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On The Interval-valued Bifuzzy ψ-ideals of ψ-algebra

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***Abstract**: The concept tripolar fuzzy subset is a generalization of fuzzy subset. In this paper, the concept bifuzzy ψ-ideals, bifuzzy sub-commutative bifuzzy ψ-ideals and sub-implicative fuzzy ψ-ideals of ψ-algebras are introduced and several properties are investigated. Also, the relations between fuzzy bifuzzy ψ-ideals, sub-commutative bifuzzy ψ-ideals and sub-implicative bifuzzy ideals are given. The image and the preimage of fuzzy ψ-algebras, bifuzzy ψ-ideals sub-commutative bifuzzy ψ-ideals and sub-implicative bifuzzy ψ-ideals under homomorphism of ψ-algebras are defined and how the image and the preimage of them are studied.*

Keywords: ψ-algebra, bifuzzy ψ-ideals, ψ- subalgebra, fuzzy ψ-subalgebra, interval-valued bifuzzy ψ-ideal.

1. INTRODUCTION

In 1965, L.A. Zadeh introduced the notion of fuzzy subset, [1]. In 1976, K. Is´eki and S. Tanaka studied the notion of BCK-algebra, [2]. In 1991, O.G. Xi studied the notion of fuzzy BCK-algebra, [3]. In 2006, A.B. Saoid introduced fuzzy QS-algebra with interval-valued membership function, [4]. Also, T. Priya and T. Ramachandran introduced anti-fuzzy ideals of CI-algebra and its lower level cuts, [5]. Jun[6,7] studied the notion of cubic set as generalization of fuzzy set and interval-valued fuzzy set. In 2015, A.T. Hameed introduced the idea of SA-algebras. She stated some concepts related to it such as SA-subalgebra, SA-ideal, fuzzy SA-subalgebra and fuzzy SA-ideal of SA-algebra. She introduced the concept of homomorphisms on SA-algebra and fuzzy homomorphisms on SA-algebra, [9]. In 2023, A.T. Hameed and N.H. Jaber introduced the notion of ψ -subalgebra, ψ -ideal, bifuzzy ψ -subalgebra, bifuzzy ψ-ideal and they introduced the concept of homomorphisms on ψ -algebra and fuzzy homomorphisms on ψ -algebra. In this paper**,** the concepts of interval-valued bifuzzy ψ-ideal, interval-valued sub-implicative bifuzzy ψ -ideals under homomorphism of ψ -algebras are defined and how the image and the preimage of them under homomorphism of ψ-algebras are studied.

2. Preliminaries

In this section, we give some basic definitions and preliminaries proprieties of ψ-subalgebras and fuzzy ψ -ideals in ψ -algebra such that we include some elementary aspects that are necessary for this paper.

Definition 2.1.([14]. Let $(X;+,-,0)$ be an algebra with two operations $(+)$ and $(-)$ and constant (0). X is called an ***ψ*-algebra** if it satisfies the following properties: for all $x, y, z \in X$,

(ψ_1) $x - x = 0$,

(ψ_2) $(0 - x) + x = 0$,

(ψ_3) $(x - y) - z = x - (z + y)$,

(ψ_4) $(y + x) - (x - z) = y + z$.

In , we can define a binary relation ($\leq$) by : $x \leq y$ if and only if $x + y = 0$ and $x - y = 0$, $x, y \in X$.

Definition 2.2. [13].

Let $(X;+,-,0)$ be a ψ-algebra and let S be a nonempty set of X. S is called **a *ψ*- subalgebra of *X*** if $x + y \in S$ *and* $x - y \in S$, whenever $x, y \in S$.

Definition 2.3. [15].

A nonempty subset I of a ψ-algebra $(X;+,-,0)$ is called **a *ψ*-ideal of *X*** if it satisfies: for $x, y, z \in X$,

(1) $0 \in I$,

(2) $(y+z) \in I$ *and* $(x-z) \in I$ *imply* $(y+x) \in I$.

Proposition 2.4.[2].

Every ψ-ideal of ψ-algebra is a ψ-subalgebra of X and the converse is not true.

Definition 2.5.[4].

Let X be a nonempty set, a fuzzy subset μ of X is a mapping $\mu: X \to [0,1]$.

Definition 2.6.[14].

For any $t \in [0,1]$ and a fuzzy subset μ in a nonempty set X, the set

$U(\mu, t) = \{x \in X \mid \mu(x) \geq t\}$ is called **an upper t-level cut of μ,** and the set $L(\mu, t) = \{x \in X \mid \mu(x) \leq t\}$ is called **a lower t-level cut of μ**.

Definition 2.7.[13].

Let $(X; +, -, 0)$ be a ψ-algebra, a fuzzy subset μ of X is called **a fuzzy ψ-subalgebra of X** if for all $x, y \in X$,

1- $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$ and
2- $\mu(x-y) \geq min\{\mu(x), \mu(y)\}$.

Definition 2.8.[15].

Let $(X; +, -, 0)$ be a ψ-algebra, a fuzzy subset μ of X is called **a fuzzy ψ-ideal of X** if it satisfies: for all $x, y, z \in X$,

(FI_1) $\mu(0) \geq \mu(x)$,

(FI_2) $\mu(y+x) \geq min\{\mu(y+z), \mu(x-z)\}$.

Proposition 2.9.[15].

Every fuzzy ψ-ideal of ψ-algebra $(X; +, -, 0)$ is a fuzzy ψ-subalgebra of X .

Proposition 2.10.[15].

1- Let μ be a fuzzy subset of ψ-algebra $(X; +, -, 0)$. If μ is a fuzzy ψ-subalgebra of X, for any t ∈ [0,1], μ_t is a ψ-subalgebra of X .

2- Let μ be a fuzzy subset of ψ-algebra $(X; +, -, 0)$. If for all t ∈ [0,1], μ_t is a ψ-subalgebra of X , then μ is a fuzzy ψ-subalgebra of X .

3- Let μ be a fuzzy ideal of ψ-algebra $(X; +, -, 0)$. If μ is a fuzzy ψ-ideal of X, then for any t ∈ [0,1], μ_t is an ψ-ideal of X .

4- Let μ be a fuzzy ideal of ψ-algebra $(X; +, -, 0)$. If for all t ∈ [0,1], μ_t is an ψ-ideal of X , then μ is a fuzzy ψ-ideal of X .

Now, we will recall the concept of anti-fuzzy subsets.

Definition 2.11. [14].

Let $(X; +, -, 0)$ be an ψ-algebra, a fuzzy subset μ of X is called **an anti-fuzzy ψ-subalgebra of X** if for all $x, y \in X$, AFψS1) $\mu(x+y) \leq max\{\mu(x), \mu(y)\}$,

AFψS2) $\mu(x-y) \leq max\{\mu(x), \mu(y)\}$.

Proposition 2.12. [4].

Let μ be an anti-fuzzy subset of an ψ-algebra $(X; +, -, 0)$.

1- If μ is an anti-fuzzy ψ-subalgebra of , then it satisfies for any t∈[0, 1], $L(\mu, t) \neq \emptyset$ implies $L(\mu, t)$ is a ψ-subalgebra of X.

2- If $L(\mu, t)$ is a ψ-subalgebra of X, for all t∈[0, 1], $L(\mu, t) \neq \emptyset$, then μ is an anti-fuzzy ψ-subalgebra of X .

Definition 2.13. [6]:

Let $(X; +, -, 0)$ be an ψ-algebra, a fuzzy subset μ of X is called **an anti-fuzzy ψ-ideal of *X*** if it satisfies the following conditions, for all $x, y \in X$,

(AFψI1) $\mu(0) \leq \mu(x)$,

(AFψI2) $\mu(y+x) \leq max\{\mu(y+z), \mu(x-z)\}$.

Proposition 2.14. [4].

Let μ be an anti-fuzzy subset of an ψ-algebra $(X; +, -, 0)$.

1- If μ is an anti-fuzzy ψ-ideal of , then it satisfies for any t∈[0, 1], $L(\mu, t) \neq \emptyset$ implies $L(\mu, t)$ is an ψ- ideal of X.

2- If $L(\mu, t)$ is an ψ- ideal of X, for all t∈[0, 1], $L(\mu, t) \neq \emptyset$, then μ is an anti-fuzzy ψ-ideal of X .

Definition 2.15. [12].

Let $f: (X; +, -, 0) \to (Y; +', -', 0')$be a mapping nonempty ψ-algebras X and Y respectively. If μ is anti-fuzzy subset of X, then the anti-fuzzy subset β of Y defined by:

$$f(\mu)(y) = \begin{cases} inf\{\mu(x): x \in f^{-1}(y)\} & if\ \ f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ 1 & otherwise \end{cases}$$

is said to be the image of μ under f.

Similarly if β is anti-fuzzy subset of , then the fuzzy subset $\mu = (\beta \circ f)$ of X (i.e the anti-fuzzy subset defined by $\mu(x) = \beta(f(x))$, for all

$x \in X$) is called the pre-image of β under f .

Now, we will recall the concept of bifuzzy subsets.

Definition 2.16. [6].

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be a bifuzzy subset of a ψ-algebra X. A is said to be **an bifuzzy ψ -subalgebra of X** if : for all x, y ∈X,

(IFS1) $\mu_A(x+y) \geq min\{\mu_A(x), \mu_A(y)\}$ and

$\mu_A(x-y) \geq min\{\mu_A(x), \mu_A(y)\}$.

(IFS2) $\nu_A(x+y) \leq max\{\nu_A(x), \nu_A(y)\}$ and

$\nu_A(x-y) \leq max\{\nu_A(x), \nu_A(y)\}$.

i.e., μ_A is fuzzy ψ-subalgebra of ψ-algebra and ν_A is anti-fuzzy ψ-subalgebra of ψ-algebra.

Definition 2.17. [5].

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be a bifuzzy subset of a ψ-algebra $(X; +, -, 0)$. A is said to be a **bifuzzy ψ-ideal of X** if for all x, y, z $\in$X,

$(IF\psi_1)$ $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$,

$(IF\psi_2)$ $\mu_A(y+x) \geq min\{\mu_A(y+z), \mu_A(x-z)\}$ and

$(IF\psi_3)$ $\nu_A(y+x) \leq \max\{\nu_A(y+z), \nu_A(x-z)\}$.

i.e., μ_A is fuzzy ψ-ideal of ψ-algebra and ν_A is anti-fuzzy ψ-ideal of ψ-algebra.

Now, we will recall the concept of interval-valued fuzzy subsets.

Remark 2.18. [7].

An interval number is $\tilde{a} = [a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. Let I be a closed unit interval, (i.e., I = [0, 1]).

Let D[0, 1] denote the family of all closed subintervals of I = [0, 1], that is, D[0, 1] = $\{\tilde{a} = [a^-, a^+] \mid a^- \leq a^+$, for $a^-, a^+ \in I\}$.

Now, we define what is known as refined minimum (briefly, rmin) of two element in D[0,1].

Definition 2.19. [3].

We also define the symbols (≽), (≼), (=), "rmin " and "rmax " in case of two elements in D[0, 1]. Consider two interval numbers (elements numbers)

$\tilde{a} = [a^-, a^+]$, $\tilde{b} = [b^-, b^+]$ in D[0, 1] : Then

(1) $\tilde{a} \succcurlyeq \tilde{b}$ if and only if, $a^- \geq b^-$ and $a^+ \geq b^+$,

(2) $\tilde{a} \preccurlyeq \tilde{b}$ if and only if, $a^- \leq b^-$ and $a^+ \leq b^+$,

(3) $\tilde{a} = \tilde{b}$ if and only if, $a^- = b^-$ and $a^+ = b^+$,

(4) rmin $\{\tilde{a}, \tilde{b}\}$= [min $\{a^-, b^-\}$, min $\{a^+, b^+\}$],

(5) rmax $\{\tilde{a}, \tilde{b}\}$= [max $\{a^-, b^-\}$, max $\{a^+, b^+\}$],

Remark 2.20. [11].

Let $\tilde{0}$ = [0, 0] as its least element and $\tilde{1}$ = [1, 1] as its greatest element. Let $\tilde{a}_i \in$D[0, 1] where i $\in \Lambda$. We define $r\, inf_{i\in\Lambda}\tilde{a}$= $[r\, inf_{i\in\Lambda} a^-, r\, inf_{i\in\Lambda} a^+]$, $r\, sup_{i\in\Lambda}\tilde{a}$= $[r\, sup_{i\in\Lambda} a^-, r\, sup_{i\in\Lambda} a^+]$.

Definition 2.21. [10].

An **interval-valued fuzzy subset** $\widetilde{\mu}_A$ on subset is defined as $\widetilde{\mu}_A = \{< x, [\mu_A^-(x), \mu_A^+(x)] >| x \in X\}$. Where $\mu_A^-(x) \leq \mu_A^+(x)$, for all x∈ X. Then the ordinary fuzzy subsets $\mu_A^-: X \to [0, 1]$ and $\mu_A^+: X \to [0, 1]$ are called a **lower fuzzy subset and an upper fuzzy subset** of $\widetilde{\mu}_A$ respectively.

Let $\widetilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$, $\widetilde{\mu}_A: X \to D[0, 1]$, then $A = \{< x, \widetilde{\mu}_A(x) >| x \in X\}$.

Definition 2.22.[7].

Let $(X ; +, -, 0)$ be a nonempty set. A interval-valued bifuzzy set Ω in a structure $\Omega = \{< x, \tilde{\mu}_\Omega(x), \tilde{v}_\Omega(x) >| x \in X\}$, which is briefly denoted by $\Omega = <\tilde{\mu}_\Omega, \tilde{v}_\Omega>$, where $\tilde{\mu}_\Omega : X \to D[0, 1]$, $\tilde{\mu}_\Omega$ is an interval-valued fuzzy subset of X and $\tilde{v}_\Omega : X \to D[0, 1]$, $\tilde{v}_\Omega$ is an interval-valued fuzzy subset of X.

Proposition 2.23.[5].

Let $(X ; +, -, 0)$ be an ψ-algebra. An interval-valued bifuzzy subset $\Omega = <\tilde{\mu}_\Omega, \tilde{v}_\Omega>$ of . If for all $\tilde{t} \in D[0, 1]$ and $\tilde{s} \in D[0, 1]$, the set $\widetilde{U}(\Omega; \tilde{t}, \tilde{s})$ is an ψ-subalgebra of X, then Ω is an interval-valued bifuzzy ψ-subalgebra of X.

3. Interval-valued Bifuzzy ψ-ideals of ψ-algebra

In this section, we will introduce a new notion called interval-valued bifuzzy ψ -ideals of ψ -algebra and study several properties of it.

Definition 3.1.

Let $(X ; +, -, 0)$ be an ψ-algebra. An interval-valued bifuzzy subset $\Omega = <\tilde{\mu}_\Omega, \tilde{v}_\Omega>$ of X is called **interval-valued bifuzzy ψ-ideal of X** if, for all $x, y, z \in X$:

(IVψI$_1$) $\tilde{\mu}_\Omega(0) \succcurlyeq \tilde{\mu}_\Omega(x)$, and $\tilde{v}_\Omega(0) \preccurlyeq \tilde{v}_\Omega(x)$,

(IVψI$_2$) $\tilde{\mu}_\Omega(y + x) \succcurlyeq rmin\{\tilde{\mu}_\Omega(y + z), \tilde{\mu}_\Omega(x - z)\}, and$

$$\tilde{v}_\Omega(y + x) \preccurlyeq max\{\tilde{v}_\Omega(y + z), \tilde{v}_\Omega(x - z)\}.$$

Example 3.2.

Let $X = \{0, a, b, c, d\}$ be a set with the following table:

+	0	a	b	c	d
0	0	a	b	c	d
a	a	b	c	d	0
b	b	c	d	0	a
c	c	d	0	a	b
d	d	0	a	b	c

-	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	a
b	b	b	0	0	a
c	c	b	d	0	a
d	d	d	d	d	0

Then $(X; +, -, 0)$ is an ψ-algebra. It is easy to show that I ={0, c} and J ={0, d} are ψ-ideals of X .

We defined two cubic set $\Omega_1 = \{(x, \tilde{\mu}_{\Omega 1}(x), \tilde{v}_{\Omega 1}(x)) \mid x \in X\}$ and $\Omega_2 = \{(x, \tilde{\mu}_{\Omega 2}(x), \tilde{v}_{\Omega 2}(x)) \mid x \in X\}$ of X by :-

$$\tilde{\mu}_{\Omega 1}(x) = \begin{cases} [0.5,0.8], & \text{if } x \in \{0, c\}, \\ [0.4,0.7], & \text{if } x \in \{a, b\}, \\ [0.3,0.8], & \text{otherwise} \end{cases} \quad \tilde{v}_{\Omega 1}(x) = \begin{cases} [0.5,0.8], & \text{if } x \in \{0, c\}, \\ [0.4,0.7], & \text{if } x \in \{a, b\}, \\ [0.3,0.8], & \text{otherwise} \end{cases}$$

$$\tilde{\mu}_{\Omega 2}(x) = \begin{cases} [0.4,0.9], & \text{if} x \in \{0, d\}, \\ [0.3,0.5], & \text{otherwise.} \end{cases} \quad \text{and} \quad \tilde{v}_{\Omega 2}(x) = \begin{cases} [0.4,0.9], & \text{if} x \in \{0, d\}, \\ [0.3,0.7], & \text{otherwise.} \end{cases}$$

Then Ω_1 and Ω_2 are interval-valued bifuzzy ψ-ideal of X .

Proposition 3.3.

Let $(X; +, -, 0)$ be an ψ-algebra. An interval-valued bifuzzy subset $\Omega = <\tilde{\mu}_\Omega, \tilde{v}_\Omega>$ of X. If Ω is an interval-valued bifuzzy ψ-ideal of X, then for all $\tilde{t}, \tilde{s} \in$ D[0, 1], the set $\tilde{U}(\Omega; \tilde{t}, \tilde{s})$ is an ψ-ideal of X .

Proof.

Assume that $\Omega = <\tilde{\mu}_\Omega, \tilde{v}_\Omega>$ is an interval-valued bifuzzy ψ-ideal of X and let $\tilde{t}, \tilde{s} \in$ D[0, 1] , be such that $\tilde{U}(\Omega; \tilde{t}, \tilde{s}) \neq \emptyset$,

$\tilde{\mu}_\Omega(\tilde{0}) \succcurlyeq \tilde{\mu}_\Omega(x) \succcurlyeq \tilde{t}$ and $\tilde{v}_\Omega(\tilde{0}) \preccurlyeq \tilde{v}_\Omega(x) \preccurlyeq \tilde{s}$, for all x $\in X$.

Let $x, y, z \in X$ such that $+z, x - z \in \tilde{U}(\Omega; \tilde{t}, \tilde{s})$, then $\tilde{\mu}_\Omega(y + z) \succcurlyeq \tilde{t}$, $\tilde{\mu}_\Omega(x - z) \succcurlyeq \tilde{t}$ and $\tilde{v}_\Omega(y + z) \preccurlyeq \tilde{s}$, $\tilde{v}_\Omega(x - z) \preccurlyeq \tilde{s}$. Since Ω is an interval-valued bifuzzy ψ-ideal of X, we get

$\tilde{\mu}_\Omega(y + x) \succcurlyeq$ rmin{ $\tilde{\mu}_\Omega(y + z), \tilde{\mu}_\Omega(x - z)$} $\succcurlyeq \tilde{t}$ and $\tilde{v}_\Omega(y + x) \preccurlyeq$ rmax {$\tilde{v}_\Omega(y + z), \tilde{v}_\Omega(x - z)$} $\preccurlyeq \tilde{s}$. Thus $+x \in \tilde{U}(\Omega; \tilde{t}, \tilde{s})$.

Hence the set $\tilde{U}(\Omega; \tilde{t}, \tilde{s})$ is an ψ-ideal of X. ⌂

Proposition 3.4.

Let $(X; +, -, 0)$ be an ψ-algebra. An interval-valued bifuzzy subset $\Omega = <\tilde{\mu}_\Omega, \tilde{v}_\Omega>$ of . If for all $\tilde{t}, \tilde{s} \in$ D[0, 1], the set $\tilde{U}(\Omega; \tilde{t}, \tilde{s})$ is an ψ-ideal of X, then Ω is an interval-valued bifuzzy ψ-ideal of X.

Proof.

Suppose that $\tilde{U}(\Omega; \tilde{t}, \tilde{s})$ is an ψ-ideal of X and let

$\tilde{\mu}_\Omega(\tilde{0}) \succcurlyeq \tilde{\mu}_\Omega(x) \succcurlyeq \tilde{t}$ and $\tilde{v}_\Omega(\tilde{0}) \preccurlyeq \tilde{v}_\Omega(x) \preccurlyeq \tilde{s}$, for all x $\in X$.

$x, y, z \in X$ be such that $\tilde{\mu}_\Omega(y + x) \prec$ rmin {$\tilde{\mu}_\Omega(y + z), \tilde{\mu}_\Omega(x - z)$}, and $\tilde{v}_\Omega(y + x) \succ$ rmax {$\tilde{v}_\Omega(y + z), \tilde{v}_\Omega(x - z)$}.

Consider $\tilde{\beta} = 1/2$ { $\tilde{\mu}_\Omega(y + x)$ + rmin{$\tilde{\mu}_\Omega(y + z), \tilde{\mu}_\Omega(x - z)$} } and

$\tilde{\gamma} = 1/2$ { $\tilde{v}_\Omega(y + x)$ + rmax{$\tilde{v}_\Omega(y + z), \tilde{v}_\Omega(x - z)$}}.

We have $\tilde{\beta} \in$ D[0, 1] and $\tilde{\gamma} \in$ D[0, 1], and $\tilde{\mu}_\Omega(y + x) \prec \tilde{\beta} \prec$ rmin {$\tilde{\mu}_\Omega(y + z), \tilde{\mu}_\Omega(x - z)$ }, and

$\tilde{v}_\Omega(y + x) \succ \tilde{\gamma} \succ r$max {$\tilde{v}_\Omega(y + z), \tilde{v}_\Omega(x - z)$} .

It follows that $y + z,\ x - z \in \tilde{U}(\Omega; \tilde{t}, \tilde{s})$, and $(y + x) \notin \tilde{U}(\Omega; \tilde{t}, \tilde{s})$. This is a contradiction.

Hence $\tilde{\mu}_{\Omega}(y+x) \succcurlyeq$ rmin{ $\tilde{\mu}_{\Omega}(y+z)$,$\tilde{\mu}_{\Omega}(x-z)$} $\succcurlyeq \tilde{t}$ and $\tilde{v}_{\Omega}(y+x$ y) $\preccurlyeq$ rmax {$\tilde{v}_{\Omega}(y+z)$,$\tilde{v}_{\Omega}(x-z)$} $\preccurlyeq \tilde{s}$.

Therefore Ω =<$\tilde{\mu}_{\Omega}$, $\tilde{v}_{\Omega}$> is an interval-valued bifuzzy ψ-ideal of X. ⌂

Theorem 3.5.

Interval-valued bifuzzy subset Ω =<$\tilde{\mu}_{\Omega}$, $\tilde{v}_{\Omega}$> is a interval-valued bifuzzy ψ-ideal of ψ-algebra X if and only if, $\mu^{-}{}_{\Omega}$, and $\mu^{+}{}_{\Omega}$ are fuzzy ψ-ideals of X and $v^{-}{}_{\Omega}$, and $v^{+}{}_{\Omega}$ are anti-fuzzy ψ-ideals of X.

Proof.

Assume that Ω is an interval-valued bifuzzy ψ-ideal of X, for any $x, y, z \in X$,

$\tilde{\mu}_{\Omega}(\tilde{0}) \succcurlyeq \tilde{\mu}_{\Omega}(x) \succcurlyeq \tilde{t}$ and $\tilde{v}_{\Omega}(\tilde{0}) \preccurlyeq \tilde{v}_{\Omega}(x) \preccurlyeq \tilde{s}$, for all x ∈ X.

$[\mu^{-}{}_{\Omega}(y+x), \mu^{+}{}_{\Omega}(y+x)]$=$\tilde{\mu}_{\Omega}(y+x) \succcurlyeq$ rmin{$\tilde{\mu}_{\Omega}(y+z)$,$\tilde{\mu}_{\Omega}(x-z)$}

= rmin{$[\mu^{-}{}_{\Omega}(y+z), \mu^{+}{}_{\Omega}(y+z)], [\mu^{-}{}_{\Omega}(x-z), \mu^{+}{}_{\Omega}(x-z)]$}

= [min{$\mu^{-}{}_{\Omega}(y+z), \mu^{-}{}_{\Omega}(x-z)$, min{$\mu^{+}{}_{\Omega}(y+z), \mu^{+}{}_{\Omega}(x-z)$}].

Thus $\mu^{-}{}_{\Omega}(y+x) \geq$ min {$\mu^{-}{}_{\Omega}(y+z), \mu^{-}{}_{\Omega}(x-z)$},$\mu^{+}{}_{\Omega}(y+x) \geq$ min{ $\mu^{+}{}_{\Omega}(y+z), \mu^{+}{}_{\Omega}(x-z)$} and

$[v^{-}{}_{\Omega}(y+x), v^{+}{}_{\Omega}(y+x)]$=$\tilde{v}_{\Omega}(y+x) \preccurlyeq$ rmax{$\tilde{v}_{\Omega}(y+z)$,$\tilde{v}_{\Omega}(x-z)$}

= rmax{$[v^{-}{}_{\Omega}(y+z), v^{+}{}_{\Omega}(y+z)], [v^{-}{}_{\Omega}(x-z), v^{+}{}_{\Omega}(x-z)]$}

= [max{$v^{-}{}_{\Omega}(y+z), v^{-}{}_{\Omega}(x-z)$, max{$v^{+}{}_{\Omega}(y+z), v^{+}{}_{\Omega}(x-z)$}].

Thus $v^{-}{}_{\Omega}(y+x) \preccurlyeq$ rmax {$v^{-}{}_{\Omega}(y+z), v^{-}{}_{\Omega}(x-z)$},$v^{+}{}_{\Omega}(y+x) \preccurlyeq$ rmax{ $v^{+}{}_{\Omega}(y+z), v^{+}{}_{\Omega}(x-z)$} .

Therefore $\mu^{-}{}_{\Omega}$, and $\mu^{+}{}_{\Omega}$ are fuzzy ψ-ideals of X and $v^{-}{}_{\Omega}$ and $v^{+}{}_{\Omega}$ are anti-fuzzy ψ-ideals of X.

Conversely, let $\mu^{-}{}_{\Omega}$, and $\mu^{+}{}_{\Omega}$ are fuzzy ψ-ideals of X and $v^{-}{}_{\Omega}$ and $v^{+}{}_{\Omega}$ are anti-fuzzy ψ-ideals of X and $x, y, z \in X$, then

$\tilde{\mu}_{\Omega}(\tilde{0}) \succcurlyeq \tilde{\mu}_{\Omega}(x) \succcurlyeq \tilde{t}$ and $\tilde{v}_{\Omega}(\tilde{0}) \preccurlyeq \tilde{v}_{\Omega}(x) \preccurlyeq \tilde{s}$, for all x ∈ X.

$\mu^{-}{}_{\Omega}(y+x) \geq$ min{$\mu^{-}{}_{\Omega}(y+z)$,$\mu^{-}{}_{\Omega}(x-z)$}, $\mu^{+}{}_{\Omega}(y+x) \geq$ min{$\mu^{+}{}_{\Omega}(y+z)$,$\mu^{+}{}_{\Omega}(x-z)$} and

$v^{-}{}_{\Omega}(y+x) \leq$ max{$v^{-}{}_{\Omega}(y+z)$,$v^{-}{}_{\Omega}(x-z)$}, $v^{+}{}_{\Omega}(y+x) \leq$ max{$v^{+}{}_{\Omega}(y+z)$,$v^{+}{}_{\Omega}(x-z)$} .

Now, $\tilde{\mu}_{\Omega}(y+x) = [\mu^{-}{}_{\Omega}(y+x), \mu^{+}{}_{\Omega}(y+x)]$

$\succcurlyeq$ [min{$\mu^{-}{}_{\Omega}(y+z)$,$\mu^{-}{}_{\Omega}(x-z)$}, min{$\mu^{+}{}_{\Omega}(y+z)$,$\mu^{+}{}_{\Omega}(x-z)$}]

= rmin{$[\mu^{-}{}_{\Omega}(y+z), \mu^{+}{}_{\Omega}(y+z)], [\mu^{-}{}_{\Omega}(x-z), \mu^{+}{}_{\Omega}(x-z)]$}

= rmin{$\tilde{\mu}_{\Omega}(y+z)$,$\tilde{\mu}_{\Omega}(x-z)$}, therefore

$\tilde{\mu}_{\Omega}(y+x) \succcurlyeq$ rmin{ $\tilde{\mu}_{\Omega}(y+z)$, $\tilde{\mu}_{\Omega}(x-zy)$} $\succcurlyeq \tilde{t}$ and

$\tilde{v}_{\Omega}(y+x) = [v^{-}{}_{\Omega}(y+x), v^{-}{}_{\Omega}(y+x)]$

$\preccurlyeq [\min\{v^-{}_\Omega(y+z), v^-{}_\Omega(x-z)\}, \min\{v^+{}_\Omega(y+z), v^+{}_\Omega(x-z)\}]$

$=$ rmax$\{[v^-{}_\Omega(y+z), v^+{}_\Omega(y+z)], [v^-{}_\Omega(x-z), v^+{}_\Omega(x-z)]\}$

= rmax$\{\tilde{v}_\Omega(y+z), \tilde{v}_\Omega(x-z)\}$, therefore

$\tilde{v}_\Omega(y+x) \preccurlyeq$ rmax$\{\tilde{v}_\Omega(y+z), \tilde{v}_\Omega(x-z)\} \preccurlyeq \tilde{s}$.

Hence Ω is an interval-valued bifuzzy ψ-subalgebra of X. ⌂

Theorem 3.6.

If a interval-valued bifuzzy set Ω =<$\tilde{\mu}_\Omega$, $\tilde{v}_\Omega$> is a interval-valued bifuzzy ψ-ideal of X, then the upper $[t_1,t_2]$-Level and Lower $[s_1,s_2]$-Level of Ω are ψ-ideals of X.

Proof.

Let x, y, $z \in U(\tilde{\mu}_\Omega|[t_1,t_2])$, then $\tilde{\mu}_\Omega(y+z) \succcurlyeq [t_1,t_2]$ and $\tilde{\mu}_\Omega(x-z) \succcurlyeq [t_1,t_2]$. It follows that $\tilde{\mu}_\Omega(y+x) \succcurlyeq$ rmin$\{\tilde{\mu}_\Omega(y+z), \tilde{\mu}_\Omega(x-z)\} \succcurlyeq [t_1,t_2]$, so that $y+x \in U(\tilde{\mu}_\Omega|[t_1,t_2])$.

Hence $U(\tilde{\mu}_\Omega|[t_1,t_2])$ is ψ-ideal of X.

Let x, y, z ∈ L($\tilde{v}_\Omega|[s_1,s_2]$), then $\tilde{v}_\Omega(y+z) \preccurlyeq [s_1,s_2]$ and $\tilde{v}_\Omega(x-z) \preccurlyeq [s_1,s_2]$. It follows that

$\tilde{v}_\Omega(y+x) \preccurlyeq$ rmax$\{\tilde{v}_\Omega(y+z), \tilde{v}_\Omega(x-z)\} \preccurlyeq \tilde{s}$, so that $y+x \in$ L($\tilde{v}_\Omega|\tilde{s}$) .

Hence L($\tilde{v}_\Omega|\tilde{s}$) is ψ-ideal of X. ⌂

Corollary 3.7.

Let Ω =<$\tilde{\mu}_\Omega$, $\tilde{v}_\Omega$> be an interval-valued bifuzzy ψ-ideal of X, then

$\Omega(\tilde{t}, \tilde{s}) = U(\tilde{\mu}_\Omega|[t_1,t_2]) \cap$ L($\tilde{v}_\Omega|[s_1, s_2]$)

$=\{x \in X | \tilde{\mu}_\Omega(x) \succcurlyeq \tilde{t}, \tilde{v}_\Omega(x) \leq \tilde{s}\}$ is an ψ-ideal of X

Remark 3.8.

The following example shows that the converse of Corollary (3.7) is not valid

Example 3.9.

Let X ={0,a,b,c,d} be ψ-algebra in example (3.2) and cubic set Ω =< $\tilde{\mu}_\Omega$ (x), $\tilde{v}_\Omega$(x) > of X by

$$\tilde{\mu}_\Omega(x)=\begin{cases}[0.6,0.8], & \text{if } x=0,\\ [0.5,0.6], & \text{if } x \in \{a,b,c\},\\ [0.3,0.4], & \text{if } x \in \{d\},\end{cases} \quad \text{and} \quad \tilde{v}_\Omega(x)=\begin{cases}[0.3,0.4], & \text{if } x=0,\\ [0.4,0.41], & \text{if } x \in \{a,b,c\},\\ [0.5,0.6], & \text{if } x \in \{d\},\end{cases}$$

We take $[t_1,t_2]=[0.41,0.48]$ and $[s_1,s_2]=[0.41,0.48]$, then

$\Omega([s_1,s_2];t) = U(\tilde{\mu}_\Omega|[t_1,t_2]) \cap$ L($\tilde{v}_\Omega|[s_1, s_2]$) $=\{x \in X | \tilde{\mu}_\Omega(x) \succcurlyeq [t_1,t_2], \tilde{v}_\Omega(x) \preccurlyeq [s_1, s_2]\}$

={0, a, b, c}∩{0, a, b, c}={0, a, b, c} is ψ-ideal of X, but Ω =<$\tilde{\mu}_\Omega$, $\tilde{v}_\Omega$> is not an interval-valued bifuzzy ψ-ideal since $\tilde{\mu}_\Omega(y+x) \not\succcurlyeq$ rmin$\{\tilde{\mu}_\Omega(y+z), \tilde{\mu}_\Omega(x-z)\}$ and $\tilde{v}_\Omega(y+x) \not\preccurlyeq$ rmax$\{\tilde{v}_\Omega(y+z), \tilde{v}_\Omega(x-z)\}$.

Theorem 3.10.

Let Ω =<$\tilde{\mu}_\Omega$, $\tilde{\nu}_\Omega$> be an interval-valued bifuzzy subset of X such that the sets $U(\tilde{\mu}_\Omega \mid [t_1, t_2])$ and $L(\tilde{\nu}_\Omega|[s_1, s_2])$ are ψ-ideals of X, for every $[t_1,t_2]$ and $[s_1, s_2]$ ∈D[0,1], then Ω =<$\tilde{\mu}_\Omega$, $\tilde{\nu}_\Omega$> is an interval-valued bifuzzy ψ-ideal of X.

Proof.

Let $U(\tilde{\mu}_\Omega |[t_1,t_2])$and $L(\tilde{\nu}_\Omega \mid [s_1,s_2])$ are ideals of X, for every $\tilde{t}, \tilde{s}$ ∈D[0,1]

on the contrary, let $x_0, y_0, z_0 \in X$ be such that

$\tilde{\mu}_\Omega(y_0 + x_0) \prec \text{rmin}\{\tilde{\mu}_\Omega(y_0 + z_0), \tilde{\mu}_\Omega(x_0 - z_0)\}$.

Let $\tilde{\mu}_\Omega(y_0 + z_0) = [\theta_1,\theta_2]$ and $\tilde{\mu}_\Omega(x_0 - z_0) = [\theta_3,\theta_4]$ and $\tilde{\mu}_\Omega(y_0 + x_0) = [t_1,t_2]$.

Then $[t_1,t_2] \prec \text{rmin}\{[\theta_1,\theta_2],[\theta_3,\theta_4]\}=[\min\{\theta_1,\theta_3\},\min\{\theta_2,\theta_4\}]$.

So, $t_1 < \min\{\theta_1,\theta_3\}$ and $t_2 < \min\{\theta_2,\theta_4\}$. Let us consider,

$[\rho_1,\rho_2] = \frac{1}{2}[\tilde{\mu}_\Omega(y_0 + x_0) + \text{rmin}\{\tilde{\mu}_\Omega(y_0 + z_0), \tilde{\mu}_\Omega(x_0 - z_0)\}]$

$= \frac{1}{2}\left[[t_1, t_2] + [min\{\theta_1, \theta_3\}, min\{\theta_2, \theta_4\}]\right]$

$=\left[\frac{1}{2}(t_1 + \min\{\theta_1, \theta_3\}), \frac{1}{2}(t_2 + \min\{\theta_2, \theta_4\})\right]$.

Therefore, $\min\{\theta_1, \theta_3\} > \rho_1 = \frac{1}{2}(t_1 + \min\{\theta_1, \theta_3\}) > t_1$ and

$\min\{\theta_2, \theta_4\} > \rho_2 = \frac{1}{2}(t_2 + \min\{\theta_2, \theta_4\}) > t_2$.

Hence $[\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] \succ [\rho_1,\rho_2] \succ [t_1,t_2]$, so that $(y_0 + x_0) \notin U(\tilde{\mu}_\Omega |[t_1,t_2])$ which is a contradiction, since $\tilde{\mu}_\Omega(y_0 + z_0)=[\theta_1, \theta_2] \succ [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] \succ [\rho_1,\rho_2]$ and $\tilde{\mu}_\Omega(x_0 - z_0)=[\theta_3, \theta_4] \succ [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] \succ [\rho_1,\rho_2]$ this implies

$(y_0 + x_0) \in U(\tilde{\mu}_\Omega |[t_1,t_2])$. Thus $\tilde{\mu}_\Omega(y + x) \succcurlyeq \text{rmin}\{\tilde{\mu}_\Omega(y + z),\tilde{\mu}_\Omega(x - z)\}$, for all $x, y, z \in X$.

And $\tilde{\nu}_\Omega(y_0 + x_0) \succ \text{rmax}\{\tilde{\nu}_\Omega(y_0 + z_0), \tilde{\nu}_\Omega(x_0 - z_0)\}$.

Let $\tilde{\nu}_\Omega(y_0 + z_0) = [\eta_1, \eta_2]$ and $\tilde{\nu}_\Omega(x_0 - z_0) = [\eta_3, \eta_4]$ and $\tilde{\nu}_\Omega(y_0 + x_0) = [s_1,s_2]$.

Then $[s_1,s_2] \succ \text{rmax}\{[\eta_1, \eta_2],[\eta_3, \eta_4]\}=[\max\{\eta_1, \eta_2\},\max\{\eta_3, \eta_4\}]$.

So, $s_1 > \max\{\eta_1, \eta_3\}$ and $s_2 > \max\{\eta_2, \eta_4\}$. Let us consider,

$[\sigma_1, \sigma_2] = \frac{1}{2}[\tilde{\nu}_\Omega(y_0 + x_0) + \text{rmax}\{\tilde{\nu}_\Omega(y_0 + z_0), \tilde{\nu}_\Omega(x_0 - z_0)\}]$

$= \frac{1}{2}\left[[s_1, s_2] + [max\{\eta_1, \eta_3\}, max\{\eta_2, \eta_4\}]\right]$

$=\left[\frac{1}{2}(s_1 + \max\{\eta_1, \eta_3\}), \frac{1}{2}(s_2 + \max\{\eta_2, \eta_4\})\right]$.

Therefore, $\max\{\eta_1, \eta_3\} < \sigma_1 = \frac{1}{2}(s_1 + \max\{\eta_1, \eta_3\}) < s_1$ and

$\max\{\eta_2,\eta_4\} < \sigma_2 = \frac{1}{2}(s_2 + \max\{\eta_2,\eta_4\}) < s_2$.

Hence $[\max\{\eta_1,\eta_3\}, \max\{\eta_2,\eta_4\}] \prec [\ _1, \sigma_2] \prec [s_1, s_2]$, so that $(y_0 + x_0) \notin U(\tilde{v}_\Omega \| [s_1,s_2])$ which is a contradiction, since

$\tilde{v}_\Omega(y_0 + z_0) = [\eta_1, \eta_2] \prec [\max\{\eta_1,\eta_3\}, \max\{\eta_2,\eta_4\}] \prec [\sigma_1, \sigma_2]$ and

$\tilde{v}_\Omega(x_0 - z_0) = [\eta_3, \eta_4] \prec [\min\{\eta_1,\eta_3\}, \min\{\eta_2,\eta_4\}] \prec [\sigma_1, \sigma_2]$ this implies

$(y_0 + x_0) \in U(\tilde{v}_\Omega \| [s_1,s_2])$. Thus $\tilde{v}_\Omega(y + x) \preccurlyeq \text{rmax}\{\tilde{v}_\Omega(y + z), \tilde{v}_\Omega(x - z)\}$, for all $x, y, z \in X$.

Hence, $\Omega = <\tilde{\mu}_\Omega, \tilde{v}_\Omega>$ is an interval-valued bifuzzy ψ-ideal of X.⌂

Theorem 3.11.

Any ψ-ideal of ψ-algebra$(X\,; +, -, 0)$ can be realized as both the upper $[t_1,t_2]$-Level and Lower $[s_1, s_2]$-Level of some interval-valued bifuzzy ψ-ideals of X.

Proof.

Let P be an interval-valued bifuzzy ψ-ideal of X and Ω be interval-valued bifuzzy subset on X defined by

$\tilde{\mu}_\Omega(x) = \begin{cases} [\alpha_1, \alpha_2], & \text{if } x \in P \\ [0,0], & \text{otherwise} \end{cases}$ and $\tilde{v}_\Omega(x) = \begin{cases} [\beta_1, \beta_2], & \text{if } x \in P \\ [1,1], & \text{othrwise} \end{cases}$

For all$[\alpha_1, \alpha_2] \in D[0,1]and[\beta_1, \beta_2] \in D[0,1]$, we consider the following cases

Case 1) If $y + z, x - z \in P$, then $\tilde{\mu}_\Omega(y + z) = [\alpha_1, \alpha_2], \tilde{v}_\Omega(y + z) = [\beta_1, \beta_2]$ and $\tilde{\mu}_\Omega(x - z) = [\alpha_1, \alpha_2]$, $\tilde{v}_\Omega(x - z) = [\beta_1, \beta_2]$.

Thus, $\tilde{\mu}_\Omega(y + x) = [\alpha_1, \alpha_2] = \text{rmin}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = \text{rmin}\{\tilde{\mu}_\Omega(y + z), \tilde{\mu}_\Omega(x - z)\}$ and

$$\tilde{v}_\Omega(y + x) = \text{rmax}\{[\beta_1, \beta_2], [\beta_1, \beta_2]\} = \text{rmax}\{\tilde{v}_\Omega(y + z), \tilde{v}_\Omega(x - z)\}.$$

Case 2) If $y + z \in P$ and $x - z \notin P$, then $\tilde{\mu}_\Omega(y + z) = [\alpha_1, \alpha_2], \tilde{v}_\Omega(y + z) = [\beta_1, \beta_2]$ and $\tilde{\mu}_\Omega(x - z) = [0,0]$, $\tilde{v}_\Omega(x - z) = [1,1]$.

Thus $\tilde{\mu}_\Omega(y + x) = [0,0] \succcurlyeq \text{rmin}\{[\alpha_1, \alpha_2], [0,0]\} = \text{rmin}\{\tilde{\mu}_\Omega(y + z), \tilde{\mu}_\Omega(x - z)\}$ and

$$\tilde{v}_\Omega(y + x) = [1,1] \preccurlyeq \max\big[[\beta_1, \beta_2], [1,1]\big] = \max\{\tilde{v}_\Omega(y + z), \tilde{v}_\Omega(x - z)\}.$$

Case 3) If $y + z \notin P$ and $x - z \in P$, then $\tilde{\mu}_\Omega(y + z) = [0,0], \tilde{v}_\Omega(y + z) = [1,1]$ and $\tilde{\mu}_\Omega(x - z) = [\alpha_1, \alpha_2], \tilde{v}_\Omega(x - z) = [\beta_1, \beta_2]$. Thus, $\tilde{\mu}_\Omega(y + x) = [0,0] \succcurlyeq \text{rmin}\{[0,0], [\alpha_1, \alpha_2]\} = \text{rmin}\{\tilde{\mu}_\Omega(y + z), \tilde{\mu}_\Omega(x - z)\}$ and $\tilde{v}_\Omega(y + x) = [1,1] \preccurlyeq \max[[1,1], [\beta_1, \beta_2]] = \max\{\tilde{v}_\Omega(y + z), \tilde{v}_\Omega(x - z)\}$.

Case 4) If $y + z \notin P, x - z \notin P$ and y, then $\tilde{\mu}_\Omega(y + z) = [0,0], \tilde{v}_\Omega(y + z) = [1,1]$ and $\tilde{\mu}_\Omega(x - z) = [0,0], \tilde{v}_\Omega(x - z) = [1,1]$.

Now, $\tilde{\mu}_\Omega(y + x) = [0,0] = \text{rmin}\{[0,0], [0,0]\} = \text{rmin}\{\tilde{\mu}_\Omega(y + z), \tilde{\mu}_\Omega(x - z)\}$ and $\tilde{v}_\Omega(y + x) = [1,1] \preccurlyeq \max[1,1] = \max\{\tilde{v}_\Omega(y + z), \tilde{v}_\Omega(x - z)\}$.

Hence, $\tilde{\mu}_\Omega(y + x) \succcurlyeq \text{rmin}\{\tilde{\mu}_\Omega(y + z), \tilde{\mu}_\Omega(x - z)\}$ and $\tilde{v}_\Omega(y + x) \preccurlyeq \text{rmax}\{\tilde{v}_\Omega(y + z), \tilde{v}_\Omega(x - z)\}$.

Therefore, Ω is an interval-valued bifuzzy ψ-ideal of X.⌂

Theorem 3.12.

Every interval-valued bifuzzy ψ -ideal of ψ-algebra $(X;\, +, -, 0)$ is an interval-valued bifuzzy ψ -subalgebra of X.

Proof:

Let $(X ; +, -, 0)$ be an ψ -algebra and $\Omega =< \tilde{\mu}_{\Omega}(x), \tilde{\nu}_{\Omega}(x) >$ is an interval-valued bifuzzy ψ-ideal of X.

Since Ω is an interval-valued bifuzzy ψ-ideal of X, then by Proposition (3.4), for every $\tilde{t}, \tilde{s} \in$ D[0, 1], $\tilde{U}(\Omega; \tilde{t}, \tilde{s}) =$ $\{ x \in X \mid \tilde{\mu}_{\Omega}(x) \succcurlyeq \tilde{t}, \tilde{\nu}_{\Omega}(x) \preccurlyeq \tilde{s}\}$, is ideal of X . By Proposition (2.4), for every $\tilde{t}, \tilde{s} \in$ D[0, 1], $\tilde{U}(\Omega; \tilde{t}, \tilde{s})$ is ψ-subgalgebra of X .

Hence Ω is an interval-valued bifuzzy ψ-subalgebra of X by Proposition (2.23). □

Remark 3.13.

The converse of Theorem (3.12) is not true as the following example:

Example 3.14.

Let X={0,1,2, 3} in which $(+, -)$ be a defined by the following table:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

-	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

Then $(X; +, -, 0)$ is an ψ -algebra. Define an interval-valued bifuzzy subset $\Omega =<\tilde{\mu}_{\Omega}, \lambda_{\Omega}>$ of X is fuzzy subset μ: $X \rightarrow [0,1]$ by:

$\tilde{\mu}_{\Omega}(x) = \begin{cases} [0.3,0.5] & if x = \{0,1,2\} \\ [0.4,0.6] & otherwise \end{cases}$ and $\tilde{\nu}_{\Omega} = \begin{cases} [0.4,0.8] & if x = \{0,1,2\} \\ [0.2,0.7] & otherwise \end{cases}$.

The set $\Omega =< \tilde{\mu}_{\Omega}(x), \tilde{\nu}_{\Omega}(x) >$is not an interval-valued bifuzzy ψ-ideal of X.

Note that $\tilde{\nu}_{\Omega}$ is not an anti-fuzzy ψ-ideal of X since

$\tilde{\nu}_{\Omega}(1 + 2) = [0.2,0.7] \not\succcurlyeq [0.4,0.8] = \max\{\tilde{\nu}_{\Omega}(1 + 0), \tilde{\nu}_{\Omega}(2 - 0)\}$

$= \max\{\tilde{\nu}_{\Omega}(1), \tilde{\nu}_{\Omega}(2)\}$.

Proposition 3.15.

If an interval-valued bifuzzy subset $\Omega =<\tilde{\mu}_{\Omega}, \tilde{\nu}_{\Omega}>$ is an interval-valued bifuzzy ψ-ideal of X, then the upper $[t_1,t_2]$-Level and Lower $[s_1,s_2]$-Level of Ω are ψ-ideals of X.

Proof.

Let $(y + z), (x - z) \in U(\tilde{\mu}_{\Omega} | [t_1,t_2])$, then $\tilde{\mu}_{\Omega}(y + z) \succcurlyeq [t_1,t_2]$ and $\tilde{\mu}_{\Omega}(x - z) \succcurlyeq [t_1,t_2]$. It follows that $\tilde{\mu}_{\Omega}(y + x) \succcurlyeq$ rmin$\{\tilde{\mu}_{\Omega}(y + z), \tilde{\mu}_{\Omega}(x - z)\} \succcurlyeq [t_1,t_2]$, so that

$(y + x) \in U(\tilde{\mu}_{\Omega} | [t_1,t_2])$. Hence $U(\tilde{\mu}_{\Omega} | [t_1,t_2])$ is an ψ-ideal of X.

Let $(y + z), (x - z) \in L(\tilde{\nu}_{\Omega} | [s_1,s_2])$, then $\tilde{\nu}_{\Omega}(y + z) \preccurlyeq [s_1,s_2]$ and $\tilde{\nu}_{\Omega}(x - z) \preccurlyeq [s_1,s_2]$. It follows that $\tilde{\nu}_{\Omega}(y + x) \preccurlyeq$ rmax$\{\tilde{\nu}_{\Omega}(y + z), \tilde{\nu}_{\Omega}(x - z)\} \preccurlyeq [s_1,s_2]$, so that

$(y+x) \in L(\tilde{v}_\Omega|[s_1,s_2])$. Hence $L(\tilde{v}_\Omega|[s_1,s_2])$ is an ψ-ideal of X. ⌂

Definition 3.16[3].

Let $f:(X;+,-,0) \rightarrow (Y;+',-',0')$ be a mapping from the set X to a set Y.

If $\Omega = <\tilde{\mu}_\Omega, \tilde{v}_\Omega>$ is an interval-valued bifuzzy subset of X, then the interval-valued bifuzzy subset $\beta = <\tilde{\mu}_\beta, \tilde{v}_\beta>$ of Y defined by:

$$f(\tilde{\mu}_\Omega)(y) = \tilde{\mu}_\beta(y) = \begin{cases} \underset{x\in f^{-1}(y)}{rsup} \tilde{\mu}_\Omega(x) & if \quad f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ 0 & otherwise \end{cases}$$

$$f(\tilde{v}_\Omega)(y) = \tilde{v}_\beta(y) = \begin{cases} \underset{x\in f^{-1}(y)}{inf} \tilde{v}_\Omega(x) & if \quad f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ 1 & otherwise \end{cases}$$

is said to be **the image of Ω under** f **.**

Similarly if $\beta = <\tilde{\mu}_\beta, \tilde{v}_\beta>$ is an interval-valued bifuzzy subset of Y, then the interval-valued bifuzzy subset $\Omega = (\beta \circ f)$ in X (i.e the interval-valued bifuzzy subset defined by $\tilde{\mu}_\Omega(x) = \tilde{\mu}_\beta(f(x))$, $\tilde{v}_\Omega(x) = \tilde{v}_\beta(f(x))$ for all $x \in X$) is called **the pre-image of β under** f).

Theorem 3.18.

A homomorphic pre-image of interval-valued bifuzzy ψ-ideal is also interval-valued bifuzzy ψ-ideal.

Proof.

Let $f:(X;+,-,0) \rightarrow (Y;+',-',0')$ be homomorphism from an ψ-algebra X into an ψ-algebra Y.

If $\beta = <\tilde{\mu}_\beta, \tilde{v}_\beta>$ is interval-valued bifuzzy ψ-ideal of Y and $\Omega = <\tilde{\mu}_\Omega, \tilde{v}_\Omega>$ the pre-image of β under f, then $\tilde{\mu}_\Omega(x) = \tilde{\mu}_\beta(f(x))$, $\tilde{v}_\Omega(x) = \tilde{v}_\beta(f(x))$, for all $x \in X$. Let $x \in X$, then

$(\tilde{\mu}_\Omega)(0) = \tilde{\mu}_\beta(f(0)) \succcurlyeq \tilde{\mu}_\beta(f(x)) = \tilde{\mu}_\Omega(x)$, and $(\tilde{v}_\Omega)(0) = \tilde{v}_\beta(f(0)) \preccurlyeq \tilde{v}_\beta(f(x)) = \tilde{v}_\Omega(x)$.

Now, let $x, y, z \in X$, then

$\tilde{\mu}_\Omega(y+x) = \tilde{\mu}_\beta(f(y+x))$

$\succcurlyeq rmin\{\tilde{\mu}_\beta(f(y+z)), \tilde{\mu}_\beta(f(x-z))\}$

$= rmin\{\tilde{\mu}_\Omega(y+z), \tilde{\mu}_\Omega(x-z)\}$, and

$\tilde{v}_\Omega(y+x) = \tilde{v}_\beta(f(y+x))$

$\preccurlyeq rmax\{\tilde{v}_\beta(f(y+z)), \tilde{v}_\beta(f(x-z))\}$

$= rmax\{\tilde{v}_\Omega(y+z), \tilde{v}_\Omega(x-z)\}$. ⌂

Definition 3.19[2].

Let $f:(X;+,-,0) \rightarrow (Y;+',-',0')$ be a mapping from a set X into a set Y.

Ω =<$\tilde{\mu}_\Omega$, $\tilde{v}_\Omega$> is an interval-valued bifuzzy subset of X **has sup and inf properties** if for any subset T of X, there exist t, s ∈ T such that $\tilde{\mu}_\Omega(t) = \underset{t0\in T}{rsup}\ \tilde{\mu}_\Omega(t_0)$ and $\tilde{v}_\Omega(s) = \underset{s0\in T}{r\,inf}\,\tilde{v}_\Omega(s_0)$.

Theorem 3.20.

Let $f: (X; +, -, 0) \to (Y; +', -', 0')$ be a epimorphism from an ψ-algebra X into an ψ-algebra Y. For every interval-valued bifuzzy ψ-ideal Ω =<$\tilde{\mu}_\Omega$, $\tilde{v}_\Omega$> of X with **sup and inf properties**, then f (Ω) is an interval-valued bifuzzy ψ-ideal of Y.

Proof.

Since rsup(∅) = [0, 0] and rinf (∅) = [1,1] , then

Note that, 0 $\in f^{-1}$ (0') where 0,0' are the zero of X and Y, respectively. Thus

$\tilde{\mu}_\beta(0') = \underset{t\in f^{-1}(0')}{rsup}\ \tilde{\mu}_\Omega(t) = \tilde{\mu}_\Omega(0) \succcurlyeq \tilde{\mu}_\Omega(x) = \underset{t\in f^{-1}(x')}{rsup}\ \tilde{\mu}_\Omega(t) = \tilde{\mu}_\beta(x')$, and

$\tilde{v}_\beta(0') = \underset{t\in f^{-1}(0')}{rinf}\ \tilde{v}_\Omega(t) = \tilde{v}_\Omega(0) \preccurlyeq \tilde{v}_\Omega(x) = \underset{t\in f^{-1}(x')}{rinf}\ \tilde{v}_\Omega(t) = \tilde{v}_\beta(x')$,for all x ∈X , which implies that $\tilde{\mu}_\beta$ (0') $\succcurlyeq \tilde{\mu}_\beta(x')$, and $\tilde{v}_\beta(0') \preccurlyeq \tilde{v}_\beta(x')$, for all x' ∈Y .

Hence ($\tilde{\mu}_\Omega$)(0) $\succcurlyeq \tilde{\mu}_\Omega$ (x) and ($\tilde{v}_\Omega$)(0)$\preccurlyeq$ $\tilde{v}_\Omega$ (x), for all x ∈X .

For any x' , y' , z' ∈ Y, let x $\in f^{-1}$(x') , y $\in f^{-1}$(y') and z $\in f^{-1}$(z') be such that

By Definition $\tilde{\mu}_\beta(y' +' z') = f(\tilde{\mu}_\Omega)(y' +' z') = \underset{y+z\,\in f^{-1}(y'+'z')}{rsup} \tilde{\mu}_\Omega(y+z)$ &

$\tilde{v}_\beta(y' +' z') = f(\tilde{v}_\Omega)(y' +' z') = \underset{y+z\,\in f^{-1}(y'+'z')}{rinf} \tilde{v}_\Omega(y+z)$ and

$\tilde{\mu}_\beta(x' -' z') = f(\tilde{\mu}_\Omega)(x' -' z') = \underset{x-z\in f^{-1}(x'-'z')}{rsup} \tilde{\mu}_\Omega(x-z)$ &

$\tilde{v}_\beta(x' -' z') = f(\tilde{v}_\Omega)(x' -' z') = \underset{x-z\,\in f^{-1}(x'-'z')}{rinf} \tilde{v}_\Omega(x-z)$ for all x', y' , z'∈ Y and

Also ,

$\tilde{\mu}_\beta(y' + x') = \underset{t\in f^{-1}(y'+'x')}{rsup} \tilde{\mu}_\Omega(t) = \tilde{\mu}_\Omega(y+x)$

$\succcurlyeq$ rmin {$\tilde{\mu}_\Omega$ $(y+z)$, $\tilde{\mu}_\Omega$ $(x-z)$},

= rmin{ $\underset{t\in f^{-1}(y'+'z')}{rsup} \tilde{\mu}_\Omega(t), \underset{t\in f^{-1}(x'-'z')}{rsup} \tilde{\mu}_\Omega(t)$}

= rmin { $\tilde{\mu}_\beta(y' + z'), \tilde{\mu}_\beta(x' - z')$ } and

$\tilde{v}_\beta(y' + x') = \underset{t\in f^{-1}(y'+'x')}{rinf} \tilde{\mu}_\Omega(t) = \tilde{v}_\Omega(y+x)$

$\preccurlyeq$ rmax {$\tilde{v}_\Omega$ $(y+z)$, $\tilde{v}_\Omega$ $(x-z)$},

= rmax { $\underset{t\in f^{-1}(y'+'z')}{rinf} \tilde{v}_\Omega(t), \underset{t\in f^{-1}(x'-'z')}{rinf} \tilde{v}_\Omega(t)$}

$= \text{rmax} \{ \tilde{v}_\beta(y' + 'z'), \tilde{v}_\beta(x' - 'z') \}$.

$\tilde{\mu}_\beta(y' + 'x') \geqslant rmin \{\tilde{\mu}_\beta(y' + 'z'), \tilde{\mu}_\beta(x' - z')\}$, and

$\tilde{v}_\beta (y' + 'x') \leqslant rmax\{\tilde{v}_\beta (y' + 'z'), \tilde{v}_\beta (x' - 'z')\}$, for all x', y', z' $\in Y$.

Hence, β is an interval-valued bifuzzy ψ-ideal of . ⌂

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