \in [0,1]$ and $s \in [0, 1]$ such that $\tilde{U}(\Omega; \tilde{t}, s) \neq \emptyset$.

Let $x, y \in X$ such that $, y \in \tilde{U}(\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}; \tilde{t}, s)$, then

$$= \begin{cases} \max\{(\mu_{\Omega}^{-})^N(x), \beta_1\} \le t_1 \text{ and } \min\{(\mu_{\Omega}^{-})^P(x), \beta_2\} \ge t_2, \\ \max\{(\mu_{\Omega}^{+})^N(x), \beta_1\} \le t_1 \text{ and } \min\{(\mu_{\Omega}^{+})^P(x), \beta_2\} \ge t_2. \end{cases}$$

and $\max\{\lambda_{\Omega}(x), \vartheta\} \le s$.And

$$= \begin{cases} \max\{(\mu_\Omega^-)^N\ (y), \beta_1\} \le t_1 \text{ and } \min\{(\mu_\Omega^-)^P\ (y), \beta_2\} \ge t_2, \\ \max\{(\ \mu_\Omega^+)^N\ (y), \beta_1\} \le t_1 \text{ and } \min\{(\ \mu_\Omega^+)^P\ (y), \beta_2\} \ge t_2. \end{cases}$$

and $\max\{\lambda_\Omega\ (y), \vartheta\}\ \le s$

Since $\Omega_{(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})}$ is a cubic bipolar fuzzy SA-subalgebra with thresholds $(\tilde{\alpha}, \tilde{\beta})$, (ω,$\vartheta$) of X, we get

1) $in\ \{(\mu_\Omega^-)^N\ (x+y), \alpha_1\}\ \le max\{\ (\mu_\Omega^-)^N(x), \left(\mu_\Omega^-\right)^N(y), \beta_1\right\} \le t_1$,

$\max\{(\mu_\Omega^-)^P\ (x+y), \alpha_2\}\ \ge min\{\ (\mu_\Omega^-)^P(x), \left(\mu_\Omega^-\right)^P(y), \beta_2\right\} \ge t_2,$

$\min\{(\ \mu_\Omega^+)^N\ (x+y), \alpha_1\}\ \le max\{\ (\ \mu_\Omega^+)^N(x), (\ \mu_\Omega^+)^N(y), \beta_1\} \le t_1$,

$\max\{(\ \mu_\Omega^+)^P\ (x+y), \alpha_2\}\ \ge min\{\ (\ \mu_\Omega^+)^P(x), \left(\ \mu_\Omega^+\right)^P(y), \beta_2\right\} \ge t_2,$

$\min\{\lambda_\Omega\ (x+y), \omega\}\ \le\ max\{\lambda_\Omega(x), \lambda_\Omega(y), \vartheta\} \le s.$

2) $\min\{(\mu_\Omega^-)^N\ (x-y), \alpha_1\}\ \le max\{\ (\mu_\Omega^-)^N(x), \left(\mu_\Omega^-\right)^N(y), \beta_1\right\} \le t_1$,

$\max\{(\mu_\Omega^-)^P\ (x-y), \alpha_2\}\ \ge min\{\ (\mu_\Omega^-)^P(x), \left(\mu_\Omega^-\right)^P(y), \beta_2\right\} \ge t_2,$

$\min\{(\ \mu_\Omega^+)^N\ (x-y), \alpha_1\}\ \le max\{\ (\ \mu_\Omega^+)^N(x), (\ \mu_\Omega^+)^N(y), \beta_1\} \le t_1,$

$\max\{(\ \mu_\Omega^+)^P\ (x-y), \alpha_2\}\ \ge min\{\ (\ \mu_\Omega^+)^P(x), \left(\ \mu_\Omega^+\right)^P(y), \beta_2\right\} \ge t_2,$

$\min\{\lambda_\Omega\ (x-y), \omega\}\ \le\ max\{\lambda_\Omega(x), \lambda_\Omega(y), \vartheta\} \le s$.Therefore,

1) $(\mu_\Omega^-)^N\ (x+y) \le t_1$, $(\mu_\Omega^-)^P\ (x+y) \ge t_2$, $(\ \mu_\Omega^+)^N\ (x+y) \le t_1$, $(\ \mu_\Omega^+)^P\ (x+y) \ge t_2$ and $\lambda_\Omega\ (x+y) \le s$.

2) $(\mu_\Omega^-)^N\ (x-y) \le t_1$, $(\mu_\Omega^-)^P\ (x-y) \ge t_2$, $(\ \mu_\Omega^+)^N\ (x-y) \le t_1$,

$(\ \mu_\Omega^+)^P\ (x-y) \ge t_2$ and $\lambda_\Omega\ (x-y) \le s$. Then , $x+y\ , x-y \in \tilde{U}\ (\ \Omega_{(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})}\ ; \tilde{t}, s)$.

Hence the set $\tilde{U}\ (\Omega_{(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})}; \tilde{t}, s)$ is a SA-subalgebra of X.

Proposition 3.6.

Let $(X\ ; +, -, 0)$ be an SA-algebra. A cubic fuzzy subset Ω =<$\tilde{\mu}_\Omega(x)$, $\lambda_\Omega(x)$> of X. If for all $\tilde{t} = [t_1, t_2]$ such that $t_1 \in [-1,0)$, $t_2 \in [0,1]$ and s ∈ [0, 1], the set $\tilde{U}\ (\Omega_{(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})}; \tilde{t}, s)$ is a SA-subalgebra of X, then Ω is a cubic bipolar fuzzy SA-subalgebra with thresholds $(\tilde{\alpha}, \tilde{\beta})$, (ω,$\vartheta$) of X.

Proof:

Suppose that $\tilde{U}\ (\Omega_{(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})}; \tilde{t}, s)$is a SA-subalgebra of X. Let $x\ , y\ \in X$ such that

$\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}$ (x+y)≺rmin {$\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}$ (x),$\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}$ (y)}, and

$\lambda_\Omega{}^{(\omega,\vartheta)}$($x$+$y$) > max {$\lambda_\Omega{}^{(\omega,\vartheta)}$ (x),$\lambda_\Omega{}^{(\omega,\vartheta)}$(y)}. Consider

$\tilde{\delta} = 1/2$ ($\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}$ $(x+y)$ + rmin{$\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}$ (x), $\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}$ (y)}) and

$\sigma = 1/2$ ($\lambda_\Omega{}^{(\omega,\vartheta)}$ $(x+y)$ + max{$\lambda_\Omega{}^{(\omega,\vartheta)}$ (x), $\lambda_\Omega{}^{(\omega,\vartheta)}$ (y)}).

We have $\tilde{\delta}\in$ D[0, 1] and $\sigma\in$ [0, 1], and $\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}$ $(x+y) \prec \tilde{\delta} \prec$ rmin {$\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}$ (x),$\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}$ (y) } and

$\lambda_\Omega{}^{(\omega,\vartheta)}$ $(x+y) > \sigma >$ max {$\lambda_\Omega{}^{(\omega,\vartheta)}$ (x),$\lambda_\Omega{}^{(\omega,\vartheta)}$ (y) }. It follows that $x, y \in \tilde{U}$ $(\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}; \tilde{t}, s)$, and $(x+y) \notin \tilde{U}$ $\left(\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}; \tilde{t}, s\right)$.

This is contradiction.

Hence $\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}$ $(x+y) \succcurlyeq$ rmin{ $\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}$ (x),$\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}$ (y)} $\succcurlyeq \tilde{t}$ and $\lambda_\Omega{}^{(\omega,\vartheta)}$ $(x+y) \leq$ max {$\lambda_\Omega{}^{(\omega,\vartheta)}(x)$,$\lambda_\Omega{}^{(\omega,\vartheta)}$(y)} ≤ s.

Similarly, $\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}$ $(x-y) \succcurlyeq$ rmin{ $\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}$ (x),$\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}$ (y)} $\succcurlyeq \tilde{t}$ and $\lambda_\Omega{}^{(\omega,\vartheta)}$ $(x-y) \leq$ max {$\lambda_\Omega{}^{(\omega,\vartheta)}$ (x),$\lambda_\Omega{}^{(\omega,\vartheta)}$ (y)} ≤ s.

Therefore $\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}$ is a cubic bipolar fuzzy SA-subalgebra with thresholds $(\tilde{\alpha}, \tilde{\beta})$, (ω,ϑ) of X.

4. Cubic Bipolar Fuzzy SA-ideals with thresholds $(\tilde{\alpha}, \tilde{\beta})$, (ω,$\vartheta$) of SA-algebra

Definition 4.1.

Let $(X\,; +, -, 0)$ be an SA-algebra, a cubic set $\Omega =< \tilde{\mu}_\Omega$ (x)$, \lambda_\Omega$(x) $>$ of X where $\tilde{\alpha} = [\alpha_1, \alpha_2]$, $\tilde{\beta} = [\beta_1, \beta_2]$ such that $\tilde{\alpha} < \tilde{\beta}$ and $\alpha_1, \beta_1 \subseteq [-1,0]$, $\alpha_2, \beta_2 \subseteq [0,1]$ and $\omega, \vartheta \in [0,1]$. If for all $x, y \in X$. $\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)} =< \tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_\Omega{}^{(\omega,\vartheta)}(x) >$ is called **a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha}, \tilde{\beta})$, (ω,$\vartheta$) of X**, if for all $x, y, z \in X$:

1- $\text{rmin}\{(\tilde{\mu}_\Omega)^N(0), \tilde{\alpha}\} \preccurlyeq \text{rmax}\{(\tilde{\mu}_\Omega)^N(x), \tilde{\beta}\}$,

$\text{rmax}\{(\tilde{\mu}_\Omega)^P(0), \tilde{\alpha}\} \succcurlyeq \text{rmin}\{(\tilde{\mu}_\Omega)^P(x), \tilde{\beta}\}$,

$\min\{\lambda_\Omega(0), \omega\} \leq \max\{\lambda_\Omega(x), \vartheta\}$.

2- $\text{rmin}\{(\tilde{\mu}_\Omega)^N(x+y), \tilde{\alpha}\} \preccurlyeq \text{rmax}\{(\tilde{\mu}_\Omega)^N(x+z), (\tilde{\mu}_\Omega)^N(y-z), \tilde{\beta}\}$,

$\text{rmax}\{(\tilde{\mu}_\Omega)^P(x+y), \tilde{\alpha}\} \succcurlyeq \text{rmin}\{(\tilde{\mu}_\Omega)^P(x+z), \tilde{\mu}_\Omega^P(y-z), \tilde{\beta}\}$,

$\min\{\lambda_\Omega(x+y), \omega\} \leq \max\{\lambda_\Omega(x+z), \lambda_\Omega(y-z), \vartheta\}$.

i.e.,

1- $\min\{(\tilde{\mu}_\Omega)^N(0), \alpha_1\} \leq \max\{(\tilde{\mu}_\Omega)^N(x), \beta_1\}$,

$\max\{(\tilde{\mu}_\Omega)^P(0), \alpha_2\} \geq \min\{(\tilde{\mu}_\Omega)^P(x), \beta_2\}$,

$\min\{\lambda_\Omega(0), \omega\} \leq \max\{\lambda_\Omega(x), \vartheta\}$.

2- $\min\{(\tilde{\mu}_\Omega)^N(x+y), \alpha_1\} \leq \max\{(\tilde{\mu}_\Omega)^N(x+z), (\tilde{\mu}_\Omega)^N(y-z), \beta_1\}$,

$\max\{(\tilde{\mu}_\Omega)^P(x+y), \alpha_2\} \geq \min\{(\tilde{\mu}_\Omega)^P(x+z), (\tilde{\mu}_\Omega)^P(y-z), \beta_2\}$,

$\min\{\lambda_\Omega(x+y), \omega\} \leq \max\{\lambda_\Omega(x+z), \lambda_\Omega(y-z), \vartheta\}$.

i.e.,

1- $\min\{(\mu_\Omega^-)^N\ (0), \alpha_1\} \le \max\{ (\mu_\Omega^-)^N(x), \beta_1\}$,

$\max\{(\mu_\Omega^-)^P\ (0), \alpha_2\} \ge \min\{ (\mu_\Omega^-)^P(x)\beta_2, \}$,

$\min\{(\mu_\Omega^+)^N\ (0), \alpha_1\} \le \max\{ (\mu_\Omega^+)^N(x), \beta_1\}$,

$\max\{(\mu_\Omega^+)^P\ (0), \alpha_2\} \ge \min\{ (\mu_\Omega^+)^P(x), \beta_2\}$,

$\min\{\lambda_\Omega\ (0), \omega\} \le \max\{\lambda_\Omega(x), \vartheta\}$.

2- $\min\{(\mu_\Omega^-)^N\ (x+y), \alpha_1\} \le \max\{ (\mu_\Omega^-)^N(x+z), (\mu_\Omega^-)^N(y-z), \beta_1\}$,

$\max\{(\mu_A^-)^P\ (x+y), \alpha_2\} \ge \min\{ (\mu_\Omega^-)^P(x+z), (\mu_\Omega^-)^P(y-z), \beta_2\}$,

$\min\{(\mu_\Omega^+)^N\ (x+y), \alpha_1\} \le \max\{ (\mu_\Omega^+)^N(x+z), (\mu_\Omega^+)^N(y-z), \beta_1\}$,

$\max\{(\mu_\Omega^+)^P\ (x+y), \alpha_2\} \ge \min\{ (\mu_\Omega^+)^P(x+z), (\mu_\Omega^+)^P(y-z), \beta_2\}$,

$\min\{\lambda_\Omega\ (x+y), \omega\} \le \max\{\lambda_\Omega(x+z), \lambda_\Omega(y-z), \vartheta\}$.

<u>Example 4.2.</u>

Let X ={ 0,a, b, c} be a set with the following tables:

Table 2: **a cubic bipolar fuzzy SA-ideal with thresholds ($\tilde{\alpha}, \tilde{\beta}$), (ω,$\vartheta$) of X**

+	**0**	**a**	**b**	**c**
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

−	**0**	**a**	**b**	**c**
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then $(X;+,-,0)$ is an SA-algebra, $I = \{0, a\}$ is an SA-ideal of X. We define cubic set $\Omega = \{(x, \tilde{\mu}_\Omega\ (x), \lambda_\Omega\ (x)) \mid x \in X\}$

$$\tilde{\mu}_\Omega(x) = \begin{cases} [[-0.6,-0.5],[0.5,0.8]] & \text{if } x \in I \\ [[-0.5,-0.4],[0.4,0.7]] & \text{otherwise} \end{cases},\quad \lambda_\Omega(x) = \begin{cases} 0.2 & \text{if } x \in I, \\ 0.6 & \text{otherwise} \end{cases}$$

, $\tilde{\alpha} = [[-0.5,-0.3],[0.4,0.8]]$, $\tilde{\beta} = [[-0.4,-0.2],[0.5,0.9]]$, $\omega = 0.3$, $\vartheta = 0.5$

Then Ω is cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha}, \tilde{\beta})$, (ω,$\vartheta$) of X .

<u>Proposition 4.3.</u>

The R-intersection of any set of cubic bipolar fuzzy SA-ideals with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,$\vartheta$) of X is also a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,$\vartheta$) of X.

Proof:

Let $\Omega_{i}{}^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)} =< \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\Omega i}{}^{(\omega,\vartheta)}(x) >$ be family of cubic fuzzy SA-ideals of X, then for any $x, y, z \in X$,

$\left(\bigcap \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}\right)(0)$ =rinf $(\tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(0)) \succcurlyeq$ rinf $(\tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x)) = (\bigcap \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})})(x)$ and

$\left(\bigvee \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(0) = \sup \lambda_{\Omega i}{}^{(\omega,\vartheta)}(0) \leq \sup \lambda_{\Omega i}{}^{(\omega,\vartheta)}(y) = \left(\bigvee \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(y)$.

$(\bigcap \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x+y))$ = rinf $(\tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x+y))$

$\succcurlyeq$ rinf (rmin $\{\tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x+z), \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(y-z)\}$)

= rmin { rinf ($\tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x+z)$,rinf ($\tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(y-z)$)}

= rmin { $(\bigcap \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})})(x+z)$,$(\bigcap \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})})(y-z)$}

$\left(\bigvee \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(x+y) = \sup \lambda_{\Omega i}{}^{(\omega,\vartheta)}(x+y)$

$\leq \sup\{\max\{\lambda_{\Omega i}{}^{(\omega,\vartheta)}(x+z), \lambda_{\Omega i}{}^{(\omega,\vartheta)}(y-z)\}\}$

= max{sup $(\lambda_{\Omega i}{}^{(\omega,\vartheta)}(x+z))$,sup $(\lambda_{\Omega i}{}^{(\omega,\vartheta)}(y-z))$}

$= \max\{\left(\bigvee \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(x+z), \left(\bigvee \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(y-z)\}$.

Hence, R-intersection of $\Omega_{i}{}^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}$ is a cubic bipolar bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,$\vartheta$) of X.

Remark 2.2.4.

The P-intresection of any set of cubic bipolar fuzzy SA-ideals with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,$\vartheta$) need not be a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,$\vartheta$), for example:

Example 4.5.

Let X = {0, 1, 2, 3, 4,5} be a set with the following tables:

Table 3: **Ω_1 and Ω_2 are cubic bipolar fuzzy SA-ideals with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,$\vartheta$) of X**

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1

−	0	1	2	3	4	5
0	0	5	4	3	2	1
1	1	0	5	4	3	2
2	2	1	0	5	4	3

3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

3	3	2	1	0	5	4
4	4	3	2	1	0	5
5	5	4	3	2	1	0

Then $(X;*,0)$ is an SA-algebra, I $=\{0, 3\}$ and J $=\{0, 2, 4\}$ are SA-ideals of X .We define two cubic set $\Omega_1 = <\tilde{\mu}_{\Omega 1}, \lambda_{\Omega 1}>$

and $\Omega_2 = <\tilde{\mu}_{\Omega 2}, \lambda_{\Omega 2}>$ of X by : $\tilde{\mu}_{\Omega 1}(x) = \begin{cases} [[-0.7,-0.6],[0.6,0.7]] & \text{if } x \in I, \\ [[-0.5,-0.3],[0.1,0.2]] & \text{if } x \in \{1,2\}, \\ [[-0.4,-0.3],[0.3,0.4]] & \text{otherwise.} \end{cases}$ $\lambda_{\Omega 1}(x) = \begin{cases} 0.1 & \text{if } x \in I, \\ 0.6 & \text{if } x \in \{1,2\}, \\ 0.2 & \text{otherwise.} \end{cases}$

$\tilde{\mu}_{\Omega 2}(x) = \begin{cases} [[-0.8,-0.4],[0.8,0.9]] & \text{if } x \in \{0,2,4\}, \\ [[-0.5,-0.4],[0.3,0.4]] & \text{otherwise.} \end{cases}$ $\lambda_{\Omega 2}(x) = \begin{cases} 0.1, & \text{if } x \in \{0,2,4\}, \\ 0.4, & \text{otherwise.} \end{cases}$,

$\tilde{\alpha} = [[-0.5,-0.3],[0.2,0.6]]$, $\tilde{\beta} = [[-0.4,-0.2],[0.5,0.9]]$, $\omega = 0.3$ and $\vartheta = 0.5$

Then Ω_1 and Ω_2 are cubic bipolar fuzzy SA-ideals with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X , but P-intersection of Ω_1 and Ω_2 are not cubic bipolar fuzzy SA-ideals with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X . Since

Let $x = 2, y = 5, z = 2$, we have: $max\{(\tilde{\mu}_{\Omega 1})^P \cap \tilde{\mu}_{\Omega 2})^P)(2+5), \alpha_2\} = max\{min\{(\tilde{\mu}_{\Omega 1})^P, \tilde{\mu}_{\Omega 2})^P)\}(1), \alpha_2\}$

$= max\{min\{[0.1,0.2],[0.3,0.4]\},[0.2,0.6]\}$

$max\{[0.1,0.2],[0.2,0.6]\} = [0.2,0.6]$

$\ngeq min\{(\tilde{\mu}_{\Omega 1})^P \cap \tilde{\mu}_{\Omega 2})^P)(2+2), (\tilde{\mu}_{\Omega 1})^P \cap \tilde{\mu}_{\Omega 2})^P)(5-2), \beta_2\}$

$min\{min\{[0.3,0.4],[0.8,0.9]\}, min\{[0.6,0.7],[0.3,0.4]\},[0.5,0.9]\}$

$min\{[0.3,0.4],[0.3,0.4],[0.5,0.9]\} = [0.3,0.4]$.

Proposition 4.6.

Let $\Omega_{i(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})} =< \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\Omega i}{}^{(\omega,\vartheta)}(x) >$ be a family of cubic bipolar fuzzy SA-ideals with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of an SA-algebra $(X;+,-,0)$, where $i \in \Lambda$, $\inf\{\max\{\lambda_{\Omega i}{}^{(\omega,\vartheta)}(x), \lambda_{\Omega i}{}^{(\omega,\vartheta)}(y)\}\} = \max\{\inf \lambda_{\Omega i}{}^{(\omega,\vartheta)}(x), \inf \lambda_{\Omega i}{}^{(\omega,\vartheta)}(y)\}$, for all $x, y \in X$, then the P-intresection of $\Omega_{i(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})}$ is also a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X.

Proof:

Let $\Omega_{i(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})} =< \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\Omega i}{}^{(\omega,\vartheta)}(x) >$ where $i \in \Lambda$, be a set of cubic bipolar fuzzy SA-ideals with thresholds

$(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of , for all $x, y, z \in X$

$\left(\cap \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}\right)(0) = \text{rinf}\, (\tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(0)) \succcurlyeq \text{rinf}\, (\tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x)) = (\cap \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})})(x)$ and

$\left(\wedge \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(0) = \inf \lambda_{\Omega i}{}^{(\omega,\vartheta)}(0) \leq \inf \lambda_{\Omega i}{}^{(\omega,\vartheta)}(x) = \left(\wedge \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(x)$.

$\left(\cap \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}\right)(x+y) = \text{rinf}\, \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x+y)$

$\succcurlyeq \text{rinf}\{\text{rmin}\{\tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x+z), \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(y-z)\}\}$

$= \text{rmin}\{\text{rinf}\, \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x+z), \text{rinf}\, \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(y-z)\}$

$= \text{rmin}\left\{\left(\cap \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}\right)(x+z), \left(\cap \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}\right)(y-z)\right\}$

and $\left(\wedge \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(x+y) = \inf \lambda_{\Omega i}{}^{(\omega,\vartheta)}(x+y)$

$\leq \inf\{\max\{\lambda_{\Omega i}{}^{(\omega,\vartheta)}(x+z), \lambda_{\Omega i}{}^{(\omega,\vartheta)}(y-z)\}\}$

$= \max\{\inf \lambda_{\Omega i}{}^{(\omega,\vartheta)}(x+z), \inf \lambda_{\Omega i}{}^{(\omega,\vartheta)}(y-z)\}$

$= \max\{\left(\wedge \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(x+z), \left(\wedge \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(y-z)\}$.

$\leq \min\{\left(\wedge \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(x+z), \left(\wedge \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(y-z)\}$.

Hence, P-intersection of $\Omega_{i}{}^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}$ is a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X.

Remark 4.7.

The P -union of any set of cubic bipolar fuzzy SA-ideals with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) need not be a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ), for example:

Example 4.8.

By using example (4.5), we can see that Ω_1 and Ω_2 are cubic bipolar fuzzy SA-ideals with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X , but P–union of Ω_1 and Ω_2 are not cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X . Since

$max\{((\tilde{\mu}_{\Omega 1})^p \cup (\tilde{\mu}_{\Omega 2})^p)(2+5), \alpha_2\}$

$= max\{((\tilde{\mu}_{\Omega 1})^p \cup (\tilde{\mu}_{\Omega 2})^p)(1), \alpha_2\}$

$= max\{max\{[0.1,0.2], [0.3,0.4]\}, [0.2,0.6]\}$

$= max\{[0.3,0.4], [0.2,0.6]\}$

$= [0.3,0.6]$

$\not\succeq min\{((\tilde{\mu}_{\Omega 1})^p \cup (\tilde{\mu}_{\Omega 2})^p)(2+2), ((\tilde{\mu}_{\Omega 1})^p \cup (\tilde{\mu}_{\Omega 2})^p)(5-2), \beta_2\}$

$= min\{max\{[0.8,0.9], [0.6,0.7]\}, [0.5,0.9]\}$

$min\{[0.8,0.9], [0.5,0.9]\} = [0.5,0.9]$.

Proposition 4.9.

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Let $\Omega_{i_{(\omega,\vartheta)}}^{(\tilde{\alpha},\tilde{\beta})} =< \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\Omega i}{}^{(\omega,\vartheta)}(x) >$ be a family of cubic bipolar fuzzy SA-ideals with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of an SA-algebra, where $i \in \Lambda$, $\text{rsup}\{\text{rmin}\{\tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(y)\}\} = \text{rmin}\{\text{rsup}\, \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \text{rsup}\, \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(y)\}$ for all $x, y \in X$, then the P-union of $\Omega_{i_{(\omega,\vartheta)}}^{(\tilde{\alpha},\tilde{\beta})}$ is also a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X.

Proof.

Let $\Omega_{i_{(\omega,\vartheta)}}^{(\tilde{\alpha},\tilde{\beta})} =< \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\Omega i}{}^{(\omega,\vartheta)}(x) >$ where $i \in \Lambda$, be a set of cubic bipolar fuzzy SA-ideals with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X and let $x, y, z \in X$, then $\left(\cup\, \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}\right)(0) = \text{rsup}\,(\tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(0)) \succcurlyeq \text{rsup}\,(\tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x)) = (\cup\, \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})})(x)$ and

$$\left(\vee\, \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(0) = \sup \lambda_{\Omega i}{}^{(\omega,\vartheta)}(0) \le \sup \lambda_{\Omega i}{}^{(\omega,\vartheta)}(x) = \left(\vee\, \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(x).$$

$$\left(\cup\, \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}\right)(x+y) = \text{rsup}\ \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x+y)$$

$$\succcurlyeq \text{rsup}\{\text{rmin}\{\tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x+z), \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(y-z)\}\}$$

$$\succcurlyeq \text{rmin}\{\text{rsup}\, \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x+z), \text{rsup}\, \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(y-z)\}$$

$$= \text{rmin}\ \left\{\left(\cup\, \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}\right)(x+z), \left(\cup\, \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}\right)(y-z)\right\}.$$

$$\left(\vee\, \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(x+y) = \sup \lambda_{\Omega i}{}^{(\omega,\vartheta)}(x+y)$$

$$\le \sup\{\max\{\lambda_{\Omega i}{}^{(\omega,\vartheta)}(x+z), \lambda_{\Omega i}{}^{(\omega,\vartheta)}(y-z)\}\}$$

$$= \max\{\sup \lambda_{\Omega i}{}^{(\omega,\vartheta)}(x+z), \sup \lambda_{\Omega i}{}^{(\omega,\vartheta)}(y-z)\}$$

$$= \max\{\left(\vee\, \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(x+z), \left(\vee\, \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(y-z)\},$$

Hence, P-union of Ω_i is a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X.

Remark 4.10.

The R-union of any sets of cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) need not be a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ), see example (4.8) .

Proposition 2.2.11.

Let $\Omega_{i_{(\omega,\vartheta)}}^{(\tilde{\alpha},\tilde{\beta})} =< \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\Omega i}{}^{(\omega,\vartheta)}(x) >$ be a family of cubic bipolar fuzzy SA-ideals with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of an SA-algebra where $i \in \Lambda$, $\text{rsup}\{\text{rmin}\{\tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(y)\}\} = \text{rmin}\{\text{rsup}\, \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \text{rsup}\, \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(y)\}$ and $inf\left\{max\{\lambda_{\Omega i}{}^{(\omega,\vartheta)}(x), \lambda_{\Omega i}{}^{(\omega,\vartheta)}(y)\}\right\} = max\{inf\, \lambda_{\Omega i}{}^{(\omega,\vartheta)}(x), inf\, \lambda_{\Omega i}{}^{(\omega,\vartheta)}(y)\}$. for all $x, y \in X$, then the R-union of $\Omega_{i_{(\omega,\vartheta)}}^{(\tilde{\alpha},\tilde{\beta})}$ is also a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X.

Proof:

Let $\Omega_{i_{(\omega,\vartheta)}}^{(\tilde{\alpha},\tilde{\beta})}$ where $i \in \Lambda$ be a family of cubic bipolar fuzzy SA-ideals with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of , then for $x, y, z \in X$,

$\left(\cup\, \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}\right)(0) = \text{rsup}\,(\tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(0)) \succcurlyeq \text{rsup}\,(\tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x)) = (\cup\, \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})})(x)$ and

$$\left(\wedge\, \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(0) = \inf \lambda_{\Omega i}{}^{(\omega,\vartheta)}(0) \le \inf \lambda_{\Omega i}{}^{(\omega,\vartheta)}(x) = \left(\wedge\, \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(x).$$

$\left(\cup \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}\right)(x+y) = \text{rsup}\, \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x+y)$

$\succcurlyeq \text{rsup}\{\text{rmin}\{\tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x+z), \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(y-z)\}\}$

$= \text{rmin}\{\text{rsup}\, \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(x+z), \text{rsup}\, \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}(y-z)\}$

$= \text{rmin}\left\{\left(\cup \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}\right)(x+z), \left(\cup \tilde{\mu}_{\Omega i}{}^{(\tilde{\alpha},\tilde{\beta})}\right)(y-z)\right\}$

and $\left(\wedge \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(x+y) = \inf \lambda_{\Omega i}{}^{(\omega,\vartheta)}(x+y)$

$\leq \inf\{\max\{\lambda_{\Omega i}{}^{(\omega,\vartheta)}(x+z), \lambda_{\Omega i}{}^{(\omega,\vartheta)}(y-z)\}\}$

$= \max\{\inf \lambda_{\Omega i}{}^{(\omega,\vartheta)}(x+z), \inf \lambda_{\Omega i}{}^{(\omega,\vartheta)}(y-z)\}$

$= \max\{\left(\wedge \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(x+z), \left(\wedge \lambda_{\Omega i}{}^{(\omega,\vartheta)}\right)(y-z)\}.$

Hence, R-union of $\Omega_{i}{}^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}$ is a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X.

Proposition 4.12.

Let $(X\,;+,-,0)$ be an SA-algebra. If a cubic bipolar fuzzy subset $\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)} = < \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\Omega}{}^{(\omega,\vartheta)}(x) >$ of , then $\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}$ is a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X, then for some $\tilde{t} = [t_1, t_2]$ such that $t_1 \in [-1,0)$, $t_2 \in [0,1]$ and s $\in [0, 1]$, the set $\widetilde{U}\,(\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}; \tilde{t}, s)$ is an SA-ideal of X.

Proof:

Assume that $\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}$ is a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X and let $\tilde{t}=[t_1, t_2]$ such that $t_1\ [-1,0)$, $t_2 \in [1,0]$ and s $\in [0, 1]$, such that $\widetilde{U}\,(\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}; \tilde{t}, s) \neq \emptyset$.

Let $x, y, z \in X$ such that $+z$, $y-z \in \widetilde{U}\,(\Omega; \tilde{t}, s)$, then $\tilde{\mu}_{\Omega}\,(x+z) \succcurlyeq \tilde{t}$, $\tilde{\mu}_{\Omega}\,(y-z) \succcurlyeq \tilde{t}$

and $\lambda_{\Omega}\,(x+z) \leq$ s, $\lambda_{\Omega}\,(y-z) \leq$ s. Since $\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}$ is a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X, we get:

$\tilde{\mu}_{\Omega}\,(x+y) \succcurlyeq \text{rmin}\{\tilde{\mu}_{\Omega}\,(x+z), \tilde{\mu}_{\Omega}\,(y-z)\} \succcurlyeq \tilde{t}$ and $\lambda_{\Omega}\,(x+y) \leq \max\{\lambda_{\Omega}\,(x+z), \lambda_{\Omega}\,(y-z)\} \leq$ s.

$\tilde{\mu}_{\Omega}(x+y) \succcurlyeq \tilde{t}$ and $\lambda_{\Omega}(x+y) \leq$ s

$\Rightarrow x+y \in \widetilde{U}\left(\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}; \tilde{t}, s\right)$. Hence the set $\widetilde{U}\,(\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}; \tilde{t}, s)$ is an SA-ideal of X.

Proposition 4.13.

Let $(X\,;+,-,0)$ be an SA-algebra. A cubic fuzzy subset $\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)} = < \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\Omega}{}^{(\omega,\vartheta)}(x) >$ of . If for all $\tilde{t} = [t_1, t_2]$ such that $t_1 \in [-1,0)$, $t_2 \in [0,1]$and s $\in [0, 1]$, the set $\widetilde{U}\,(\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}; \tilde{t}, s)$ is an SA-ideal of X, then $\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}$ is a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X.

Proof:

Suppose that $\widetilde{U}\ (\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)};\tilde{t},s)$ is an SA-ideal of X and $\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(0) \succcurlyeq \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x) \succcurlyeq \tilde{t}$ and $\lambda_{\Omega}{}^{(\omega,\vartheta)}(0) \leq \lambda_{\Omega}{}^{(\omega,\vartheta)}(x) \leq s$. Let $x, y, z \in X$ be such that $\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x+y) \prec \text{rmin}\{\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x+z), \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(y-z)\}$, and

$\lambda_{\Omega}{}^{(\omega,\vartheta)}(x+y) > \max\{\lambda_{\Omega}{}^{(\omega,\vartheta)}(x+z), \lambda_{\Omega}{}^{(\omega,\vartheta)}(y-z)\}$.

Consider $\tilde{\delta} = 1/2\ (\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x+y) + \text{rmin}\{\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x+z), \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(y-z)\})$ and

$\sigma = 1/2\ (\lambda_{\Omega}{}^{(\omega,\vartheta)}(x+y) + \max\{\lambda_{\Omega}{}^{(\omega,\vartheta)}(x+z), \lambda_{\Omega}{}^{(\omega,\vartheta)}(y-z)\})$. We have $\tilde{\delta} \in D[0,1]$ and

$\sigma \in [0,1]$, and $\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x+y) \prec \tilde{\delta} \prec \text{rmin}\{\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x+z), \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(y-z)\}$, and

$\lambda_{\Omega}{}^{(\omega,\vartheta)}(x+y) > \sigma > \max\{\lambda_{\Omega}{}^{(\omega,\vartheta)}(x+z), \lambda_{\Omega}{}^{(\omega,\vartheta)}(y-z)\}$.

It follows that $x, y \in \widetilde{U}\ (\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)};\tilde{t},s)$, and $(x+y) \notin \widetilde{U}\left(\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)};\tilde{t},s\right)$.

This is contradiction. Hence $\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x+y) \succcurlyeq \text{rmin}\{\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x+z), \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(y-z)\} \succcurlyeq \tilde{t}$

And $\lambda_{\Omega}{}^{(\omega,\vartheta)}(x+y) \leq \max\{\lambda_{\Omega}{}^{(\omega,\vartheta)}(x+z), \lambda_{\Omega}{}^{(\omega,\vartheta)}(y-z)\} \leq s$.

Therefore $\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)}$ is a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X.

Theorem 4.14.

Every cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of a SA-algebra $(X;\ +,-,0)$ is a cubic bipolar fuzzy SA-subalgebra with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X.

Proof:

Let $(X\ ;+,-,0)$ be an SA-algebra and $\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)} = < \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\Omega}{}^{(\omega,\vartheta)}(x) >$ is a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X, then by Proposition (4.12), for every $\tilde{t}=[t_1,\ t_2]$ such that $t_1 \in [-1,0)$, $t_2 \in [0,1]$

and $s \in [0,1]$, $\widetilde{U}\ (\Omega;\tilde{t},s) = \left\{x \in X \mid \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x) \succcurlyeq \tilde{t}, \lambda_{\Omega}{}^{(\omega,\vartheta)}(x) \leq s\right\}$, is SA-ideal of X.

By Proposition (2.5), for some $\tilde{t}=[t_1,\ t_2]$ such that $t_1 \in [-1,0)$, $t_2 \in [0,1]$ and $s \in [0,1]$, $\widetilde{U}\left(\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)};\tilde{t},s\right)$ is SA-subgalgebra of X. Hence $\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)} = < \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\Omega}{}^{(\omega,\vartheta)}(x) >$ is a cubic bipolar fuzzy SA-subalgebra with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X by Proposition(3.6).

Remark 4.15.

The converse of Theorem (4.14) is not true as the following example:

Example 4.16.

By using example (3.2) . Define a cubic fuzzy subset

$\Omega = < \tilde{\mu}_{\Omega}, \lambda_{\Omega} >$ of X of fuzzy subset $\mu_{\Omega}^{-}: X \to [-1,0]$ and $\mu_{\Omega}^{+}: X \to [0,1]$ by:

$$\tilde{\mu}_{\Omega}(x) = \begin{cases} [[-0.6\,,-0.3],[0.3,0.9]] & \text{if } x = \{0,2\} \\ [[-0.5\,,-0.4],[0.1,0.9]] & \text{otherwise} \end{cases},\quad \lambda_{\Omega} = \begin{cases} 0.1 & if x = \{0,2\} \\ 0.6 & \text{otherwise} \end{cases},$$

$\tilde{\alpha} = \left[[-0.5,-0.3],[0.4,0.7]\right]$, $\tilde{\beta} = \left[[-0.4,-0.2],[0.5,0.8]\right]$, ω=0.4 and $\vartheta = 0.5$

$\Omega_{(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})} =< \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\Omega}{}^{(\omega,\vartheta)}(x) >$ is a cubic bipolar fuzzy SA-subalgebra with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,$\vartheta$) of X, but it is not a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,$\vartheta$) of X. Since

Let $x = 1, y = 2, z = 3$

$max\{(\tilde{\mu}_{\Omega})^p(x+y), \alpha_2\} \geq min\{(\tilde{\mu}_{\Omega})^p(x+z), (\tilde{\mu}_{\Omega})^p(y-z), \beta_2\}$

$max\{(\tilde{\mu}_{\Omega})^p(1+2), \alpha_2\} \geq min\{(\tilde{\mu}_{\Omega})^p(1+3), (\tilde{\mu}_{\Omega})^p(2-3), \beta_2\}$

$max\{[0.1,0.6],[0.2,0.8]\} \geq min\{[0.3,0.9],[0.3,0.9],[0.4,0.9]\}$.But $[0.2,0.8] \ngeq [0.1,0.9]$.

5.Homomorphism of Cubic Bipolar Fuzzy SA-subalgebras (ideals) with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,$\vartheta$) on SA-algebra

In this part, we will present some results on images and preimages of cubic bipolar fuzzy SA-subalgebra with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,$\vartheta$) and cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,$\vartheta$) of SA-algebra.

Definition 5.1.

Let $f: (X; +, -, 0) \to (Y; +', -', 0')$ be a mapping from the set X to a set Y. If $\Omega_{(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})} =< \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\Omega}{}^{(\omega,\vartheta)}(x) >$ is a cubic bipolar fuzzy subset of X, then the cubic bipolar fuzzy subset $\pi_{(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})} =< \tilde{\mu}_{\pi}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\pi}{}^{(\omega,\vartheta)}(x) >$ with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,$\vartheta$) of Y defined by: $f\left(\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}\right)(y) = \tilde{\mu}_{\pi}{}^{(\tilde{\alpha},\tilde{\beta})}(y) = \begin{cases} \underset{x\in f^{-1}(y)}{rsup}\ \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x) if\ f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ 0 \quad \text{otherwise} \end{cases}$

$$f(\lambda_{\Omega}{}^{(\omega,\vartheta)})(y) = \lambda_{\pi}{}^{(\omega,\vartheta)}(y) = \begin{cases} \underset{x\in f^{-1}(y)}{rinf}\ \lambda_{\Omega}{}^{(\omega,\vartheta)}(x) if\ f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ 1 \quad \text{otherwise} \end{cases}$$

is said to be **the image of $\Omega_{(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})}$ under f.** Similarly, if $\pi_{(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})} =< \tilde{\mu}_{\pi}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\pi}{}^{(\omega,\vartheta)}(x) >$ is a cubic bipolar fuzzy subset of Y, then the cubic bipolar fuzzy subset $\Omega_{(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})} = (\pi \circ f)$ in X. (i.e., the cubic bipolar fuzzy subset defined by $\tilde{\mu}_{f^{-1}(\pi)}(x) = \tilde{\mu}_{\pi}(f(x))$, $\lambda_{f^{-1}(\pi)}(x) = \lambda_{\pi}(f(x))$ for all $x \in X$) is called **the preimage of β under f**).

Theorem 5.2.

A homomorphic preimage of cubic bipolar fuzzy SA-subalgebra with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,$\vartheta$) is also a cubic bipolar fuzzy SA-subalgebra with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,$\vartheta$) of SA-algebra.

Proof:

Let $f: (X; +, -, 0) \to (Y; +', -', 0')$be homomorphism from an SA -algebra X into an SA-algebra Y. If $\pi_{(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})} =< \tilde{\mu}_{\pi}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\pi}{}^{(\omega,\vartheta)}(x) >$ is cubic bipolar fuzzy SA-subalgebra with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,$\vartheta$) of Y and

$\Omega_{(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})} =< \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\Omega}{}^{(\omega,\vartheta)}(x) >$ the preimage of π under f, then $\tilde{\mu}_{f^{-1}(\pi)}{}^{(\tilde{\alpha},\tilde{\beta})}(x) = \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(f(x))$,

$\lambda_{f^{-1}(\pi)}{}^{(\omega,\vartheta)}(x) = \lambda_{\Omega}{}^{(\omega,\vartheta)}(f(x))$,for all $x \in X$. Let $x \in X$ by Definition (5.1). Then

$(\tilde{\mu}_{f^{-1}(\pi)}{}^{(\tilde{\alpha},\tilde{\beta})}0) = \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(f(0)) \succcurlyeq \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(f(x)) = \tilde{\mu}_{f^{-1}(\pi)}{}^{(\tilde{\alpha},\tilde{\beta})}(x)$ and

$(\lambda_{f^{-1}(\pi)}{}^{(\omega,\vartheta)})(0) = \lambda_{\Omega}{}^{(\omega,\vartheta)}\ (f\ (0)) \leq \lambda_{\Omega}{}^{(\omega,\vartheta)}\ (f\ (x)) = \lambda_{f^{-1}(\pi)}{}^{(\omega,\vartheta)}\ (x)$.

Now, let $x, y \in X$, then $\tilde{\mu}_{f^{-1}(\pi)}{}^{(\tilde{\alpha},\tilde{\beta})}\ (x+y) = \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}\ (f\ (x+y)) = \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}\ (f\ (x)+' f(y))$

$\succcurlyeq$ rmin $\{\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}\ (f\ (x)), \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}\ (f\ (y))\}$ = rmin $\{\tilde{\mu}_{f^{-1}(\pi)}{}^{(\tilde{\alpha},\tilde{\beta})}\ (x), \tilde{\mu}_{f^{-1}(\pi)}{}^{(\tilde{\alpha},\tilde{\beta})}\ (y)\}$

And $\lambda_{f^{-1}(\pi)}{}^{(\omega,\vartheta)}(x+y) = \lambda_{\Omega}{}^{(\omega,\vartheta)}(f\ (x+y)) = \lambda_{\Omega}{}^{(\omega,\vartheta)}(f\ (x)+' f(y))$

$\leq$ max $\{\lambda_{\Omega}{}^{(\omega,\vartheta)}\ (f\ (x)), \lambda_{\Omega}{}^{(\omega,\vartheta)}\ (f\ (y))\}$

= max $\{\lambda_{f^{-1}(\pi)}{}^{(\omega,\vartheta)}\ (x), \lambda_{f^{-1}(\pi)}{}^{(\omega,\vartheta)}\ (y)\}$.

Similarly, $\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}{}_{f^{-1}(\pi)}(x-y) \succcurlyeq$ rmin$\{\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}{}_{f^{-1}(\pi)}\ (x), \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}{}_{f^{-1}(\pi)}\ (y)\}$ and

$\lambda_{f^{-1}(\pi)}{}^{(\omega,\vartheta)}\ (x- y) \leq$ max $\{\lambda_{f^{-1}(\pi)}{}^{(\omega,\vartheta)}\ (x), \lambda_{f^{-1}(\pi)}{}^{(\omega,\vartheta)}\ (y)\}$.

Definition 5.3.

Let $f: (X; +, -, 0) \rightarrow (Y; +', -', 0')$be a mapping from a set X into a set Y. $\Omega_{(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})} =< \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}\ (x), \lambda_{\Omega}{}^{(\omega,\vartheta)}(x) >$ is a cubic subset of X **has sup and inf properties** if for any subset T of X, there exist t, s ∈ T such that

$\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}\ (t) = rsup_{t_0 \in T}\ \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}\ (t_0)$ and $\lambda_{\Omega}{}^{(\omega,\vartheta)}(s) = rinf_{s_0 \in T}\ \lambda_{\Omega}{}^{(\omega,\vartheta)}\ (s_0)$

Theorem 5.4.

Let $f: (X; +, -, 0) \rightarrow (Y; +', -', 0')$ be an epimorphism from an SA-algebra X into an SA-algebra Y. For every

$\Omega_{(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})} =< \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}\ (x), \lambda_{\Omega}{}^{(\omega,\vartheta)}(x) >$cubic bipolar fuzzy SA-subalgebra with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X with **sup**

and inf properties , then $f\ (\Omega_{(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})})$ is a cubic bipolar fuzzy SA-subalgebra with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of Y.

Proof:

By Definition (5.3), $f(\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})})(y') = \underset{x \in f^{-1}(y')}{rsup}\ \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}\ (x)$ and

$f(\lambda_{\Omega}{}^{(\omega,\vartheta)})(y') = \underset{x \in f^{-1}(y')}{rinf}\ \lambda_{\Omega}{}^{(\omega,\vartheta)}(x)$ for all y' ∈ Y and

$rsup(\emptyset) = [0, 0]$ and $rinf\ (\emptyset) = 1$. We prove that

$f(\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})})\ (x' +' y') \succcurlyeq$ rmin $\{f(\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})})(x'), \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}\ (y')\}$, and

$f(\lambda_{\Omega}{}^{(\omega,\vartheta)})\ (x' +' y') \leq$ rmax$\{f(\lambda_{\Omega}{}^{(\omega,\vartheta)})(x'), f(\lambda_{\Omega}{}^{(\omega,\vartheta)})\ (y')\}$, for all $x', y' \in Y$

and $x_0, y_0 \in X$ such that $x_0 = f^{-1}(x')$, $y_0 = f^{-1}(y')$

$f(\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})})(x' + 'y') = \underset{x_0+y_0 \in f^{-1}(x'+'y')}{rsup}\ \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}\ (x_0 + y_0) = \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}\ (x_0 + y_0)$.

$\succcurlyeq$ rmin $\{\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}\ (x_0),\ \tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}\ (y_0)\}$,

$= \text{rmin}\ \{ \underset{x_0\in f^{-1}(x')}{rsup}\ \tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}\ (x_0),\ \underset{y_0\in f^{-1}(y')}{rsup}\ \tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}\ (y_0)\ \}$

$= \text{rmin}\ \{ f(\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})})(x'), f(\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})})(y') \}$ and

$f(\lambda_\Omega{}^{(\omega,\vartheta)})(x' +' y') = \underset{x_0+y_0\in f^{-1}(x'+'y')}{inf} \lambda_\Omega{}^{(\omega,\vartheta)}(x_0+y_0) = \lambda_\Omega{}^{(\omega,\vartheta)}(x_0+y_0)$

$\leq \max\ \{\lambda_\Omega{}^{(\omega,\vartheta)}(x_0), \lambda_\Omega{}^{(\omega,\vartheta)}(y_0)\ \}$

$= \max\ \{ \underset{x_0\in f^{-1}(x')}{inf} \lambda_\Omega{}^{(\omega,\vartheta)}(x_0),\ \underset{y_0\in f^{-1}(y')}{inf} \lambda_\Omega{}^{(\omega,\vartheta)}(y_0)\ \}$

$= \max\ \{ f(\lambda_\Omega{}^{(\omega,\vartheta)})(x'), f(\lambda_\Omega{}^{(\omega,\vartheta)})(y')\ \}$. Similarly,

$f(\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})})(x-y) \succcurlyeq \text{rmin}\{ f(\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})})(x), f(\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})})(y)\}$ and $f(\lambda_\Omega{}^{(\omega,\vartheta)})(x\text{-}y) \leq \max\ \{f(\lambda_\Omega{}^{(\omega,\vartheta)})(x), f(\lambda_\Omega{}^{(\omega,\vartheta)})(y)\}$.

Hence, $f(\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)})$ is a cubic bipolar fuzzy SA-subalgebra with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of Y.

Theorem 5.5.

A homomorphic pre-image of cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) is also cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of SA-algebra.

Proof:

Let $f:(X;+,-,0) \to (Y;+',-',0')$ be homomorphism from an SA-algebra X into an SA-algebra Y. If $\pi^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)} =< \tilde{\mu}_\pi{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_\pi{}^{(\omega,\vartheta)}(x) >$ is a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of Y and

$\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)} =< \tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_\Omega{}^{(\omega,\vartheta)}(x) >$ the pre-image of π under f, then

$\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}(x) = \tilde{\mu}_\pi{}^{(\tilde{\alpha},\tilde{\beta})}(f(x))$, $\lambda_\Omega{}^{(\omega,\vartheta)}(x) = \lambda_\pi{}^{(\omega,\vartheta)}(f(x))$, for all $x \in X$, by Definition (2.9). Let $x \in X$,

then $(\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})})(0) = \tilde{\mu}_\pi{}^{(\tilde{\alpha},\tilde{\beta})}(f(0)) \succcurlyeq \tilde{\mu}_\pi{}^{(\tilde{\alpha},\tilde{\beta})}(f(x)) = \tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}(x)$, and

$(\lambda_\Omega{}^{(\omega,\vartheta)})(0) = \lambda_\pi{}^{(\omega,\vartheta)}(f(0)) \leq \lambda_\pi{}^{(\omega,\vartheta)}(f(x)) = \lambda_\Omega{}^{(\omega,\vartheta)}(x)$. Now,

let $x,y,z \in X$, then $\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}(x+y) = \tilde{\mu}_\pi{}^{(\tilde{\alpha},\tilde{\beta})}(f(x+y))$

$\succcurlyeq \text{rmin}\ \{\tilde{\mu}_\pi{}^{(\tilde{\alpha},\tilde{\beta})}(f(x+z)), \tilde{\mu}_\pi{}^{(\tilde{\alpha},\tilde{\beta})}(f(y-z))\ \}$

$= \text{rmin}\ \{\tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}(x+z), \tilde{\mu}_\Omega{}^{(\tilde{\alpha},\tilde{\beta})}(y-z)\}$ and

$\lambda_\Omega{}^{(\omega,\vartheta)}(x+y) = \lambda_\pi{}^{(\omega,\vartheta)}(f(x+y)) \leq \max\ \{\lambda_\pi{}^{(\omega,\vartheta)}(f(x+z)), \lambda_\pi{}^{(\omega,\vartheta)}(f(y-z))\}$

$= \max\ \{\lambda_\Omega{}^{(\omega,\vartheta)}(x+z), \lambda_\Omega{}^{(\omega,\vartheta)}(y-z)\}$.

Theorem 5.6.

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Let $f:(X;+,-,0) \to (Y;+',-',0')$ be an epimorphism from an SA-algebra X into an SA-algebra Y. For every $\Omega_{(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})} =< \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\Omega}{}^{(\omega,\vartheta)}(x) >$ cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X with **sup and inf properties**, then $f\,(\Omega_{(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})})$ is a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of Y.

Proof:

Since $\Omega_{(\omega,\vartheta)}^{(\tilde{\alpha},\tilde{\beta})} =< \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x), \lambda_{\Omega}{}^{(\omega,\vartheta)}(x) >$ is a cubic bipolar fuzzy SA-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of X, we have $(\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})})(0) \succcurlyeq \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x)$, and $(\lambda_{\Omega}{}^{(\omega,\vartheta)})(0) \le \lambda_{\Omega}{}^{(\omega,\vartheta)}(x)$,

for all $x \in X$. Note that, $0 \in f^{-1}(0')$ where 0,0' are the zero of X and Y, respectively.

Thus $\tilde{\mu}_{\pi}{}^{(\tilde{\alpha},\tilde{\beta})}(0') = \underset{0\in f^{-1}(0')}{rsup}\, \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(0) = \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(0)$

$$\succcurlyeq \underset{x\in f^{-1}(x')}{rsup}\, \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x) = \tilde{\mu}_{\pi}{}^{(\tilde{\alpha},\tilde{\beta})}(x'),$$

$$\lambda_{\pi}{}^{(\omega,\vartheta)}(0') = \underset{0\in f^{-1}(0')}{inf}\, \lambda_{\Omega}{}^{(\omega,\vartheta)}(0) = \lambda_{\Omega}{}^{(\omega,\vartheta)}(0)$$

$\le \underset{x\in f^{-1}(x')}{inf}\, \lambda_{\Omega}{}^{(\omega,\vartheta)}(x) = \lambda_{\pi}{}^{(\omega,\vartheta)}(x')$, for all $x \in X$, by Definition (2.15). Which implies that

$\tilde{\mu}_{\pi}{}^{(\tilde{\alpha},\tilde{\beta})}(0') \succcurlyeq \tilde{\mu}_{\pi}{}^{(\tilde{\alpha},\tilde{\beta})}(x')$, and $\lambda_{\pi}{}^{(\omega,\vartheta)}(0') \le \lambda_{\pi}{}^{(\omega,\vartheta)}(x')$, for all $x' \in Y$. By Definition (5.1),

let $, y, z \in X$ such that $x = f^{-1}(x')$, $y = f^{-1}(y')$ and $z = f^{-1}(z')$, then

$$\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x' +' z') = f(\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})})(x' +' z') = \underset{x+z\,\in f^{-1}(x'+'z')}{rsup}\, \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x+z) \;\&$$

$$\lambda_{\pi}{}^{(\omega,\vartheta)}(x'+'z') = f\big(\lambda_{\Omega}{}^{(\omega,\vartheta)}\big)(x'+'z') = \underset{x+z\,\in f^{-1}(x'+'z')}{inf}\, \lambda_{\Omega}{}^{(\omega,\vartheta)}(x+z) \text{ and}$$

$$\tilde{\mu}_{\pi}{}^{(\tilde{\alpha},\tilde{\beta})}(y' -' z') = f(\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})})(y' -' z') = \underset{y-z\in f^{-1}(y'-'z')}{rsup}\, \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(y-z) \;\&$$

$$\lambda_{\pi}{}^{(\omega,\vartheta)}(y'-'z') = f\big(\lambda_{\Omega}{}^{(\omega,\vartheta)}\big)(y'-'z') = \underset{y-z\,\in f^{-1}(y'-'z')}{inf}\, \lambda_{\Omega}{}^{(\omega,\vartheta)}(y-z),$$

for all $x', y', z' \in Y$ and rsup(∅) = [0, 0] and rinf (∅) = 1. We have prove that

$\tilde{\mu}_{\pi}{}^{(\tilde{\alpha},\tilde{\beta})}(x' + y') \succcurlyeq$ rmin $\{\tilde{\mu}_{\pi}{}^{(\tilde{\alpha},\tilde{\beta})}(x' + z'), \tilde{\mu}_{\pi}{}^{(\tilde{\alpha},\tilde{\beta})}(y' - z')\}$, and

$\lambda_{\pi}{}^{(\omega,\vartheta)}(x' + y') \le \max\{\lambda_{\pi}{}^{(\omega,\vartheta)}(x' + z'), \lambda_{\pi}{}^{(\omega,\vartheta)}(y' - z')\}$, for all $x', y', z' \in Y$.

$$\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x+y) = \tilde{\mu}_{\pi}{}^{(\tilde{\alpha},\tilde{\beta})}(x'+'y') = \underset{x+y\in f^{-1}(x'+'y')}{rsup}\, \tilde{\mu}_{\pi}{}^{(\tilde{\alpha},\tilde{\beta})}(x+y)$$

Also, $\tilde{\mu}_{\pi}{}^{(\tilde{\alpha},\tilde{\beta})}(x' + y') = \underset{x+y\in f^{-1}(x'+y')}{rsup}\, \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x+y) = \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}(x+y)$

$\geqslant$ rmin $\{\tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}\ (x+z),\ \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}\ (y-z)\}$,

$= \text{rmin}\{\underset{x+z\in f^{-1}(x'+'z')}{rsup}\ \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}\ (x+z),\ \underset{y-z\in f^{-1}(y'-'z')}{rsup}\ \tilde{\mu}_{\Omega}{}^{(\tilde{\alpha},\tilde{\beta})}\ (y-z)\}$

= rmin $\{\ \tilde{\mu}_{\pi}{}^{(\tilde{\alpha},\tilde{\beta})}(x'+z'),\ \tilde{\mu}_{\pi}{}^{(\tilde{\alpha},\tilde{\beta})}(y'-z')\ \}$and

$\lambda_{\Omega}{}^{(\omega,\vartheta)}(x+y) = \underset{x+y\in f^{-1}(x'+\prime z\prime)}{inf}\lambda_{\Omega}{}^{(\omega,\vartheta)}(x+y)$,

$\lambda_{\Omega}{}^{(\omega,\vartheta)}(x'+y') = \underset{(x+y)\in f^{-1}(y')}{inf}\lambda_{\Omega}{}^{(\omega,\vartheta)}(x+y)$

$\leq$ max $\{\lambda_{\Omega}{}^{(\omega,\vartheta)}\ (x+z),\ \lambda_{\Omega}{}^{(\omega,\vartheta)}\ (y-z)\}$,

$= \max\{\underset{x+z\in f^{-1}(x'+'z')}{inf}\lambda_{\Omega}{}^{(\omega,\vartheta)}(x+z),\ \underset{y-z\in f^{-1}(y'-'z')}{inf}\lambda_{\Omega}{}^{(\omega,\vartheta)}(y-z)\}$

= max $\{\lambda_{\Omega}{}^{(\omega,\vartheta)}(x'+z'),\ \lambda_{\Omega}{}^{(\omega,\vartheta)}(y'-z')\ \}$.

Hence, $f\ (\Omega^{(\tilde{\alpha},\tilde{\beta})}_{(\omega,\vartheta)})$ is a cubic bipolar fuzzy *SA*-ideal with thresholds $(\tilde{\alpha},\tilde{\beta})$, (ω,ϑ) of Y.

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