On Copositive Approximation

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Abstract: This research is a review of copositive approximation, where we presented some basic theories and results on this topic within different spaces and criteria.

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1. Introduction

Numerous articles have been written recently on the notions of copositive approximation see [2-61 We say that two function f and g are copositive on an interval $[a, b]$ if $f(x)$ $g(x) \geq 0$ for all x in $[a, b]$. Let Π_n denote the set of algebraic polynomials of degree less than or equal to n and let $\|.\|$ be the uniform norm on $[a, b]$. Given a continuous function f on $[a, b]$ we define the degree of copositive approximation $\bar{E}_n(f)$ as $inf \{\|f - p\|: p \in \Pi_n\}$ and p copositive with . The degree of approximation to f is

$$\bar{E}_n(f) = inf \{\|f - p\|: p \in \Pi_n\}.$$

2. The main result on copositive approximation .

Theorem 2.1. [**3**] Let $f \in C[0, 1]$ change sign r times in [0, 1]. Then for each $n \geq 1$, there is a polynomial p_n of degree $\leq n$ which is copositive with f and such that

$$\|f - p_n\| \leq K\omega\left(f, \frac{1}{n}\right)$$

where K is an absolute constant independent of f and n.

Lemma 2.2. [**3**] Let f be as in the Theorem. If for each $n > \frac{4}{\delta}$, there is a polynomial P_n of degree $\leq n$ which is copositive with f in $\bigcup_{i=1}^{r}\left(y_i - \frac{1}{2n}, y_i + \frac{1}{2n}\right)$, then there exists a polynomial P_n of degree $\leq C_n$ which is copositive with f in $[0, 1]$ and

$$\|f - P_n\| \leq C\|f - p_n\|.$$

Lemma 2.3. [**3**] Let f be as in the theorem. Then for each $n > \frac{4}{\delta}$ there is a function $f_n \in C^2[0, 1]$ copositive with f in $\bigcup_{i=1}^{r}\left(y_i - \frac{1}{2n}, y_i + \frac{1}{2n}\right)$, and such that

$$\|f - f_n\| \leq K\omega_2\left(f, \frac{1}{n}\right),$$

$$\|f_n''\| \leq Kn^2\omega_2\left(f, \frac{1}{n}\right),$$

and $|f_n'(x)| \geq n\omega_2\left(f, \frac{1}{n}\right)$, $x \in \bigcup_{i=1}^{r}\left(y_i - \frac{1}{2n}, y_i + \frac{1}{2n}\right)$.

Lemma 2.4. [**3**] Let $f \in C^2[0, 1]$ and assume $y_0 = 0 < y_1 < \cdots < y_r < y_{r+1} = 1$.

If $= \min_{0 \leq i \leq r}|y_{i+1} - y_i|$, then for each $n > \frac{4}{\delta}$ there is a polynomial p_n of degree $\leq n$ interpolating f at $y_1, \dots, y_r$ and such that

$$\left\|f^{(j)} - p_n^{(j)}\right\| \leq C\left\|\frac{f''}{n^{2-j}}\right\| , \qquad j = 0, 1$$

where C is a constant depending only on δ and r .

Lemma 2.5. [3] Let f be as in the theorem. Then for each $n > \frac{4}{\delta}$ there is a polynomial p_n of degree $\leq C_n$ which is copositive with f in $\bigcup_{i=1}^{r} \left(y_i - \frac{1}{2n}, y_i + \frac{1}{2n} \right)$ and such that

$$\|f - p_n\| \leq C\omega_2\left(f, \frac{1}{n}\right).$$

Theorem 2.6. [3] Let $f \in C[0,1]$ change sign r times at $0 < y_1 < \cdots < y_r < 1$ and let $\delta = \min_{0 \leq i \leq r} |y_{i+1} - y_i|$ where $y_0 = 0$ and $y_{r+1} = 1$. Then there exists a constant $C = C(r,s)$ but otherwise independent of f and n such that for each $n > \frac{4}{\delta}$ there is a polynomial p_n of degree $\leq C_n$, copositive with f, satisfying

$$\|f - p_n\| \leq C\omega_2\left(f, \frac{1}{n}\right).$$

Theorem 2.7. [4] For every $n \in N, 0 < p < \infty, 0 < \varepsilon \leq 2$ and $A > 0$, there exists a nonnegative function $f \in C^\infty[-1,1]$ such that for every polynomial $P_n \in \pi_n$ that is nonnegative at $x = 1$, the following inequality holds:

$$\|f - P_n\|_{L_p[1-\varepsilon,1]} > A\omega^2(f,1)_P.$$

Lemma 2.8. [4] For every $n \in N, 1 \leq j \leq n - 1$ and $\mu \geq 10$ there exist polynomials T_j and $\bar{T}_j$ of degree $\leq n$ satisfying for $x \in [-1,1]$:

$$0 \leq \chi_j(x) - T_j(x)C(\mu)\,\psi_j^\mu$$

$$0 \leq \bar{T}_j(x) - \chi_j(x)C(\mu)\,\psi_j^\mu$$

Where

$$\chi_j(x) = \begin{cases} 1, & if\ x \geq \chi_j \\ 0, & \text{otherwise} \end{cases}$$

Lemma 2.9. [4] For any function $g \in L_p[-1,1], 0 < p < \infty$, the following inequality holds:

$$\left(\sum_{j=1}^{n} \omega(g, h_j, \rho_j)_p^p \right)^{1/p} \leq C C_0\, \omega_\varphi(g, (n+1)^{-1})_p$$

where, for every $j, \rho_j \supset I_j$ is such that $|\rho_j| \leq C_0|I_j|$ and C depends on p if $0 < p < 1$

Theorem 2.10. [4] If $f \in L_p[-1,1], 0 < p < \infty$ and $f(x) \geq 0, x \in [-1,1]$, then for every $n \in N_0$

$$E_n^{(0)}(f)_p \leq C\omega_\varphi(f, (n+1)^{-1})_p.$$

Theorem 2.11. [4] Let $Y_s = \{y_1, \ldots, y_s\}$ be given, and let $\delta = \min_{0 \leq i \leq r} |y_{i+1} - y_i|$. If $f \in L_p[-1,1] \cap \Delta^0(Y_s), 0 < p < \infty$, then for every $n \in N_0$

$$E_n^{(0)}(f)_p \leq C\omega_\varphi(f, (n+1)^{-1})_p.$$

where C depends on s, δ and also on p in the case for $0 < p < 1$.

Theorem 2.12. [4] Let $f \in L_p[-1,1] \cap \Delta^0(Y_s), 0 < p < \infty$, $s \geq 0$ and let $r \geq 1$ be an integer. Let $T_n = \{z_0, \ldots, z_n \mid -1 = z_0 < z_1 < \cdots < z_{n-1} < z_n = 1\}$ be a given knot sequence such that there are at least $max(2, 4(r-1)^2)$ knots in each open interval (y_j, y_{j+1}), $j = 0, \ldots, s$. Then there exists a spline $s_n \in C^{r-2}[-1,1] \cap \Delta^0(Y_s)$ of order r on knot sequence T_n such that

$$\|f - s_n\| \leq C\omega(f, d)_p .$$

Theorem 2.13. [7] Given $f \in C[a,b]$, then there exists $r^* \in R_f[a,b]$ such that

$$\|f - r^*\| \leq \inf_{r \in R_f[a,b]} \|f - r\| .$$

Lemma 2. 14. [7] Assume that $x_i, y_i \in X_{r^*}$, $(x_i, y_i) \cap S = \{z_{i+1}, \ldots, z_{i+r}\}$ and $f - r^*$ alternates ω_i times between x_i and y_i . Let $r \in R_f[a,b]$ satisfy $\|f - r\| < \|f - r^*\|$, then

i.   If $r^*(x_i) = r(x_i) = 0$ then $r^* - r$ has at least $v + \omega_i + 1$ zeros in $[x_i, y_i]$
ii. If $r^*(x_i) \neq r(x_i)$, $r^*(y_i) \neq r(y_i)$ and $\omega_i = 1$, then $r^* - r$ has at least $v + \omega_i$ zeros in (x_i, y_i)
iii. If $\omega_i = 2$, then $r^* - r$ has at least $v + \omega_i$ zeros in (x_i, y_i).

Theorem 2.15. [7] Let $f \in C[a,b] \sim R_n^m[a,b]$ be an admissible function and $S = (z_, , \ldots, z_k), k \leq m$ as described earlier . Then $r^* \in R_f[a,b]$ is a best approximation to f if and only if there exists a set of open intervals (x_i, y_i) which is an alternant of length $N - k$ for $f - r^*$, where

$$N = 1 + max\{n + \partial p^*, m + \partial q^*\}, \qquad r^* = p^*/q^*$$

Theorem 2.16. [7] Let $f \in C[a,b] \sim R_n^m[a,b]$ and $r^* \in R_f[a,b]$, then r^* is a best approximation to f from $R_f[a,b]$ if and only if for each $h \in \bar{s}_{r^*}$, $min_{x \in S_1 \cup S_2} \sigma(x)h(x) \leq 0$.

Lemma 2. 17. [7] Let $r^* = p^*/q^* \in R_f[a,b]$ be a best approximation to $\in C[a,b] \sim R_n^m[a,b]$. If there exists $r = p/q \in R_f[a,b]$ such that $\sigma(x)[p(x) - r^*(x)q(x)] \geq 0$ for all $x \in X_*$ then $p - r^*q = 0$.

Corollary 2.18. [7] Let $f \in C[a,b] \sim R_n^m[a,b]$ then f has a unique best approximation from $R_f[a,b]$.

Corollary 2.19. [7] Let r^* be the best approximation to f from $R_f[a,b]$, then for every $h \in \bar{s}_{r^*}$ with $\|h\| = 1$, $min_{x \in X_*} \sigma(x)h(x) < 0$.

Definition 2.20. [7] Let $f \in C[a,b]$ and $r^* = p^*/q^*$ be a best approximation to f from $R_f[a,b]$ f is said to be copositive normal if either $\partial p^* = m$ or $q^* = n$.

Theorem 2.21. [7] Let $r^* \in R_f[a,b]$ be the best approximation to f from $R_f[a,b]$ with $(p^*, q^*) = \mathbf{1}$ where $f \in C[a,b]$ If f is copositive normal then there exists a constant $y = y(f) > 0$ such that for all

$r \in R_f[a,b]$

$$\|f - r\| \geq \|f - r^*\| + y\|r - r^*\|.$$

Theorem 2.22. [1] Let $m, k \in N, m < k$ and $f \in \Delta^k \cap W_p^m (I)$. Then for any, $n \geq k - 1$, there exists a polynomial $p_n \in \Pi_n$ such that :

$$\left\|f^{(j)} - p^{(j)}\right\|_p \leq c(p,k)\omega_{k-j}^\varphi \left(f^{(j)}, n^{-1}, I\right)_p \quad for \quad j = 1, \ldots, m.$$

Lemma 2. 23. [1] Let $J_i \subset \ell_i$ and $f \in L_{\psi,p}(\ell_i) \cap \Delta^0(\ell_i), p < 1$. Then there exist $p_{k-1}(f) \in \Pi_{k-1} \cap \Delta^0(\ell_i)$ interpolate f at $k - 1$ points in side J_i , then for any constant $\mu > 0$, we have two cases:

Case (1): For $\breve{\alpha} = \dfrac{J_i - J_i^{(k-1)}}{k-1} + \mu|J_i| < J_i^{(k-1)}$, we have

$$\|p_{k-1}(f)\|_{L_{\psi,p}\left[\frac{J_i - J_i^{(v)}}{k-1}, \breve{\alpha}\right]} \leq C(p,\mu)\| f \|_{L_{\psi,p}\left[\frac{J_i - J_i^{(v)}}{k-1}, \breve{\alpha}\right]}.$$

Case (2) : For $\breve{b} = \frac{J_i - J_i^{(v)}}{k-1} - +\mu|J_i| > J_i^{(v)}$, we have

$$\|p_{k-1}(f)\|_{L_{\psi,p}\left[\breve{b}, \frac{J_i - J_i^{(k-1)}}{k-1}\right]} \leq C(p,\mu)\|f\|_{L_{\psi,p}\left[\breve{b}, \frac{J_i - J_i^{(k-1)}}{k-1}\right]}.$$

Lemma 2. 24. [1] Let $\in L_{\psi,p}(\ell_i) \cap \Delta^0(\ell_i)$, then there exist $p_{k-1}(f) \in \Pi_{k-1} \cap \Delta^0(\ell_i)$ interpolate f at $k-1$ points in side J_i , such that

$$\|f - p_{k-1}(f)\|_{L_{\psi,p}(\ell_i)} \leq c(p,k)\omega_k^\varphi(f, |\ell_i|, \ell_i)_p .$$

Lemma 2. 25. [1] Let $\in L_{\psi,p}(\ell_i) \cap \Delta^0(\ell_i)$, then there exist a polynomial $q_{k-1}(f) \in \Pi_{k-1} \cap \Delta^0(\ell_i)$ interpolate f at $k-1$ points in side ℓ_i , such that

$$\|f - q_{k-1}(f)\|_{L_{\psi,p}(\ell_i)} \leq c(p,k)\tau_k(f, |\ell_i|, \ell_i)_{\psi,p} .$$

Theorem 2.26. [1] Let $\in L_{\psi,p}(I) \cap \Delta^0(J_s)$, $p < 1$, then there exist a polynomial $p_{k-1}(f) \in \Pi_{k-1} \cap \Delta^0(J_s)$, $k > 1$ such that

$$\|f - p_{k-1}\|_{L_{\psi,p}(I)} \leq c(p,k)\tau_k(f, |I|, I)_{\psi,p}.$$

Lemma 2.27. [8] If F $\in L_{\psi,p}(I) \cap \Delta^0(J_s)$ $0 < p < 1$,then there is copositive a piecewise spline

S_{k-1} with F on the knots sequence $\{tr\}_{r=1}^n$ that satisfies:

$$\|f - S_{k-1}\|L\psi, p(I) \leq c(p,k)\,\mathcal{W}k\varphi(F, n-1)\psi, p \text{ , if tr } = \text{Xn} - r, 0 \leq r \leq n.$$

Theorem 2.28. [8] If F $\in L_{\psi,p}(I) \cap \Delta^0(J_s)$ $0 < p < 1, k > 1$, where $J_s = \{j_1, \dots, j_s\}$, then

$$\mathbb{E}_n^{(0)}(F, J_s)\psi, p \leq c\,\mathcal{W}k\varphi(F, n-1)\psi, p, 0 \leq n \leq k - 1.$$

Where $c = A(p, k, b, \mathfrak{m})$ and $A = \left(\frac{|x - j_r|}{1 + |x - j_r|}\right)^m c(p, k), \mathfrak{m} \geq 2$, r $= 0, \dots$ s.

Theorem 2.29. [5] $P(\boldsymbol{x}) = (\boldsymbol{x} \circ \boldsymbol{x})^T M(\boldsymbol{x} \circ \boldsymbol{x})$ *allows for a polynomial s.o.s if and only if* $M \in K_n^0$ *, i.e., if and only if* $M = S + T$ *for matrices* $S \in S_n^+$ *and* $T \in Nn$.

Higher order sufficient conditions can be derived by the polynomial

$$P^{(r)}(\boldsymbol{x}) = P(\boldsymbol{x})\left(\sum_{k=1}^n x_k^2\right)^r = \sum_{i,j=1}^n x_i^2 Mijx_i^2x_j^2\left(\sum_{k=1}^n x_k^2\right)^r. \quad (1)$$

Definition 2.30. [2] The convex cone K_n^r consists of the matrices for which $P^{(r)}(\boldsymbol{x})$ in (1) allows a polynomial sum of squares decomposition.

Theorem 3.1. [4] M $\in K_n^2$ if and only if there are n symmetric n $\times$ n matrices $M^{(ij)} \in Sn$ for i = 1, ..., n and j $= 1, \dots, n$ such that the following system of linear inequalities has a solution:

$$M - M^{(ii)} \in S_n^+ , i = 1,\dots,n,$$
$$M_{ii}^{(ii)} \geq 0 , i = 1,\dots,n,$$

$$2M_{ii}^{(ij)} + 2M_{ij}^{(ii)} \geq 0 . i \neq j ,$$

$$M_{ii}^{(jj)} + M_{jj}^{(ii)} + 4M_{ij}^{(ij)} \geq 0 . i \neq j ,$$

where $M^{(ij)} \in S_n$ for $i = 1,\dots,n$ and $j = 1,\dots,n$.

References

[1] E. S.Bhaya and N. Zuhair , Whitney theorem for copositive approximation in $L_{\psi,p}$ $(I), p < 1$, J. AL-Qadisiyah for computer science and mathematics,(2015), 111-125.

[2] E. De Klerk and D.V. Pasechnik, Approximation of the stability number of graph via copositive programming, SIAM J. Opt., (2002) ,875-892 .

[3] Y.Hu, D. Leviatan and X. Yu ,Copositive polynomial approximation in C[0; 1], J. Anal (1993), 85-90.

[4] T. Mouktonglang and A. Suebsriwichai , Approximation of Copositive Programming via Linear Programming Using Second Order Sum of Square Decomposition, Applied Mathematical Sciences ,(2011) , 3627 – 3635.

[5] P.A. Parrilo, Structured Semidefinite Programs and Semi-algebraic Geometry Methods in Robustness and Optimization, PhD thesis, California Institute of Technology, Pasadena, California, USA. Available at: http: / /www.cds.caltech.edu/ —pablo/, 2000 .

[6] K. A. Kopotun, On Positive and Copositive Polynomial and Spline Approximation in $Lp[-1, 1]$, $0 < p < \infty$, J. Approx. Theory 80 (1996), 320 - 334.

[7] N. PH. Seif and K. A. Taylor, Copositive Rational Approximation , J. Approx. Theory 35 (1982), 225-242.

[8] N. Zuhair , Pointwise Estimates for Finding the Error of Best Approximation by Spline, Positive Algebraic Polynomials and Copositive , Iraqi Journal of Science, (2022), 4967-4986 .