Fuzzy Inverse Odd Neighbor in D^{-1^c} Domination Number

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Abstract— *In the paper, a new domination parameter for the fuzzy graph, called the fuzzy inverse odd neighbor domination number was described. This parameter was studied in a strong fuzzy graph, and constraints were identified for various graphs. The inverse odd neighbor domination number depends on the odd neighbors of the vertices and the values of these vertices such that these vertices do not belong to minimum odd neighbor dominating sets.*

Keywords— *graph, inverse odd neighbor domination, fuzzy inverse odd neighbor domination*.

1. INTRODUCTION

After the discovery of a significant branch of mathematics known as graph theory, its importance in solving various problems was established. The details of this field are extensively covered in [1,2]. In recent years, the concept of fuzzy sets, introduced by Zadeh [3], was studied several years after the development of graph theory. Shortly thereafter, the connection between graphs and fuzzy sets was made to better align with real-life requirements, leading to the creation of a new type of graph known as fuzzy graphs, first studied by Rosenfeld [4]. The branch of domination in graph theory has gained significant attention due to its crucial role in various applications [5]. More of researches introduced some types of domination [6,7].

Researchers have increasingly recognized the importance of fuzzy graphs and their relevance to everyday life, prompting significant research efforts in this area. Due to the critical concept of domination within graph theory, which has been widely studied, the notion of fuzzy domination was introduced to bridge these two branches. The concept of domination in fuzzy graphs was initially introduced by A. Somasundaram and S. Somasundaram [10,11]. This domination number has been characterized in various manners. Mahioub and Soner defined it by considering the minimum fuzzy cardinality of an all-dominating set. Alternatively, Xavier et al.[12] defined it by identifying the smallest dominating set and summing all its elements [13,14]and [15]. In this research, we adopt the definition provided by Xavier et al. for specific expansions.

Let $G = (V, E)$ simple graph. The fuzzy graph (V, σ) is defined as a mapping $\sigma: V \to [0,1]$ and $\mu: E \to [0,1]$ [0, where V is vertex set and for all $u, v \in V$, $\mu(u,v) \leq min(\sigma(u)$ .An edge (u,v) is called effective if $\mu(u,v) = min(\sigma(u), \sigma(v))$ and the fuzzy graph of effective edge for all his edges is called strong. In this study, we examine the strong fuzzy graph, where every two adjacent vertices are connected by an effective edge.

Definition 1.1. Let $G_f = (\sigma, \rho)$ be a fuzzy graph. A subset D^{-1} of $V(G_f)$ is called a *fuzzy inverse odd neighbor of D^c dominating set ($FIOD^cS$)* of G_f if for every vertex $v \in D^{-1}$, v is adjacent to odd number of vertices in D^{c-1} (or $V - D^{-1}$) and it does not belong to D such that D is $MFOD^cS$ and has the minimum sum of membership values of the vertices for all other FOD^cS.

Definition 1.2. A fuzzy inverse odd neighbor of D^{c-1} dominating set D^{-1} of $G_f = (\sigma, \rho)$ is called a *minimal fuzzy odd neighbor of D^{c-1} dominating set* if there is no subset of D^{-1} is a $FIOD^cS$ of G_f.

Definition 1.3. A fuzzy inverse odd neighbor of D^{c-1} dominating set of a fuzzy graph $G_f = (\sigma, \rho)$ with minimum number of vertices is called a *minimum fuzzy inverse odd neighbor of D^{c-1} dominating set* (MFIOD^{c}S) of G_f.

Definition 1.4. A *fuzzy inverse odd neighbor of D^{c-1} domination number* of a fuzzy graph $G_f = (\sigma, \rho)$ is the minimum sum of membership values of the vertices for all minimum fuzzy inverse odd neighbor of D^{-1} dominating sets and denoted by $\gamma_{fodc}^{-1}(G_f)$ or simply $\gamma_{\gamma_{fodc}^{-1}}$.

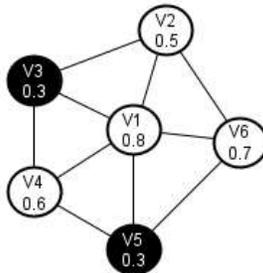**Example 1.5.** For the following wheel graph of order 6

Fig. 1. MFIOD^cS of W_{n_f}.

There are five MFIOD^cS as follows.

$D^{c-1}{}_1 = \{v_2, v_4\}$

$D^{c-1}{}_2 = \{v_2, v_5\}$

$D^{c-1}{}_3 = \{v_3, v_5\}$

$D^{c-1}{}_4 = \{v_3, v_6\}$

$D^{c-1}{}_5 = \{v_4, v_6\}$

since all sets have the same order equal to 2, then we take the set of minimum sum of membership values of the vertices. So,

$\gamma_{fodc}^{-1}\left(W_{n_f}\right) = \sigma(v\ ) + \sigma(v\ )$.

2. MAIN RESULTS

2.1 Proposition 2.1. If G_f is a strong fuzzy path graph, $\gamma_{fodc}^{-1}\left(P_{n_f}\right) =$

$$
\begin{cases}
\begin{cases}
\textbf{has no}, \ \textbf{\textit{if n is odd}} \\
\begin{cases}
\{\sigma(v_1), \sigma(v_2)\}, \quad \textbf{\textit{if }} n = 2 \\
\{\sigma(v_1) + \sigma(v_n), \sigma(v_2) + \sigma(v_3)\}, \textbf{\textit{if }} n = 4 \\
\max\begin{cases}
\begin{cases}
\sum_{i=0}^{\frac{n}{4}-2}(\sigma(v_{4+4i}) + \sigma(v_{5+4i})) + \sigma(v_1) + \sigma(v_n)), \\
\sum_{i=0}^{\frac{n}{4}-1}((\sigma(v_{2+4i}) + \sigma(v_{3+4i})) + \sigma(v_n)).
\end{cases}, \textbf{\textit{if }} n \equiv 0 \ (mod\ 4) \\
\begin{cases}
(\sum_{i=0}^{\left|\frac{n}{4}\right|-1}(\sigma(v_{4+4i}) + \sigma(v_{5+4i})) + \sigma(v_1)), \\
\sum_{i=0}^{\frac{n}{4}-1}((\sigma(v_{2+4i}) + \sigma(v_{3+4i})) + \sigma(v_n))
\end{cases}, \textbf{\textit{if }} n \equiv 2 \ (mod\ 4)
\end{cases} \\
\textbf{\textit{if n is even}}
\end{cases}
\end{cases}
\end{cases}
$$

Proof:
There are cases.
Case1. If $n = 2$, then it is clear there are only two $MFIOD^{-1^c}S$, so $D_1 = \{v_1\}$ and $D_2 = \{v_2\}$. Thus, $\gamma_{fodc}^{-1}(P_n) = \max\{\sigma(v_1), \sigma(v_2)\}$.
Case 2. If $n = 4$, since the $MFIOD^c S$ are two sets $|D_1| = |D_2| = 2$ because every two adjacent vertices is adjacent to one vertex in D^c, so $\gamma_{fodc}^{-1}\left(P_{n_f}\right) = \max\{\sigma(v_1) + \sigma(v_n), \sigma(v_2) + \sigma(v_3)\}$.
Case 3. If $n \geq 6$ and even, by proposition there are just two sets as $MFOD^c S$ according to modulo n to following cases.

Subcase1. If $n \equiv 0 \bmod 4$, $D_1 = \left\{v_{4+4i}, v_{5+4i}; i = 0, \dots, \frac{n}{4} - 2\right\} \cup \{v_1, v_n\}$ and $D_2 = \left\{v_{2+4i}, v_{3+4i}; i = 0, \dots, \frac{n}{4} - 1\right\}$ are only $MFOD^cS$. Since $|D_1| = |D_2|$, then the $MFIOD^cS$ has the maximum values of vertices, so $\gamma_{fodc}^{-1}(P_{nf}) = \max\{(\sum_{i=0}^{\frac{n}{4}-2}(\sigma(v_{4+4i}) + \sigma(v_{5+4i})) + \sigma(v_1) + \sigma(v_n)), \sum_{i=0}^{\frac{n}{4}-1}((\sigma(v_{2+4i}) + \sigma(v_{3+4i})) + \sigma(v_n))\}$.

Subcase2. If $n \equiv 2 \bmod 4$, $D_1 = \left\{v_{4+4i}, v_{5+4i}; i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_1\}$ and $D_2 = \left\{v_{2+4i}, v_{3+4i}; i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_n\}$ are only $MFOD^cS$ and it is clear $|D_1| = |D_2|$. So, $\gamma_{fodc}^{-1}(P_{nf}) = \max\{(\sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}(\sigma(v_{4+4i}) + \sigma(v_{5+4i})) + \sigma(v_1)), \sum_{i=0}^{\frac{n}{4}-1}((\sigma(v_{2+4i}) + \sigma(v_{3+4i})) + \sigma(v_n))\}$. □

Proposition 2.2. If G_f is a strong fuzzy Cycle has V as set of vertices that has order n and D is $MFOD^cS$, $\gamma_{fodc}^{-1}(C_{nf}) =$

$$\begin{cases} \min\limits_{v_{j+4i}, v_{j+4i+1} \in V/D} \left\{\sum_{i=0}^{\frac{n}{4}-1}\left(\sigma(v_{j+4i}) + \sigma(v_{j+4i+1})\right); j = 1, \dots, n-1\right\}, \\ \qquad\qquad if \ n \equiv 0 \ (mod \ 4) \\ \qquad has \ no \qquad , otherwise \end{cases}$$

Proof:
If $n \equiv 0 \ (mod \ 4)$, by proposition there are j number of $MFIOD^cS$ as
$D_j = \left\{v_{j+4i}, v_{j+4i+1}, i = 0, \dots, \frac{n}{4} - 1\right\}; j = 1, \dots, n-1$. Since D_j $MFOD^cS$ is the set has minimum of values of vertices, the $MFIOD^{-1^c}S$ is complement set of D_j. Then D_j is also $MFIOD^{-1^c}S$ such that $j + 4i + 1, j + 4i + 2 \ (mod \ n)$. So, $\gamma_{fodc}^{-1}(C_{nf}) = \min\limits_{v_{j+4i}, v_{j+4i+1} \in V/D} \{\sum_{i=0}^{\frac{n}{4}-1}(\sigma(v_{j+4i}) + \sigma(v_{j+4i+1})); j = 1, \dots, n-1\}$. □

Proposition 2.3. If G_f is a strong fuzzy Complete has V as set of vertices that has order n and D is $MFOD^cS$, $\gamma_{fodc}^{-1}(K_{nf}) =$

$$\begin{cases} has \ no \qquad, \quad if \ n = 3 \\ \min\limits_{v_i \in V/D}\{\sigma(v_i)\} \qquad i = 1, \dots, n, \quad if \ n \ is \ even \\ \min\limits_{v_i, v_j \in V/D}\{\sigma(v_i) + \sigma(v_j)\} \ i, j = 1, \dots, n; j \neq i \quad, if \ n \neq 3 \ and \ odd \end{cases}$$

Proof:
There are two cases as follows.
Case1. If n is an even, this graph has n sets represent FOD^cS has the same order equal to 1. That means a vertex has the minimum value (say $v_k, k = 1, \dots, n$) is $MFOD^cS$, then the $MFIOD^{-1^c}S$ $D^{-1} = \{v_i\}; i = 1, \dots, n$ and $i \neq k$. So, $\gamma_{fodc}^{-1}(K_{nf}) = \min\{\sigma(v_i), i = 1, \dots, n \ and \ i \neq k\}$
Case2. If n is odd, if $n = 3$, it is clear K_3 has no $FIOD^{-1^c}S$.
Other cases $n \geq 5$, by same away in case1 the set of order two is $MFOD^cS$ (say $D = \{v_k, v_r\}$). Then the $MFIOD^{-1^c}S$ $D^{-1} = \{v_i, v_j\}; i, j = 1, \dots, n$ and $i, j \neq k, r$. So, $\gamma_{fodc}^{-1}(K_{nf}) = \min\{\sigma(v_i), \sigma(v_j), i, j = 1, \dots, n \ and \ i \neq j \neq k \neq r\}$. □

Proposition 2.4. If G_f is a strong fuzzy Complete has V as set of vertices that has order n even and D is $MFOD^cS$, $\gamma_{fodc}^{-1}(W_{nf}) =$

$$\begin{cases} \begin{cases} \min\begin{cases}\left\{\sigma(v_{j+3i}); i = 1, \dots, \left\lfloor\frac{n}{3}\right\rfloor - 1\right\}; j = 2, \dots, n \\ such \ that \ j + 3i \ (mod \ n) \\ if \ n \equiv 0,1 \ (mod \ 3)\end{cases}, \\ \min\begin{cases}\left\{\sigma(v_{j+3i}); i = 1, \dots, \left\lfloor\frac{n}{3}\right\rfloor - 1\right\}; j = 2, \dots, n \\ such \ that \ j + 3i \ (mod \ n) \\ if \ n \equiv 2 \ (mod \ 3)\end{cases}, \end{cases} & if \ n \ is \ even \end{cases}$$

Proof: The wheel graph represents $W_{nf} = C_{n-1} + K_1$ and v_1 is center vertex. There are two cases as.

Case1. If $n \equiv 0,1 \ (mod \ 3)$, the set $D^{-1}{}_j = \left\{v_{j+3i}, i = 0, \dots, \left\lfloor\frac{n}{3}\right\rfloor - 1\right\}; j = 2, \dots, n$ are $FIOD^{-1^c}S$ because each vertex in $D^{-1}{}_j$ dominates three vertices in $V - D^{-1}{}_j$ and removing any vertex from $D^{-1}{}_j$ makes it not dominating set. So, $D^{-1}{}_j$ is $MFIOD^{-1^c}S$ and $\gamma_{fodc}^{-1}(W_{nf}) = \min\left\{\left\{\sigma(v_{j+3i}); i = 1, \dots, \left\lfloor\frac{n}{3}\right\rfloor - 1\right\}; j = 2, \dots, n\right\}; j + 3i \ (mod \ n)$.

Case2. If $n \equiv 2 \ (mod \ 3)$, by the same manner in case1 $D^{-1}{}_j = \left\{v_{j+3i}, i = 0, \dots, \left\lceil\frac{n}{3}\right\rceil - 1\right\}; j = 2, \dots, n$ are $MFIOD^{-1^c}S$. So, $\gamma_{fodc}^{-1}(W_{nf}) = \min\left\{\left\{\sigma(v_{j+3i}); i = 1, \dots, \left\lceil\frac{n}{3}\right\rceil - 1\right\}; j = 2, \dots, n\right\}; j + 3i \ (mod \ n)$. □

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