International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 8 Issue 6 June - 2024, Pages: 90-98

Using Richardson acceleration to improve values of double integrals

Rana Hasan hilal

Department of Mathematics / Faculty of education for Girls /University of Kufa,Najaf ,Iraq

Abstract: *The main goal of this research is to numerically calculate double integrals with continuous integrals.Using the TS method ((the combined method of using the Richardson acceleration method with the trapezoidal rule on the outer dimension Y and Simpson's rule on the inner dimensionX,when the number of partial periods into which the external integration period is divided is equal to the number of partial periods into which the internal integration period is divided, it gives good results in terms of accuracy and speed of approaching the real value, and thus the method of calculating double integrals numerically can be relied upon.*

Keyword: Richardson acceleration, trapezoidal rule

1. Introduction

The subject of numerical analysis is characterized by devising various methods to find approximate solutions to specific mathematical problems in an effective manner. The efficiency of these methods depends on both the accuracy and ease with which they can be implemented. Modern numerical analysis is the numerical interface to the broad field of applied analysis

In this research, we discuss a numerical method composed of the trapezoidal base and the Simpson rule, then improving the results using the Richardson acceleration. We will symbolize this method with the Richardson acceleration with the symbol $\bar{R}ST$: S represents the Simpson rule, T represents the trapezoidal base, and $\bar{R}$ the Richardson acceleration represents the following. The derivation of the method follows.

2.2Simpson's rule on the outer dimension and the trapezoid rule on the inner dimension (ST) :

Applying Simpson's rule to the one-way integral $\int_c^d f(x,y)dy$ we get:

$$\int_c^d f(x,y)dy = \frac{h}{3}\left(f(x,c) + f(x,d) + 4\sum_{j=1}^{n/2} f(x,y_{(2j-1)}) + 2\sum_{j=1}^{(n/2)-1} f(x,y_{(2j)}) \right)$$
$$-\frac{(d-c)h^4}{180}\frac{\partial^4 f(x,\lambda_1)}{\partial y^4} + \frac{(d-c)h^6}{1512}\frac{\partial^6 f(x,\lambda_2)}{\partial y^6} - \cdots \qquad ...(1)$$

whereas

$$j = 1,2,...,(n/2)\text{-}1 \ , \ y_{2j} = c + 2jh \quad \lambda_1, \lambda_2, \ldots \in (c,d) \text{ و } j = 1,2,...,n/2 \ , \ y_{(2j\text{-}1)} = c + (2j-1)h$$

By numerically integrating both sides of formula (1) and taking each of its parts over the interval $[a,b]$ using the trapezoidal rule over the distance, we obtain:

$$i) \quad \int_a^b f(x,c)dx = \frac{h}{2}\left(f(a,c) + f(b,c) + 2\sum_{i=1}^{n-1} f(x_i,c) \right)$$
$$-\frac{(b-a)h^2\partial^2 f(\eta_1,c)}{12\ \partial x^2} + \frac{(b-a)h^4\partial^4 f(\eta_2,c)}{720\ \partial x^4} - \ldots$$

$$ii)\quad \int_a^b f(x,d)dx = \frac{h}{2}\left(f(a,d)+f(b,d)+2\sum_{i=1}^{n-1} f(x_i,d)\right)$$

$$-\frac{(b-a)h^2\partial^2 f(\mu_1,d)}{12\,\partial x^2}+\frac{(b-a)h^4\partial^4 f(\mu_2,d)}{720\,\partial x^4}-\ldots$$

$$iii)\quad \int_a^b f(x,y_{(2j-1)})dx = \frac{h}{2}\left(f(a,y_{(2j-1)})+f(b,y_{(2j-1)})+2\sum_{i=1}^{n-1} f(x_i,y_{(2j-1)})\right)$$

$$-\frac{(b-a)h^2\partial^2 f(\xi_{1j},y_{(2j-1)})}{12\,\partial x^2}+\frac{(b-a)h^4\partial^4 f(\xi_{2j},y_{(2j-1)})}{720\,\partial x^4}-\ldots$$

$$iv)\quad \int_a^b f(x,y_{2j})dx = \frac{h}{2}\left(f(a,y_{2j})+f(b,y_{2j})+2\sum_{i=1}^{n-1} f(x_i,y_{2j})\right)$$

$$-\frac{(b-a)h^2\partial^2 f(\tau_{1j},y_{2j})}{12\,\partial x^2}+\frac{(b-a)h^4\partial^4 f(\tau_{2j},y_{2j})}{720\,\partial x^4}-\ldots$$

$$\eta_i,\mu_i,\xi_{ij},\tau_{ij}\in(a,b)\quad ,i,j=1,2,\ldots \qquad :$$

Compensating everyone i ، ii ، iii و iv In formula (1) we get:

$$\int_c^d\int_a^b f(x,y)dxdy = \frac{h^2}{6}\Bigg[f(a,c)+f(a,d)+f(b,c)+f(b,d)+4\sum_{i=1}^{n-1}\left(f(x_i,c)+f(x_i,d)\right)$$

$$+4\sum_{j=1}^{n/2}\left(f(a,y_{(2j-1)})+f(b,y_{(2j-1)})+2\sum_{i=1}^{n-1} f(x_i,y_{(2j-1)})\right)$$

$$+2\sum_{j=1}^{(n/2)-1}\left(f(a,y_{2j})+f(b,y_{2j})+2\sum_{i=1}^{n-1} f(x_i,y_{2j})\right)\Bigg]$$

$$+\frac{h}{3}\Bigg[-\frac{(b-a)h^2\partial^2 f(\eta_1,c)}{12\,\partial x^2}+\frac{(b-a)h^4\partial^4 f(\eta_2,c)}{720\,\partial x^4}-\ldots$$

$$+4\sum_{j=1}^{n/2} -\frac{(b-a)h^2\partial^2 f(\xi_{1j},y_{(2j-1)})}{12\,\partial x^2}+\frac{(b-a)h^4\partial^4 f(\xi_{2j},y_{(2j-1)})}{720\,\partial x^4}-\ldots$$

$$+2\sum_{j=1}^{(n/2)-1} -\frac{(b-a)h^2\partial^2 f(\tau_{1j},y_{2j})}{12\,\partial x^2}+\frac{(b-a)h^4\partial^4 f(\tau_{2j},y_{2j})}{720\,\partial x^4}-\ldots\Bigg]$$

$$+\int_a^b\left[-\frac{(d-c)h^4}{180}\frac{\partial^4 f(x,\lambda_1)}{\partial y^4}+\frac{(d-c)h^6}{1512}\frac{\partial^6 f(x,\lambda_2)}{\partial y^6}-\cdots\right]dx$$

Since $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^4 f}{\partial x^4}, \ldots$ and $\frac{\partial^4 f}{\partial y^4}, \frac{\partial^6 f}{\partial y^6}, \ldots$ are continuous functions at every point of $[a,b]\times[c,d]$. The error formula (correction limits) becomes than

$$E_{ST}(h) = -(d-c)(b-a)\frac{h^2 \partial^2 f(\overline{n_1}, \overline{\mu_1})}{12\ \partial x^2} + (d-c)(b-a)\frac{h^4}{180}\left(\frac{\partial^4 f(\overline{n_2}, \overline{\mu_2})}{4\ \partial x^4} - \frac{\partial^4 f(\hat{n}_1, \hat{\mu}_1)}{\partial y^4}\right) + \ldots$$

$$E_{ST}(h) = I - ST(h) = A_{ST}h^2 + B_{ST}h^4 + C_{ST}h^6 + \ldots.$$

We finally get it

$$ST = \frac{h^2}{6}\Bigg[f(a,c) + f(a,d) + f(b,c) + f(b,d) + 4\sum_{i=1}^{n-1}\big(f(x_i,c) + f(x_i,d)\big)$$
$$+4\sum_{j=1}^{n/2}\left(f(a, y_{(2j-1)}) + f(b, y_{(2j-1)}) + 2\sum_{i=1}^{n-1} f(x_i, y_{(2j-1)})\right) .$$
$$+2\sum_{j=1}^{(n/2)-1}\left(f(a, y_{2j}) + f(b, y_{2j}) + 2\sum_{i=1}^{n-1} f(x_i, y_{2j})\right)\Bigg] + A_{ST}h^2 + B_{ST}h^4 + C_{ST}h^6 + \ldots.$$

3.

3.Complete Richardson

In numerical integration problems, numerical solutions to differential equations, and problems in which the range is divided into a limited number of equal subintervals of equal length, there is a method to improve the results. It is the Richardson interpolation method, which dates back to the English scientist Lewis Fry Richardson (1881-1953), which he introduced. At the beginning of the twentieth century, specifically in 1910.

It is a method applied to problems in which the error formula can be written in the form of a power series in.

We assume in formula (1-1) that the subperiod $[x_0, x_m]$ is divided into m subperiods of the length of each $h = \frac{(x_m - x_0)}{m}$. For example, at the base of the trapezoid

$$J = \int_{x_0}^{x_m} f(x)\ dx \approx h\left[\frac{1}{2}f_0 + f_1 + f_2 + \ldots + f_{m-1} + \frac{1}{2}f_m\right] \ldots(9)$$ ..

The truncated error is defined as follows:

$$R_T = -\left[(x_m - x_0)\left(\frac{h^2}{12}\right)\right] f''(\lambda) = -\left[\frac{(x_m - x_0)^3}{12m^2}\right] f''(\lambda) \ldots(10)$$

wheares $x_0 < \lambda < x_m$

It is also possible to make a value very R_T small by making it large m , assuming f'' that it is specified in (x_0, x_m) .

$$J - I_1 \left[\frac{(x_m - x_0)^3}{12m_1^2}\right] f''(\xi_1). x_0 < \xi_1 < x_m \dots\dots (1)$$
$$J - I_2 \left[\frac{(x_m - x_0)^3}{12m_1^2}\right] f''(\xi_2). x_0 < \xi_2 < x_m \dots\dots (2)$$
frome (2)and (3)

$$J = I_2 + \frac{m_1^2(I_2 - I_1)}{m_2^2 - m_1^2}$$
The Richardson external adjustment process is a special case of the more general case, in which the analytically calculated error is generated over a period of length, and we express it by the formula

$$E = \sum_{j=k}^{\infty} a_j h^j \quad , a_k \neq 0 \qquad (h \to 0)$$

The value of this approximation depends on the extent to which the two derivatives are assumed to be equal. A process of this type that uses two approximate values to calculate a third value is called Richardson’s external modification (or Richardson’s completion). Ralston [1]. We will choose

The Richardson external adjustment process is a special case of the more general case, in which the error calculated analytically is generated over a period of length h

$$E = \sum_{j=k}^{\infty} a_j h^j \quad , a_k \neq 0 \qquad (h \to 0)$$

a_j 's constant

Now let us assume that we have calculated the formula (1-1) for two different values of h_1, h_2 Let us assume that one of these two values is ϕ_1 When to use h_1 and ϕ_2 the other when to use h_2 The real value can be expressed as J follows:

$$J = \phi_1 + \sum_{j=k}^{\infty} a_j h_1^j \quad \dots(4)$$

$$J = \phi_2 + \sum_{j=k}^{\infty} a_j h_2^j \quad \dots(5)$$

By solving formulas (4) and (5) with respect to, it results that $J = \frac{\left(h_1^k \phi_2 - h_2^k \phi_1\right)}{\left(h_1^k - h_2^k\right)} + \sum_{j=k+1}^{\infty} a_j \frac{\left(h_1^k h_2^j - h_2^k h_1^j\right)}{\left(h_1^k - h_2^k\right)}$

,In the special case when $h_1 = 2h_2$, then

$$J = \frac{\left(2^k \phi_2 - \phi_1\right)}{\left(2^k - 1\right)} + \sum_{j=k+1}^{\infty} \left[\frac{\left(2^k - 2^j\right)}{\left(2^k - 1\right)} \right] a_j h^j$$

$$h_2 = h \, .$$

The effect of Richardson's external adjustment is to increase the force so that the error is very small, that is, this process is used to accelerate the approach of the approximate values to the true integral value. Ralston [1]

4.Example:

Real value	integral
2	$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin(x+y)\, dxdy$
1.08913865206603	$\int_1^2\int_1^2 \ln\left(x+y\right) dxdy$
0.0614477281973 3	$\int_1^2\int_0^1 xe^{-(x+y)} dxdy$

1-The function $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin(x+y)\, dxdy$ is defined and can be differentiated for each $(x, y) \in \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$. Therefore, the error formulas for the aforementioned integral when applying the rule ST

$$E_{ST}(h) = I - ST(h) = A_{ST} h^2 + B_{ST} h^4 + C_{ST} h^6 + \ldots.$$

We conclude from Table (1) that n=44the value of the integration using the rule ST is correct to three decimal places, while the value using $\bar{R}ST$ the method is correct to eleven decimal places.

n	ST	$k=2$	$k=4$	$k=6$	$k=8$
2	1.90044181673500				
4	1.97449730385475	1.99918246622800			
6	1.98861610001364	1.99991113694075	2.00009050203927		
8	1.99358688148291	1.99997788622911	2.00000878161401	1.99999108806188	
10	1.99589274319483	1.99999205290490	2.00000188127619	1.99999942973800	2.000 00111

12	1.99714666001216	1.99999647096066	2.00000058613927	1.99999993400062	2.000 00008
14	1.99790319111780	1.99999820033342	2.00000022862945	1.99999999367701	2.000 00001
16	1.99839439350943	1.99999898798875	2.00000010371762	2.00000000201341	2.000 00000
18	1.99873124418746	1.99999938791650	2.00000005246174	2.00000000256731	2.000 00000
20	1.99897223337274	1.99999960832052	2.00000002881218	2.00000000198877	2.000 00000
22	1.99915056059216	1.99999973782752	2.00000001687728	2.00000000140877	2.000 00000
24	1.99928620586450	1.99999981794942	2.00000001041288	2.00000000098282	2.000 00000
26	1.99939177743241	1.99999986966357	2.00000000670440	2.00000000068890	2.000 00000
28	1.99947555000380	1.99999990424692	2.00000000447373	2.00000000048996	2.000 00000
30	1.99954313651158	1.99999992808143	2.00000000307745	2.00000000035456	2.000 00000
32	1.99959845315708	1.99999994493896	2.00000000217271	2.00000000025953	2.000 00000
34	1.99964429963189	1.99999995713340	2.00000000156906	2.00000000019309	2.000 00000
36	1.99968272039566	1.99999996613081	2.00000000115630	2.00000000014737	2.000 00000
38	1.99971523657892	1.99999997288635	2.00000000086700	2.00000000011203	2.000 00000
40	1.99974299887120	1.99999997803819	2.00000000066003	2.00000000008571	2.000 00000
42	1.99976689072880	1.99999998202251	2.00000000051068	2.00000000007154	2.000 00000
44	1.99978759953003	1.99999998514263	2.00000000039853	2.00000000005022	2.000 00000
table (1)					2.000000000 00000

2- The function $f(x,y)=\ln(x+y)$ is defined and can be differentiated for each $(x,y) \in [1,2]\times[1,2]$. Therefore, the error formulas for the aforementioned integral when applying the rule ST

$$E_{ST}(h)=I-ST(h)=A_{ST}h^2+B_{ST}h^4+C_{ST}h^6+\ldots$$

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 8 Issue 6 June - 2024, Pages: 90-98

We conclude from Table (2) that $n = 36$ the value of the integration using the rule ST is correct to five decimal places, while the value using $\bar{R}ST$ the method is correct to thirteen decimal places.

n	ST	$k=2$	$k=4$	$k=6$	$k=8$
2	1.08666194294075				
4	1.08852373935799	1.08914433816373			
6	1.08886571678687	1.08913929872998	1.08913805825398		
8	1.08898519706037	1.08913881455487	1.08913859045097	1.08913870567865	
10	1.08904046196187	1.08913871067563	1.08913863860765	1.08913865571666	1.08913864564462
12	1.08907047246059	1.08913867813951	1.08913864783388	1.08913865247955	1.08913865149856
14	1.08908856445855	1.08913866537597	1.08913865040624	1.08913865209677	1.08913865193940
16	1.08910030549687	1.08913865955537	1.08913865131038	1.08913865204654	1.08913865202024
18	1.08910835449332	1.08913865659762	1.08913865168283	1.08913865204539	1.08913865204465
20	1.08911411158326	1.08913865496670	1.08913865185519	1.08913865205068	1.08913865205467
22	1.08911837101235	1.08913865400801	1.08913865194233	1.08913865205526	1.08913865205928
24	1.08912161056273	1.08913865341474	1.08913865198962	1.08913865205861	1.08913865206194
26	1.08912413163803	1.08913865303174	1.08913865201680	1.08913865206088	1.08913865206341
28	1.08912613199881	1.08913865277557	1.08913865203314	1.08913865206234	1.08913865206413
30	1.08912774576506	1.08913865259899	1.08913865204340	1.08913865206341	1.08913865206487
32	1.08912906649936	1.08913865247409	1.08913865205004	1.08913865206407	1.08913865206506
34	1.08913016108131	1.08913865238374	1.08913865205448	1.08913865206460	1.08913865206545
36	1.08913107834443	1.08913865231707	1.08913865205757	1.08913865206512	1.08913865206600
table (2)					1.08913865206603

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 8 Issue 6 June - 2024, Pages: 90-98

3-The function $f\left(x,y\right)=xe^{-(x+y)}$ is defined and can be differentiated for each $\left(x,y\right)\in\left[0,1\right]\times\left[1,2\right]$. Therefore, the error formulas for the aforementioned integral when applying the rule ST

$$E_{ST}(h)=I-ST(h)=A_{ST}h^{2}+B_{ST}h^{4}+C_{ST}h^{6}+....$$

n	ST	$k=2$	$k=4$	$k=6$	$k=8$
2	0.05666744324245				
4	0.06024070835403	0.06143179672456			
6	0.06091025547075	0.06144589316413	0.06144936305695		
8	0.06114519754098	0.06144726591698	0.06144790130545	0.06144758481693	
10	0.06125404849147	0.06144756129235	0.06144776621402	0.06144771821901	0.06144774511207
12	0.06131320570331	0.06144765391202	0.06144774018220	0.06144772707443	0.06144772975805
14	0.06134888528124	0.06144769026630	0.06144773290445	0.06144772812157	0.06144772855210
16	0.06137204658655	0.06144770685053	0.06144773034240	0.06144772825636	0.06144772832692
18	0.06138792767027	0.06144771527958	0.06144772928582	0.06144772825730	0.06144772825791
20	0.06139928819925	0.06144771992807	0.06144772879656	0.06144772824164	0.06144772822981
22	0.06140769418018	0.06144772266078	0.06144772854899	0.06144772822813	0.06144772821631
24	0.06141408788818	0.06144772435203	0.06144772841462	0.06144772821859	0.06144772820912
26	0.06141906385797	0.06144772544392	0.06144772833740	0.06144772821216	0.06144772820498
28	0.06142301223827	0.06144772617427	0.06144772829089	0.06144772820783	0.06144772820249
30	0.06142619765491	0.06144772667769	0.06144772826174	0.06144772820490	0.06144772820092
32	0.06142880472813	0.06144772703380	0.06144772824284	0.06144772820286	0.06144772819985
34	0.06143096543605	0.06144772729143	0.06144772823023	0.06144772820147	0.06144772819924
36	0.06143277615084	0.06144772748154	0.06144772822157	0.06144772820043	0.06144772819863
38	0.06143430857333	0.06144772762428	0.06144772821552	0.06144772819973	0.06144772819845
40	0.06143561694141	0.06144772773315	0.06144772821120	0.06144772819920	0.06144772819814
42	0.06143674289586	0.06144772781735	0.06144772820804	0.06144772819874	0.06144772819780
44	0.06143771883483	0.06144772788329	0.06144772820573	0.06144772819855	0.06144772819811
46	0.06143857027062	0.06144772793553	0.06144772820396	0.06144772819819	0.06144772819737
				table(3)	0.06144772819733

.

We conclude from Table (3) that $n = 46$ the value of the integration using the rule ST is correct to for decimal places, while the value using $\bar{R}ST$ the method is correct to thirteen decimal places

.

5.Conclusion

It is clear from the tables of this chapter that when calculating double integrals with continuous integrals when using the Richardson interpolation with the rule ST (Resulting from applying the trapezium rule on the outer dimension gave better results in terms of speed of convergence with a small number of subintervals to the values of Y and Simpson's rule on the inner dimension of true integrals, as the tables showed that the Regardson interpolation method gave better results in the first example when $n = 44$

The value of the integration using the rule STis correct to three decimal places, while the value using the two methods is correct to eleven decimal places $\bar{R}ST$.

In the second example, when $n = 36$ *the value of the integration using the rule* ST *is correct to five decimal places, while the value using the method* $\bar{R}ST$ *is correct to thirteen decimal places, likewise in the third example, when the value of the integration using the rule* ST *is correct to four decimal places, while the value using the method is* $\bar{R}ST$ *correct to thirteen decimal places. Thus, it is possible to rely on a method for calculating double integrals with continuous integrals*

Sources

[1] Anthony Ralston, "A First Course in Numerical Analysis " McGraw –Hil Book Company ,1965.

[2] Al-Sharifi, Fouad Hamza, "Derivation of composite methods from the trapezoid and Simpson rules to calculate binary integrals numerically and their error formulas and improve the results using acceleration methods," Master's thesis submitted to the University of Kufa, 2012.

[3] Saifi, Ali Muhammad Sadiq, "Principles of Numerical Analysis," University of Baghdad, College of Science, 1985.