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CFSA-subalgebras and CFSA-ideals of SA-algebra

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Abstract. *In this paper, we define the notion of cubic fuzzy SA-subalgebras and cubic fuzzy SA-ideals of SA-algebra and prove their generalizations. Further, we display the relation between them their level cuts, several theorems, properties are studied and proved.*

Keywords. SA-subalgebra , SA-ideal, Cubic fuzzy SA-subalgebra, cubic fuzzy SA-ideal.

1. Introduction

In 1965, L.A. Zadeh introduced the notion of fuzzy subset, [5]. In 1976, K. Is´eki and S. Tanaka studied the notion of BCK-algebra, [4]. In 1991, O.G. Xi studied the notion of fuzzy BCK-algebra, [6]. Jun studied the notion of cubic set as generalization of fuzzy set and interval-valued fuzzy set [7]. In 2015, Mustafa and Hameed introduced the idea of SA-algebras. She stated some concepts related to it such as SA-subalgebra, SA-ideal, fuzzy SA-subalgebra and fuzzy SA-ideal with degree(λ, κ) of SA-algebra [1].In 2021, A.T. Hameed and N.J. Raheem introduced the notion of anti-fuzzy SA-subalgebra, anti-fuzzy SA-ideals with degree$(\lambda, \kappa)$and the notion of interval-valued fuzzy SA-subalgebra,fuzzy SA-ideals with degree(λ, κ) [2,3]. The paper aims to introduce the notion of cubic fuzzy SA-subalgebras of SA-algebra and prove their generalization, Also, we discuss the relation between them and their level cuts.

2. Preliminaries

In this section, we give some elementary aspects in SA-algebra such that we deem it necessary for these papers.

Definition 2.1.[1]. Let $(X;+,-,0)$ be an algebra with two binary operations $(+)$ and $(-)$ and constant (0). X is called a ***SA*-algebra** if it satisfies the following identities: for any $x, y, z \in X$.

(SA_1) $x - x = 0$, (SA_2) $x - 0 = x$,

(SA_3) $(x - y) - z = x - (z + y)$, (SA_4) $(x + y) - (x + z) = y - z$.

In X we can define a binary relation ($\leq$) by : $x \leq y$ if and only if $x + y = 0$ and $x - y = 0$, $x, y \in X$. And we will code it by Ꙏ

Definition 2.2.[1].

Let S be a nonempty set of Ꙏ, S is called a ***SA*-subalgebra** of Ꙏ if $x + y \in S$ *and* $x - y \in S$, whenever $x, y \in S$.

Definition 2.3.[1].

A nonempty subset I of Ꙏ is called a ***SA*-ideal** of Ꙏ if it satisfies: for $x, y, z \in$ Ꙏ ,

(1) $0 \in I$,

(2) $(x + z) \in I$ *and* $(y - z) \in I$ *imply* $(x + y) \in I$.

Proposition 2.4.[1].

Every SA-ideal of Ꙏ is a SA-subalgebra of Ꙏ and the converse is not true.

Definition 2.5.[5].

Let X be a nonempty set, a fuzzy subset μ of X is a mapping $\mu: X \to [0,1]$.

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Definition 2.6.[5].

For any $t \in [0,1]$ and a fuzzy subset μ in a nonempty set X, the set

$U(\mu, t) = \{x \in X \mid \mu(x) \geq t\}$ is called **an upper t-level cut of μ,** and the set $L(\mu, t) = \{x \in X \mid \mu(x) \leq t\}$ is called **a lower t-level cut of μ**.

Definition 2.7.[8].

A fuzzy subset μ of Ϊђ is called **a fuzzy *SA*-subalgebra of Ϊђ** if for all $x, y \in$ Ϊђ ,

1- $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$ and
2- $\mu(x-y) \geq min\{\mu(x), \mu(y)\}$.

Definition 2.8.[1].

A fuzzy subset μ of Ϊђ is called **a fuzzy *SA*-ideal of Ϊђ** if it satisfies: for all $x, y, z \in$ Ϊђ ,

- $\mu(0) \geq \mu(x)$,
- $\mu(x+y) \geq min\{\mu(x+z), \mu(y-z)\}$.

Proposition 2.9.[1].

Every fuzzy SA-ideal of Ϊђ is a fuzzy SA-subalgebra of Ϊђ**.**

Proposition 2.10.[1].

1- Let μ be a fuzzy subset of Ϊђ. If μ is a fuzzy SA-subalgebra of Ϊђ, for any t ∈ [0,1], μ_t is a SA-subalgebra of Ϊђ.

2- Let μ be a fuzzy subset of Ϊђ. If for all t ∈ [0,1], μ_t is a SA-subalgebra of Ϊђ, then μ is a fuzzy SA-subalgebra of Ϊђ.

3- Let μ be a fuzzy ideal of Ϊђ **.** If μ is a fuzzy SA-ideal of Ϊђ , then for any t ∈ [0,1], μ_t is an SA-ideal of Ϊђ.

4- Let μ be a fuzzy ideal of Ϊђ **.** If for all t ∈ [0,1], μ_t is an SA-ideal of Ϊђ , then μ is a fuzzy SA-ideal of Ϊђ.

Definition 2.11.[1].

Let $\{\mu_i : i \in \Lambda\}$ be a collection of fuzzy subsets of a set X. Define the fuzzy subset of X **(intersection)** by: $(\bigcap_{i\in\Lambda} \mu_i)(x) = \inf\{\mu_i(x) : i \in \Lambda\}$, for all $x \in$X.

Define the fuzzy subset of X **(union)** by: $(\bigcup_{i\in\Lambda} \mu_i)(x) = \sup\{\mu_i(x) : i \in \Lambda\}$, for all $x \in$X.

Proposition 2.12.[1].

1- The intersection of any set of fuzzy SA-subalgebras of Ϊђ is also fuzzy SA-subalgebra.

2- The union of any set of fuzzy SA-subalgebras of Ϊђ is also fuzzy SA-subalgebra of Ϊђ where is chain (Noetherian).

Definition 2.13.[7].

An **interval-valued fuzzy subset** $\widetilde{\mu}_A$ on Ϊђ is defined as $\widetilde{\mu}_A = \{< x, [\mu_A^-(x), \mu_A^+(x)] > \mid x \in X\}$. Where $\mu_A^-(x) \leq \mu_A^+(x)$, for all x∈ X. Then the ordinary fuzzy subsets $\mu_A^- : X \to [0, 1]$ and $\mu_A^+ : X \to [0, 1]$ are called a **lower fuzzy subset and an upper fuzzy subset** of $\widetilde{\mu}_A$ respectively.

Let $\widetilde{\mu}_A$ (x) = [μ_A^- (x) , μ_A^+ (x)] , $\widetilde{\mu}_A$:X ⟶ D[0, 1], then A = {< x, $\widetilde{\mu}_A$ (x) >| x∈ X} .

Proposition 2.14.[3].

Let $\{A_i$ | i∈Λ} be a family of interval-valued fuzzy SA-ideal of Ꮒ, then $\bigcap_{i\in\wedge} A_i$ is also an interval-valued fuzzy SA-ideal of Ꮒ .

Definition 2.15.[3].

An interval-valued fuzzy subset A = [μ_A^- , μ_A^+] in Ꮒ is called an **interval-valued fuzzy SA-subalgebra of** Ꮒ if for all x, y∈ Ꮒ

1- $\tilde{\mu}_A$ (x +y) ≽ rmin{$\tilde{\mu}_A$ (x), $\tilde{\mu}_A$ (y)},

2- $\tilde{\mu}_A$ (x -y) ≽ rmin{$\tilde{\mu}_A$ (x), $\tilde{\mu}_A$ (y)}.

Definition 2.16.[3].

An interval-valued fuzzy subset A = {< x, $\tilde{\mu}_A$ (x) >| x∈ Ꮒ } = [μ_A^- , μ_A^+] of Ꮒ is called **an interval-valued fuzzy SA-ideal** (i-v fuzzy ideal, in short) if it satisfies the following conditions, for all $x, y, z \in$ Ꮒ ,

- $\tilde{\mu}_A$ (0) ≽ $\tilde{\mu}_A$ (x) ,
- $\tilde{\mu}_A$ (x+y) ≽ rmin{$\tilde{\mu}_A$ $(x+z)$, $\tilde{\mu}_A$ (y-z)}.

Definition 2.17.[2].

A fuzzy subset μ of Ꮒ is called **an anti-fuzzy SA-subalgebra** of Ꮒ if for all $x, y \in$ Ꮒ ,

- $\mu\,(x+y) \leq\ max\,\{\mu\,(x), \mu\,(y)\}$,
- $\mu\,(x-y)\ \leq\ max\,\{\mu\,(x), \mu\,(y)\}$.

Proposition 2.18.[2].

Let μ be an anti-fuzzy subset of Ꮒ :

1- If μ is an anti-fuzzy SA-subalgebra of Ꮒ, then it satisfies for any t∈[0, 1], $L(\mu, t) \neq \emptyset$ implies $L(\mu, t)$ is a SA-subalgebra of Ꮒ.

2- If $L(\mu, t)$ is a SA-subalgebra of Ꮒ, for all t∈[0, 1], $L(\mu, t) \neq \emptyset$, then μ is an anti-fuzzy SA-subalgebra of Ꮒ .

Proposition 2.19.[2].

1- The union of any set of anti-fuzzy SA-subalgebras of Ꮒ is also anti-fuzzy SA-subalgebra.

2- The intersection of any set of anti-fuzzy SA-subalgebras of Ꮒ is also anti-fuzzy SA-subalgebra of Ꮒ where is chain (Artinian).

Proposition 2.20.[3].

Let A be an interval-valued fuzzy subset of Ƕ. If the nonempty set $\widetilde{U}$ (A;[δ_1,δ_2]):={ x∈ Ƕ |$\tilde{\mu}_A$ (x) ≥ [δ_1,δ_2]} is an SA-subalgebra of Ƕ, for all [δ_1, δ_2]∈ D[0, 1] , then A is an interval-valued fuzzy SA-subalgebra of Ƕ.

Proposition 2.21.[8].

Let A be an interval-valued fuzzy subset of Ƕ and A is an interval-valued fuzzy SA-subalgebra of Ƕ, then the nonempty set $\widetilde{U}$ (A;[δ_1,δ_2]) is SA-subalgebra of Ƕ, for all [δ_1, δ_2]∈ D[0, 1].

Proposition 2.22.[3].

An interval-valued fuzzy subset A = [μ_A^- , μ_A^+] of Ƕ is an interval-valued fuzzy SA-subalgebra of Ƕ if and only if μ_A^- and μ_A^+ are fuzzy SA-subalgebras of Ƕ.

Proposition 2.23.[3].

Let $(X; +, -, 0)$ be an SA-algebra and A be an interval-valued fuzzy subset of X. If the nonempty set $\widetilde{U}$ (A;[δ_1,δ_2]):={ x∈ X |$\tilde{\mu}_A$ (x) ≥ [δ_1,δ_2]} is an SA-ideal of X, for all [δ_1, δ_2]∈ D[0, 1] , then A is an interval-valued fuzzy SA-ideal of X.

Definition 2.24.[2].

A fuzzy subset μ of Ƕ is called **an anti-fuzzy SA-ideal of Ƕ** if it satisfies the following conditions, for all $x, y \in$ Ƕ ,

- $\mu(0) \leq \mu(x)$,
- $\mu(x + y) \leq max\{\mu(x + z), \mu(y - z)\}$.

Proposition 2.25.[3].

Let A = [μ_A^- , μ_A^+] be interval-valued fuzzy subset of Ƕ, μ_A^- and μ_A^+ are fuzzy SA-ideals of Ƕ if and only if A is an interval-valued fuzzy SA-ideal of Ƕ.

Proposition 2.26.[2].

Let μ be an anti-fuzzy subset of Ƕ

1- If μ is an anti-fuzzy SA-ideal of Ƕ, then it satisfies for any t∈[0, 1], $L(\mu, t) \neq \emptyset$ implies $L(\mu, t)$ is a SA- ideal of Ƕ.

2- If $L(\mu, t)$ is a SA- ideal of Ƕ, for all t∈[0, 1], $L(\mu, t) \neq \emptyset$, then μ is an anti-fuzzy SA-ideal of Ƕ.

Proposition 2.27.[2].

1- The union of any set of anti-fuzzy SA-ideals of Ƕ is also anti-fuzzy SA-ideal.

2- The intersection of any set of anti-fuzzy SA-subalgebras of Ƕ is also anti-fuzzy SA-ideal of Ƕ where is chain (Artinian).

Definition 2.27.[2].

Let $f: (X; +, -, 0) \rightarrow (Y; +', -', 0')$be a mapping nonempty SA-algebras X and Y respectively. If μ is anti-fuzzy subset of X, then the anti-fuzzy subset β of Y defined by:

$$f(\mu)(y) = \begin{cases} inf\{\mu(x): x \in f^{-1}(y)\} & if\ \ f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ 1 & otherwise \end{cases}$$

is said to be the image of μ under f.

Similarly if β is anti-fuzzy subset of , then the fuzzy subset $\mu = (\beta \circ f)$ of X (i.e the anti-fuzzy subset defined by $\mu(x) = \beta(f(x))$, for all

$x \in X$) is called the pre-image of β under f .

Theorem 2.29.[2].

A homomorphism pre-image of anti-fuzzy SA-subalgebra is also anti- fuzzy SA-subalgebra.

Definition 2.30.[2].

An anti fuzzy subset μ of Ҧ has inf proporty if for any subset Tof Ҧ,there exist $t_0 \in T$ such that $\mu(t_0) = inf_{t \in T} \in \mu(t)$. And it has sup proporty if for any subset Tof Ҧ,there exist $t_0 \in T$ such that $\mu(t_0) = sup_{t \in T} \in \mu(t)$.

Theorem 2.31.[2].

Let $: (X; +, -, 0) \rightarrow (Y; +', -', 0')$ be an epimorphism between SA-algebras X and Y respectively and f has inf proporety. For every μ is anti-fuzzy SA-subalgebra of X, then $f(\mu)$ is an anti-fuzzy SA-subalgebra of Y.

Theorem 2.32.[2].

A homomorphism pre-image of anti-fuzzy SA-ideal is also anti- fuzzy SA-ideal.

Theorem 2.33.[2].

Let $: (X; +, -, 0) \rightarrow (Y; +', -', 0')$ be an epimorphism between SA-algebras X and Y respectively and f has inf proporety. For every μ is anti-fuzzy SA-ideal of X, then $f(\mu)$ is an anti-fuzzy SA-idealof Y.

Proposition 2.34.[2].

For agiven fuzzy subset β of Ҧ. Let R_β be the fuzzy relation on Ҧ.If β is anti fuzzy SA-ideal of X× X ,then $R_\beta(x, x) \geq R_\beta(0,0) \forall x \in X$.

Theorem 2.35.[3].

Let $: (X; +, -, 0) \rightarrow (Y; +', -', 0')$ be an homomorphism between SA-algebras X and Y. If B is an interval-valued fuzzy SA-subalgebra of Y, then pre-image $f^{-1}(B)$ is an interval-valued fuzzy SA-subalgebra of X.

Theorem 2.36.[3].

Let $: (X; +, -, 0) \rightarrow (Y; +', -', 0')$ be an homomorphism between SA-algebras X and Y. If A is an interval-valued fuzzy SA-subalgebra of X with sup property, then $f(B)$ is an interval-valued fuzzy SA-subalgebra of Y.

Theorem 2.37.[3].

Let $: (X; +, -, 0) \rightarrow (Y; +', -', 0')$ be an homomorphism between SA-algebras X and Y. If B is an interval-valued fuzzy SA-ideal of Y, then pre-image $f^{-1}(B)$ is an interval-valued fuzzy SA-ideal of X.

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Theorem 2.38.[3].

Let : $(X;+,-,0) \to (Y;+',-',0')$ be an homomorphism between SA-algebras X and Y. If A is an interval-valued fuzzy SA-ideal of X with sup property, then $f(B)$ is an interval-valued fuzzy SA-idealof Y.

Remark 2.39.[7].

An interval number is $\tilde{a} = [a^-, a^+]$, where $0 \le a^- \le a^+ \le 1$. Let I be a closed unit interval, (i.e., I = [0, 1]).

Let D[0, 1] denote the family of all closed subintervals of I = [0, 1] , that is, D[0, 1] = { $\tilde{a} = [a^-, a^+]$ | $a^- \le a^+$, for $a^-, a^+ \in$ I} .

Now, we define what is known as refined minimum (briefly, rmin) of two element in D[0,1].

Definition 2.40.[7].

We also define the symbols (≽) , (≼) , (=) ,"rmin " and "rmax " in case of two elements in D[0, 1]. Consider two interval numbers (elements numbers)

$\tilde{a} = [a^-, a^+]$, $\tilde{b} = [b^-, b^+]$in D[0, 1] : Then

(1) $\tilde{a} \succcurlyeq \tilde{b}$ if and only if, $a^- \ge b^-$ and $a^+ \ge b^+$, (2) $\tilde{a} \preccurlyeq \tilde{b}$ if and only if, $a^- \le b^-$ and $a^+ \le b^+$,

(3) $\tilde{a} = \tilde{b}$ if and only if, $a^- = b^-$ and $a^+ = b^+$, (4) rmin $\{\tilde{a}, \tilde{b}\}$= [min $\{a^-, b^-\}$, min $\{a^+, b^+\}$],

(5) rmax $\{\tilde{a}, \tilde{b}\}$= [max $\{a^-, b^-\}$, max $\{a^+, b^+\}$],

Remark 2.41.[7].

It is obvious that (D[0, 1] , ≼ , ∨ , ∧) is a complete lattice with $\tilde{0}$ = [0, 0] as its least element and $\tilde{1}$ = [1, 1] as its greatest element. Let $\tilde{a}_i \in$D[0, 1] where i ∈Λ. We define $r\,inf_{i\in\Lambda}\tilde{a}$= [$r\,inf_{i\in\Lambda} a^-$, $r\,inf_{i\in\Lambda} a^+$], $r\,sup_{i\in\Lambda}\tilde{a}$= [$r\,sup_{i\in\Lambda} a^-$, $r\,sup_{i\in\Lambda} a^+$].

Definition 2.42.[7].

Let $(X;+,-,0)$ be a nonempty set. A cubic set Ω in a structure Ω = {< x, $\tilde{\mu}_\Omega$ (x),λ_Ω (x)>| x∈ X }, which is briefly denoted by Ω =<$\tilde{\mu}_\Omega$, λ_Ω>, where $\tilde{\mu}_\Omega : X \to$ D[0, 1], $\tilde{\mu}_\Omega$ is an interval-valued fuzzy subset of X and λ_Ω: $X \to$ [0, 1], λ_Ω is a fuzzy subset of X.

Definition 2.43.[7].

For a family $\Omega_i = \{\langle x, \tilde{\mu}_{\Omega i}(x)\rangle | x \in X\}$ on fuzzy subsets of X, where i∈Λ and Λ is index set, we define the join (⋁)and meet (⋀) operations as follows:

$\bigvee_{i\in\Lambda} \Omega_i = (\bigvee_{i\in\Lambda} \tilde{\mu}_{\Omega i})(x) = sup\{\tilde{\mu}_{\Omega i}(x) | i \in \Lambda\}$,

$\bigwedge_{i\in\Lambda} \Omega_i = (\bigwedge_{i\in\Lambda} \tilde{\mu}_{\Omega i})(x) = inf\{\tilde{\mu}_{\Omega i}(x) | i \in \Lambda\}$,

Definition 2.44.[7].

Let $: (X;+,-,0) \rightarrow (Y;+',-',0')$ be a mapping from the SA-algebra X to SA-algebra Y. If A is an interval-valued fuzzy subset of X, then the subset B is an interval-valued fuzzy subset of Y defined by:

$$f(\tilde{\mu}_A)(y) = \begin{cases} \underset{x\in f^{-1}(y)}{rsup} \tilde{\mu}_A(x) if f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ 0 \qquad otherwise \end{cases}$$

(i.e., the cubic subset defined by $\tilde{\mu}_A$ (x) = $\tilde{\mu}_A$ (f (x)), for all x $\in$X), is said to be **the image of Ω under f.** Then the inverse image of B, denoted by f^{-1} (B) , is an interval-valued fuzzy subset of X with the membership function given by $\mu_{f^{-1}(B)}$ (x) =$\tilde{\mu}_B$ (f (x)), for all $x \in X$), is called **the preimage of β under f**.

3. Cubic Bipolar SA-subalgebras of SA-algebra

In this section, we will introduce a new notion called cubic fuzzy SA-subalgebra of SA-algebra and study several properties of it.

Definition 3.1.

A cubic fuzzy subset $\Omega = < \tilde{\mu}_\Omega$ (x)$, \lambda_\Omega$(x) $>$ of Ƕ is called **cubic fuzzy SA-subalgebra of** Ƕ if, for all $x, y \in$ Ƕ:

1- $\tilde{\mu}_\Omega (x+y) \succcurlyeq rmin\{\tilde{\mu}_\Omega (x), \tilde{\mu}_\Omega (y)\}$, and

$\lambda_\Omega (x+y) \leq max\{\lambda_\Omega(x), \lambda_\Omega(y)\}$.

2- $\tilde{\mu}_\Omega (x-y) \succcurlyeq rmin\{\tilde{\mu}_\Omega (x), \tilde{\mu}_\Omega (y)\}$, and $\lambda_\Omega (x-y) \leq max\{\lambda_\Omega(x), \lambda_\Omega(y)\}$.

Example 3.2.

Let Ƕ = {0, 1, 2, 3} define by the following table:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

−	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Define a cubic fuzzy subset $\Omega = < \tilde{\mu}_\Omega , \lambda_\Omega >$ of Ƕ of fuzzy subset μ: Ƕ→ [0,1] by: $\tilde{\mu}_\Omega$ (x) = $\begin{cases} [0.3,0.9] & if x = \{0,2\} \\ [0.1,0.6] & otherwise \end{cases}$ and $\lambda_\Omega = \begin{cases} 0.2 & if x = \{0,2\} \\ 0.3 & otherwise \end{cases}$

The cubic fuzzy subset $\Omega = < \tilde{\mu}_\Omega , \lambda_\Omega >$ is a cubic fuzzy SA-subalgebra of Ƕ.

Proposition 3.3.

Let $\Omega = < \tilde{\mu}_\Omega , \lambda_\Omega >$ be a cubic fuzzy SA-subalgebra of Ƕ then

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$\tilde{\mu}_{\Omega}(\tilde{0}) \succcurlyeq \tilde{\mu}_{\Omega}(\tilde{x})$ and $\lambda_{\Omega}(0) \leq \lambda_{\Omega}(x)$, for all $x \in \Pi\mathfrak{h}$.

Proof.

For all $x \in \Pi\mathfrak{h}$, we have

$$\tilde{\mu}_{\Omega}(\tilde{0}) = \tilde{\mu}_{\Omega}(0 * x) \succcurlyeq rmin\{\tilde{\mu}_{\Omega}((0 * x) * 0), \tilde{\mu}_{\Omega}(x)\}$$

$$= rmin\{[\mu_A^-((0 * x) * 0), \mu_A^-(x)], [\mu_A^+((0 * x) * 0), \mu_A^+(x)]\}$$

$$= rmin\{[\mu_A^-(0), \mu_A^-(x)], [\mu_A^+(0), \mu_A^+(x)]\}$$

$$= [\mu_A^-(x), \mu_A^+(x)] = \tilde{\mu}_{\Omega}(\tilde{x}).$$

Similarly, we can show that $\lambda_{\Omega}(0) \leq max\{[\lambda_{\Omega}(0), \lambda_{\Omega}(x)]\} = \lambda_{\Omega}(x)$.⌂

Proposition 3.4.

If a cubic fuzzy subset $\Omega = <\tilde{\mu}_{\Omega}, \lambda_{\Omega}>$ of SA-algebra $(X; +, -, 0)$ is a cubic fuzzy SA-subalgebra, then $\Omega(x + y) = \Omega(x + ((y + 0) + 0))$, for all $x, y \in X$.

Proof.

Let X be an SA-algebra and $x, y \in X$, then we know that $y = (y + 0) + 0$. Hence,

$\tilde{\mu}_{\Omega}(x + y) = \tilde{\mu}_{\Omega}(x + ((y + 0) + 0))$ and $\lambda_{\Omega}(x + y) = \lambda_{\Omega}(x + ((y + 0) + 0))$. Therefore

$\Omega(x + y) = \Omega(x + ((y + 0) + 0))$. ⌂

Definition 3.5.

Let $\Omega_i = <\tilde{\mu}_{\Omega i}, \lambda_{\Omega i}>$ be a cubic fuzzy subset of a SA-algebra $(X; +, -, 0)$ where $i \in \Lambda$ such that $\tilde{\mu}_{\Omega i}$ are a fuzzy SA-subalgebras of X and $\lambda_{\Omega i}$ are anti-fuzzy SA-subalgebras of X, for any $x \in X$, then

1-The R-intersection of any set of cubic fuzzy subset of X is $(\cap \tilde{\mu}_{\Omega i})(x) = \inf(\tilde{\mu}_{\Omega i})(x)$ and $(\vee \lambda_{\Omega i})(x) = \sup(\lambda_{\Omega i})(x)$.

2-The P-intersection of any set of cubic fuzzy subset of X is $(\cap \tilde{\mu}_{\Omega i})(x) = \inf(\tilde{\mu}_{\Omega i})(x)$ and $(\wedge \lambda_{\Omega i})(x) = \inf(\lambda_{\Omega i})(x)$.

3-The R-union of any set of cubic fuzzy subset of X is $(\cup \tilde{\mu}_{\Omega i})(x) = \sup(\tilde{\mu}_{\Omega i})(x)$ and $(\wedge \lambda_{\Omega i})(x) = \inf(\lambda_{\Omega i})(x)$.

4-The P-union of any set of cubic fuzzy subset of X is $(\cup \tilde{\mu}_{\Omega i})(x) = \sup(\tilde{\mu}_{\Omega i})(x)$ and $(\vee \lambda_{\Omega i})(x) = \sup(\lambda_{\Omega i})(x)$.

Proposition 3.6.

The R-intersection of any set of cubic fuzzy SA-subalgebra of $(X; +, -, 0)$ is also cubic fuzzy SA-subalgebra of X.

Proof.

Let $\Omega_i = <\tilde{\mu}_{\Omega i}, \lambda_{\Omega i}>$ where $i \in \Lambda$, be a set of cubic fuzzy SA-subalgebra of X and $x, y \in X$, then

$$(\cap \tilde{\mu}_{\Omega i})(x + y) = inf(\tilde{\mu}_{\Omega i})(x + y)$$
$$\succcurlyeq inf\{rmin\{(\tilde{\mu}_{\Omega i})(x), (\tilde{\mu}_{\Omega i})(y)\}\}$$
$$= r\,min\{inf(\tilde{\mu}_{\Omega i}(x)), inf(\tilde{\mu}_{\Omega i}(y))\}$$
$$= rmin\{(\cap \tilde{\mu}_{\Omega i})(x), (\cap \tilde{\mu}_{\Omega i})(y)\}$$ and

Summarily, $(\cap \tilde{\mu}_{\Omega i})(x - y) \succcurlyeq rmin\{(\cap \tilde{\mu}_{\Omega i})(x), (\cap \tilde{\mu}_{\Omega i})(y)\}$.

Hence $(\cap \tilde{\mu}_{\Omega i})$ is a fuzzy ψ-subalgebra of X.

$$(\vee \lambda_{\Omega i})(x + y) = \sup(\lambda_{\Omega i})(x + y)$$
$$\leq \sup\{\max\{(\lambda_{\Omega i})(x), (\lambda_{\Omega i})(y)\}\}$$

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$= \max\{\sup(\lambda_{\Omega i}(x)), \sup(\lambda_{\Omega i}(y))\}$
$= \max\{(\vee\lambda_{\Omega i})(x), (\vee\lambda_{\Omega i})(y)\}$.

Summarily, $(\vee\lambda_{\Omega i})(x-y) \leq \max\{(\vee\lambda_{\Omega i})(x), (\vee\lambda_{\Omega i})(y)\}$.

Hence $(\vee\lambda_{\Omega i})$ is anti-fuzzy SA-subalgebra of X.
Hence, R-intersection of $\Omega_i = <\tilde{\mu}_{\Omega i}, \lambda_{\Omega i}>$ is a cubic fuzzy SA-subalgebra of X. ⌂

Remark 3.7.

The P-intresection of any sets of cubic fuzzy SA-subalgebra need not be a cubic fuzzy SA-subalgebra, for example:

Example 3.8.

Let X = {0, a, b, c, d} be a set with the following table:

+	0	a	b	c	d
0	0	a	b	c	d
a	a	b	c	d	0
b	b	c	d	0	a
c	c	d	0	a	b
d	d	0	a	b	c

-	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	a
b	b	b	0	0	a
c	c	b	d	0	b
d	d	d	d	d	0

Then $(X; +, -, 0)$ is an SA-algebra. It is easy to show that I ={0, c} and J ={0, d} are SA-subalgebras of X .We defined two cubic set $\Omega_1 = \{(x, \tilde{\mu}_{\Omega 1}(x), \lambda_{\Omega 1}(x)) \mid x \in X\}$ and $\Omega_2 = \{(x, \tilde{\mu}_{\Omega 2}(x), \lambda_{\Omega 2}(x)) \mid x \in X\}$ of X by :-

$$\tilde{\mu}_{\Omega 1}(x) = \begin{cases} [0.5,0.8], & \text{if } x \in \{0, c\}, \\ [0.4,0.7], & \text{if } x \in \{a, b\}, \\ [0.3,0.8], & \text{otherwise} \end{cases} \quad \lambda_{\Omega 1}(x) = \begin{cases} 0.2, & \text{if } x \in \{0, c\}, \\ 0.6, & \text{if } x \in \{a, b\}, \\ 0.4, & \text{otherwise} \end{cases}$$

$$\tilde{\mu}_{\Omega 2}(x) = \begin{cases} [0.4,0.9], & \text{if} x \in \{0, d\}, \\ [0.3,0.7], & \text{otherwise.} \end{cases} \quad \text{and} \quad \lambda_{\Omega 2}(x) = \begin{cases} 0.1, & \text{if} x \in \{0, d\}, \\ 0.4, & \text{otherwise.} \end{cases}$$

Then Ω_1 and Ω_2 are cubic fuzzy SA-subalgebra of X , but P-intersection of $\Omega_1 \cap \Omega_2$ are not cubic fuzzy SA-subalgebras of X . Since

$$(\tilde{\mu}_{\Omega 1} \cap \tilde{\mu}_{\Omega 2})(c-d) = \min\{[0.4,0.7], [0.3,0.5]\} = [0.3,0.5] \ngeq$$
$[0.3,0.7] = \min\{(\tilde{\mu}_{\Omega 1} \cap \tilde{\mu}_{\Omega 2})(c), (\tilde{\mu}_{\Omega 1} \cap \tilde{\mu}_{\Omega 2})(d)\} = \min\{\min\{[0.5,0.8], [0.3,0.7]\}, \min\{[0.3,0.8], [0.4,0.9]\}\} = \min\{[0.3,0.7], [0.3,0.8]\}$
and $(\lambda_{\Omega 1} \vee \lambda_{\Omega 2})(c-d) = \max\{0.6, 0.4\} = 0.6 \nleq 0.4 = max\{(\lambda_{\Omega 1} \vee \lambda_{\Omega 2})(c), (\lambda_{\Omega 1} \vee \lambda_{\Omega 2})(d)\}$
$= \max\{\max\{0.2, 0.4\}, \max\{0.4, 0.1\}\}$

Proposition 3.9.

Let $\Omega_i = <\tilde{\mu}_{\Omega i}, \lambda_{\Omega i}>$ where $i \in \Lambda$, be a set of cubic fuzzy SA-subalgebra of SA-algebra $(X; +, -, 0)$, where $i \in \Lambda$, $inf\{max\{\lambda_{\Omega i}(x), \lambda_{\Omega i}(y)\}\} = max\{inf\ \lambda_{\Omega i}(x), inf\ \lambda_{\Omega i}(y)\}$, for all x ∈ X, then the P-intresection of Ω_i is also a cubic fuzzy SA-subalgebra of X.

Proof.

Let $\Omega_i = <\tilde{\mu}_{\Omega i}, \lambda_{\Omega i}>$ where $i \in \Lambda$, be a set of cubic fuzzy SA-subalgebra of X and $x, y \in X$, then
$(\cap\tilde{\mu}_{\Omega i})(x+y) = inf\ (\tilde{\mu}_{\Omega i})(x+y)$
$\succcurlyeq inf\ \{rmin\{(\tilde{\mu}_{\Omega i})(x), (\tilde{\mu}_{\Omega i})(y)\}\}$
$= rmin\{inf(\tilde{\mu}_{\Omega i}(x)), inf(\tilde{\mu}_{\Omega i}(y))\}$
$= \text{rmin}\{(\cap\tilde{\mu}_{\Omega i})(x), (\cap\cap\tilde{\mu}_{\Omega i})(y)\}$ and

Summarily, $(\cap\tilde{\mu}_{\Omega i})(x-y) \succcurlyeq r\text{min}\{(\cap\tilde{\mu}_{\Omega i})(x), (\cap\tilde{\mu}_{\Omega i})(y)\}$.

Hence $(\cap\tilde{\mu}_{\Omega i})$ is a fuzzy SA-subalgebra of X.
$(\wedge\lambda_{\Omega i})(x+y) = inf\ (\lambda_{\Omega i})(x+y)$
$\leq \inf\{\max\{(\lambda_{\Omega i})(x), (\lambda_{\Omega i})(y)\}\}$
$= \max\{\inf(\lambda_{\Omega i}(x)), \inf(\lambda_{\Omega i}(y))\}$
$= \max\{(\wedge\lambda_{\Omega i})(x), (\wedge\lambda_{\Omega i})(y)\}$.

Similarly, $(\wedge \lambda_{\Omega i})\ (x-y) \le \max\{(\wedge \lambda_{\Omega i})\ (x), (\wedge \lambda_{\Omega i})(y)\}$.

Hence $(\wedge \lambda_{\Omega i})$ is anti-fuzzy SA-subalgebra of X.

Hence, P-intersection of Ω_i is a cubic fuzzy SA-subalgebra of X. ⌂

Proposition 3.10.

Let $\Omega_i = <\tilde{\mu}_{\Omega i}, \lambda_{\Omega i}>$ where $i \in \Lambda$, be a set of cubic fuzzy SA-subalgebra of SA-algebra $(X; +, -, 0)$, where $i \in \Lambda$, $sup\{rmin\{\tilde{\mu}_{\Omega i}(x), \tilde{\mu}_{\Omega i}(y)\}\} = rmin\{sup\ \tilde{\mu}_{\Omega}(x), sup\ \tilde{\mu}_{\Omega}(y)\}$, for all $x \in X$, then the P-union of Ω_i is also a cubic fuzzy SA-subalgebra of X.

Proof.

Let $\Omega_i = <\tilde{\mu}_{\Omega i}, \lambda_{\Omega i}>$ where $i \in \Lambda$, be a set of cubic fuzzy SA-subalgebra of X and $x, y \in X$, then

$$\begin{aligned}(\cup\, \tilde{\mu}_{\Omega i})(x+y) &= sup\ (\tilde{\mu}_{\Omega i})(x+y)\\ &\succcurlyeq sup\ \{rmin\{(\tilde{\mu}_{\Omega i})(x), (\tilde{\mu}_{\Omega i})(y)\}\}\\ &= rmin\{sup(\tilde{\mu}_{\Omega i}(x)), sup(\tilde{\mu}_{\Omega i}(y))\}\\ &= rmin\{(\cup\, \tilde{\mu}_{\Omega i})(x), (\cup\, \tilde{\mu}_{\Omega i})(y)\} \quad \text{and}\end{aligned}$$

Summarily, $(\cup\, \tilde{\mu}_{\Omega i})\ (x-y) \succcurlyeq rmin\{(\cup\, \tilde{\mu}_{\Omega i})\ (x), (\cup\, \tilde{\mu}_{\Omega i})(y)\}$.

Hence $(\cap\, \tilde{\mu}_{\Omega i})$ is a fuzzy SA-subalgebra of X.

$$\begin{aligned}(\vee \lambda_{\Omega i})(x+y) &= \sup\ (\lambda_{\Omega i})(x+y)\\ &\le \sup\{\max\{(\lambda_{\Omega i})(x), (\lambda_{\Omega i})(y)\}\}\\ &= \max\{\sup\ (\lambda_{\Omega i})(x), \sup\ (\lambda_{\Omega i})(y)\}\\ &= \max\{(\vee \lambda_{\Omega i})(x), (\vee \lambda_{\Omega i})(y)\}.\end{aligned}$$

Summarily, $(\vee \lambda_{\Omega i})\ (x-y) \le \max\{(\vee \lambda_{\Omega i})\ (x), (\vee \lambda_{\Omega i})(y)\}$.

Hence $(\vee \lambda_{\Omega i})$ is anti-fuzzy SA-subalgebra of X.

Hence, P-union of Ω_i is a cubic fuzzy SA-subalgebra of X. ⌂

Remark 3.11.

The R-union of any sets of cubic fuzzy SA -subalgebra need not be a cubic fuzzy SA -subalgebra, for example:

Example 3.12.

Let X = {0, 1, 2, 3, 4} be a set with the following table:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

-	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	1
2	2	2	0	0	1
3	3	2	4	0	1
4	4	4	4	4	0

Then $(X; +, -, 0)$ is an SA-algebra. It is easy to show that I ={0, 3} and J ={0, 4} are SA-subalgebras of X .

We defined two cubic set $\Omega_1 = \{(x, \tilde{\mu}_{\Omega 1}(x), \lambda_{\Omega 1}(x)) \mid x \in X\}$ and $\Omega_2 = \{(x, \tilde{\mu}_{\Omega 2}(x), \lambda_{\Omega 2}(x)) \mid x \in X\}$ of X by :-

$$\tilde{\mu}_{\Omega 1}(x) = \begin{cases}[0.5, 0.8], & \text{if } x \in \{0, c\},\\ [0.4, 0.7], & \text{if } x \in \{a, b\},\\ [0.3, 0.8], & \text{otherwise}\end{cases} \qquad \lambda_{\Omega 1}(x) = \begin{cases}0.2, & \text{if } x \in \{0, c\},\\ 0.6, & \text{if } x \in \{a, b\},\\ 0.4, & \text{otherwise}\end{cases}$$

$$\tilde{\mu}_{\Omega 2}(x) = \begin{cases}[0.4, 0.9], & \text{if} x \in \{0, d\},\\ [0.3, 0.7], & \text{otherwise.}\end{cases} \quad \text{and} \quad \lambda_{\Omega 2}(x) = \begin{cases}0.1, & \text{if} x \in \{0, d\},\\ 0.4, & \text{otherwise.}\end{cases}$$

Then Ω_1 and Ω_2 are cubic fuzzy SA-subalgebra of X , but P-intersection of $\Omega_1 \cap \Omega_2$ are not cubic fuzzy SA-subalgebras of X . Since
$(\tilde{\mu}_{\Omega 1} \cap \tilde{\mu}_{\Omega 2})(c - d) = \min\{[0.4,0.7],[0.3,0.5]\} = [0.3,0.5] \not\geq [0.3,0.7] = \min\{(\tilde{\mu}_{\Omega 1} \cap \tilde{\mu}_{\Omega 2})(c),(\tilde{\mu}_{\Omega 1} \cap \tilde{\mu}_{\Omega 2})(d)\} = \min\{\min\{[0.5,0.8],[0.3,0.7]\},\min\{[0.3,0.8],[0.4,0.9]\}\} = \min\{[0.3,0.7],[0.3,0.8]\}$ and
$(\lambda_{\Omega 1} \vee \lambda_{\Omega 2})(c - d) = \max\{0.6,0.4\} = 0.6 \not\leq 0.4 = \max\{(\lambda_{\Omega 1} \vee \lambda_{\Omega 2})(c),(\lambda_{\Omega 1} \vee \lambda_{\Omega 2})(d)\} = \max\{\max\{0.2,0.4\},\max\{0.4,0.1\}\}$.

Proposition 3.13.

Let $\Omega_i = < \tilde{\mu}_{\Omega i}, \lambda_{\Omega i} >$ where $i \in \Lambda$, be a set of cubic fuzzy SA-subalgebra of SA-algebra $(X;+,-,0)$, where $i \in \Lambda$, $sup\{min\{\tilde{\mu}_{\Omega i}(x),\tilde{\mu}_{\Omega i}(y)\}\} = min\{sup\,\tilde{\mu}_{\Omega i}(x), sup\,\tilde{\mu}_{\Omega i}(y)\}$ and $inf\{max\{\lambda_{\Omega i}(x),\lambda_{\Omega i}(y)\}\} = max\{inf\,\lambda_{\Omega i}(x), inf\,\lambda_{\Omega i}(y)\}$, for all $x \in X$, then the R-union of Ω_i is also a cubic fuzzy SA-subalgebra of X.

Proof.

Let $\Omega_i = < \tilde{\mu}_{\Omega i}, \lambda_{\Omega i} >$ where $i \in \Lambda$, be a set of cubic fuzzy SA-subalgebra of X and $x, y \in X$, then

$(\cup\,\tilde{\mu}_{\Omega i})(x + y) = sup\,(\tilde{\mu}_{\Omega i})(x + y)$
$\succcurlyeq sup\,\{rmin\{(\tilde{\mu}_{\Omega i})(x),(\tilde{\mu}_{\Omega i})(y)\}\}$
$= r\,min\{sup(\tilde{\mu}_{\Omega i}(x)), sup(\tilde{\mu}_{\Omega i}(y))\}$
$= \text{rmin}\{(\cup\,\tilde{\mu}_{\Omega i})(x),(\cup\,\tilde{\mu}_{\Omega i})(y)\}$ and

Similarly, $(\cup\,\tilde{\mu}_{\Omega i})\,(x - y) \succcurlyeq r\text{min}\{(\cap\,\tilde{\mu}_{\Omega i})\,(x),(\cap\,\tilde{\mu}_{\Omega i})(y)\}$.

Hence $(\cup\,\tilde{\mu}_{\Omega i})$ is a fuzzy SA-subalgebra of X.

$(\wedge\,\lambda_{\Omega i})(x + y) = \inf\,(\lambda_{\Omega i})(x + y)$
$\leq \inf\{\max\{(\lambda_{\Omega i})(x),(\lambda_{\Omega i})(y)\}\}$
$= \max\{\inf\,(\lambda_{\Omega i})(x), \inf\,(\lambda_{\Omega i})(y)\}$
$= \max\{(\wedge\,\lambda_{\Omega i})(x),(\wedge\,\lambda_{\Omega i})(y)\}$.

Similarly, $(\wedge\,\lambda_{\Omega i})\,(x - y) \leq \max\{(\wedge\,\lambda_{\Omega i})\,(x),(\wedge\,\lambda_{\Omega i})(y)\}$.

Hence $(\wedge\,\lambda_{\Omega i})$ is anti-fuzzy SA-subalgebra of X.

Hence, R-union of Ω_i is a cubic fuzzy SA-subalgebra of X. ⌂

Definition 3.14.

Let $(X\,;+,-,0)$ be an SA-algebra. A cubic fuzzy subset $\Omega = <\tilde{\mu}_{\Omega}, \lambda_{\Omega}>$ of X , for all $\tilde{t} \in D[0, 1]$ and $s \in [0, 1]$, the set $\widetilde{U}\,(\Omega;\tilde{t},s) = \{\, x \in X \mid \tilde{\mu}_{\Omega}(x) \succcurlyeq \tilde{t},\ \lambda_{\Omega}(x) \leq s\}$ is an level set of X.

Proposition 3.15.

Let $(X\,;+,-,0)$ be an SA-algebra. A cubic fuzzy subset $\Omega = <\tilde{\mu}_{\Omega}, \lambda_{\Omega}>$ of X. If Ω is a cubic fuzzy SA-subalgebra of X, then for all $\tilde{t} \in D[0, 1]$ and $s \in [0, 1]$, the set $\widetilde{U}\,(\Omega;\tilde{t},s)$ is an SA-subalgebra of X .

Proof.

Assume that $\Omega = <\tilde{\mu}_{\Omega}, \lambda_{\Omega}>$ is a cubic fuzzy SA-subalgebra of X and let $\tilde{t} \in D[0, 1]$ and

$s \in [0, 1]$, be such that $\widetilde{U}\,(\Omega;\tilde{t},s) \neq \emptyset$, and let $x , y \in X$ such that

$x , y \in \widetilde{U}\,(\Omega;\tilde{t},s)$, then $\tilde{\mu}_{\Omega}(x) \succcurlyeq \tilde{t}$, $\tilde{\mu}_{\Omega}(y) \succcurlyeq \tilde{t}$ and $\lambda_{\Omega}(x) \leq s$, $\lambda_{\Omega}(y) \leq s$. Since Ω is a cubic fuzzy SA-subalgebra of X, we get

$\tilde{\mu}_{\Omega}(x+y) \succcurlyeq \text{rmin}\{\tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\} \succcurlyeq \tilde{t}$ and $\lambda_{\Omega}(x+y) \leq \max\{\lambda_{\Omega}(x),\lambda_{\Omega}(y)\} \leq s$.

Similarly, $\tilde{\mu}_{\Omega}(x-y) \succcurlyeq \text{rmin}\{\tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\} \succcurlyeq \tilde{t}$ and $\lambda_{\Omega}(x-y) \leq \max\{\lambda_{\Omega}(x),\lambda_{\Omega}(y)\} \leq s$.

Hence the set $\widetilde{U}\,(\Omega;\tilde{t},s)$ is an SA-subalgebra of X. ⌂

Proposition 3.16.

Let $(X ; +, -, 0)$ be an SA-algebra. A cubic fuzzy subset Ω =<$\tilde{\mu}_\Omega$, λ_Ω> of . If for all $\tilde{t}$ ∈ D[0, 1] and s ∈ [0, 1], the set $\tilde{U}$ $(\Omega; \tilde{t}, s)$ is an SA-subalgebra of X, then Ω is a cubic fuzzy SA-subalgebra of X.

Proof.

Suppose that $\tilde{U}$ $(\Omega; \tilde{t}, s)$is an SA-subalgebra of X and let $x , y \in X$ be such that

$\tilde{\mu}_\Omega$ (x+y)≺rmin {$\tilde{\mu}_\Omega$ (x),$\tilde{\mu}_\Omega$ (y)}, and λ_Ω (x+y)> max {λ_Ω (x),λ_Ω (y)}.

Consider $\tilde{\beta}$ = 1/2 { $\tilde{\mu}_\Omega$ (x+y) + rmin{$\tilde{\mu}_\Omega$ (x), $\tilde{\mu}_\Omega$ (y)} } and

β= 1/2 { λ_Ω (x+y) + max{λ_Ω (x), λ_Ω (y)}}.

We have $\tilde{\beta}$ ∈ D[0, 1] and β∈ [0, 1], and $\tilde{\mu}_\Omega$ (x+y) ≺ $\tilde{\beta}$ ≺ rmin {$\tilde{\mu}_\Omega$ (x),$\tilde{\mu}_\Omega$ (y) }

and λ_Ω (x+y) >β> max {λ_Ω (x),λ_Ω (y) } .

It follows that $x, y \in \tilde{U}$ $(\Omega; \tilde{t}, s)$, and (x+y) ∉ $\tilde{U}$ $(\Omega; \tilde{t}, s)$. This is a contradiction.

Hence $\tilde{\mu}_\Omega$(x+y) ≽ rmin{ $\tilde{\mu}_\Omega$ (x),$\tilde{\mu}_\Omega$ (y)} ≽$\tilde{t}$ and λ_Ω (x+y) ≤ max {λ_Ω (x),λ_Ω (y)} ≤ s.

Similarly, $\tilde{\mu}_\Omega$(x−y) ≽ rmin{ $\tilde{\mu}_\Omega$ (x),$\tilde{\mu}_\Omega$ (y)} ≽$\tilde{t}$ and λ_Ω (x-y) ≤ max {λ_Ω (x),λ_Ω (y)} ≤ s.

Therefore Ω =<$\tilde{\mu}_\Omega$, λ_Ω> is a cubic fuzzy SA-subalgebra of X. ⌂

Definition 3.17.

Let Ω =<$\tilde{\mu}_\Omega$, λ_Ω> be a cubic set of X. For $[t_1, t_2]$∈D[0,1]and s ∈[0,1],the set $U(\tilde{\mu}_\Omega \mid [t_1,t_2])$={x∈X| $\tilde{\mu}_\Omega$(x) ≽ $[t_1,t_2]$}is called upper$[t_1,t_2]$-Level of Ω and $L(\lambda_\Omega|t)$={x∈X|λ_Ω(x)≤t}is called Lower t-Level of Ω.

And $\Omega(\tilde{t}, s)$=$\Omega([t_1,t_2], s)$ = $U(\tilde{\mu}_\Omega \mid [t_1,t_2])$∩ $L(\lambda_\Omega|s)$

$$= \tilde{U}\ (\Omega; \tilde{t}, s) = \{x \in X \mid\ \tilde{\mu}_\Omega(x) \succcurlyeq [t_1, t_2] \text{ and } \lambda_\Omega(x) \le s\}.$$

Theorem 3.18.

Cubic fuzzy subset Ω =<$\tilde{\mu}_\Omega$, λ_Ω> is a cubic fuzzy SA-subalgebra of SA-algebra X if and only if, μ^-_Ω and μ^+_Ω are fuzzy SA-subalgebras of X and λ_Ω is anti-fuzzy SA-subalgebra of X.

Proof.

Assume that Ω is a cubic fuzzy SA-subalgebra of X, for any $x , y \in X$,

$$[\mu^-_\Omega (x + y), \mu^+_\Omega (x + y)] = \tilde{\mu}_\Omega(x + y) \succcurlyeq \text{rmin}\{\tilde{\mu}_\Omega (x), \tilde{\mu}_\Omega (y)\}$$

$$= \text{rmin}\{[\mu^-_\Omega (x), \mu^+_\Omega (x)], [\mu^-_\Omega (y), \mu^+_\Omega (y)]\}$$

$= [\min\{\mu^-_\Omega(x), \mu^-_\Omega(x), \min\{\mu^+_\Omega(y), \mu^+_\Omega(y)\}]$.

Thus $\mu^-_\Omega(x + y) \geq \min\{\mu^-_\Omega(x), \mu^-_\Omega(x)\}, \mu^+_\Omega(x + y) \geq \min\{\mu^+_\Omega(x), \mu^+_\Omega(x)\}$ and

$\lambda_\Omega(x + y) \leq \max\{\lambda_\Omega(x), \lambda_\Omega(y)\}$,

Similarly, $\mu^-_\Omega(x - y) \geq \min\{\mu^-_\Omega(x), \mu^-_\Omega(x)\}, \mu^+_\Omega(x - y) \geq \min\{\mu^+_\Omega(x), \mu^+_\Omega(x)\}$ and

$\lambda_\Omega(x - y) \leq \max\{\lambda_\Omega(x), \lambda_\Omega(y)\}$ .Therefore μ^-_Ω and μ^+_Ω are fuzzy SA-subalgebras of X and λ_Ω are anti-fuzzy SA-subalgebra of X.

Conversely, let μ^-_Ω and μ^+_Ω are fuzzy SA-subalgebras of X and λ_Ω are anti-fuzzy SA-subalgebra of X and $x, y \in X$, then

$\mu^-_\Omega(x + y) \geq \min\{\mu^-_\Omega(x), \mu^-_\Omega(y)\}, \mu^+_\Omega(x + y) \geq \min\{\mu^+_\Omega(x), \mu^+_\Omega(y)\}$ and

$\lambda_\Omega(x + y) \leq \max\{\lambda_\Omega(x), \lambda_\Omega(y)\}$.

Now, $\tilde{\mu}_\Omega(x + y) = [\mu^-_\Omega(x + y), \mu^+_\Omega(x + y)]$

$\succcurlyeq [\min\{\mu^-_\Omega(x), \mu^-_\Omega(y)\}, \min\{\mu^+_\Omega(x), \mu^+_\Omega(y)\}]$

$= \text{rmin}\{[\mu^-_\Omega(x), \mu^+_\Omega(x)], [\mu^-_\Omega(y), \mu^+_\Omega(y)]\}$

$= \text{rmin}\{\tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(y)\}$, therefore

$\tilde{\mu}_\Omega(x-y) \succcurlyeq \text{rmin}\{\tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(y)\} \succcurlyeq \tilde{t}$ and $\lambda_\Omega(x\text{-}y) \leq \max\{\lambda_\Omega(x), \lambda_\Omega(y)\} \leq s$.

Similarly, $\tilde{\mu}_\Omega(x-y) \succcurlyeq \text{rmin}\{\tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(y)\}$ and $\lambda_\Omega(x\text{-}y) \leq \max\{\lambda_\Omega(x), \lambda_\Omega(y)\}$.

Hence Ω is a cubic fuzzy SA-subalgebra of X. ⌂

Theorem 3.19.

Let B a nonempty subset of X and $\Omega = <\tilde{\mu}_\Omega, \lambda_\Omega>$ be a cubic set of X defined by $\tilde{\mu}_\Omega(x) = \begin{cases} [\alpha_1, \alpha_2], & \text{if } x \in B \\ [\beta_1, \beta_2], & \text{otherwise} \end{cases}$ and

$v_\Omega(x) = \begin{cases} \gamma, & \text{if } x \in B \\ \delta, & \text{otherwise} \end{cases}$

for all $[\alpha_1, \alpha_2], [\beta_1, \beta_2] \in D[0,1]$ and $\gamma, \delta \in [0,1]$ with $[\alpha_1, \alpha_2] \geq [\beta_1, \beta_2]$ and $\gamma \leq \delta$.

Then Ω is a cubic fuzzy SA-subalgebra of X if and only if, B is an SA-subalgebra of X.

Proof.

Let Ω be a cubic fuzzy SA-subalgebra of X and x, y∈B, then

$\tilde{\mu}_\Omega(x+y) \succcurlyeq \text{rmin}\{\tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(y)\} = \text{rmin}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2]$ and

$\lambda_\Omega(x+y) \leq \max\{\lambda_\Omega(x), \lambda_\Omega(y)\} = \{\gamma, \gamma\} = \gamma$.

So $x+y \in B$. Summarily,

$\tilde{\mu}_\Omega(x-y) \succcurlyeq \text{rmin}\{\tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(y)\} = [\alpha_1, \alpha_2]$ and $\lambda_\Omega(x\text{-}y) \leq \max\{\lambda_\Omega(x), \lambda_\Omega(y)\} = \gamma$.

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Hence B is an SA-subalgebra of X.

Conversely, suppose that B is SA-subalgebra of X and let $x, y \in X$. Consider three cases.

Case1 If $x, y \in B$ then $x+y \in B$, thus $\tilde{\mu}_{\Omega}(x+y)=[\alpha_1, \alpha_2]=\text{rmin}\{\tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\}$ and

$\lambda_{\Omega}(x+y)=\gamma=\max\{\lambda_{\Omega}(x),\lambda_{\Omega}(y)\}=\max\{\gamma,\gamma\}$.

Case2 If $x\in B$ and $y\notin B$, thus $\tilde{\mu}_{\Omega}(x+y) \geq [\beta_1, \beta_2] =\text{rmin}\{\tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\}$ and

$\lambda_{\Omega}(x+y)\leq \delta =\max\{\lambda_{\Omega}(x),\lambda_{\Omega}(y)\}$.

Case 3 if $x\notin B$ or $y\notin B$, then $\tilde{\mu}_{\Omega}(x+y)\geq [\beta_1, \beta_2] =\text{rmin}\{\tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\}$ and

$\lambda_{\Omega}(x+y)\leq \delta =\max\{\lambda_{\Omega}(x),\lambda_{\Omega}(y)\}$.

Similarly, $\tilde{\mu}_{\Omega}(x-y) \succcurlyeq \text{rmin}\{ \tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\}$ and $\lambda_{\Omega}(x-y) \leq \max\{\lambda_{\Omega}(x),\lambda_{\Omega}(y)\}$.

Hence, Ω is cubic fuzzy SA-subalgebra of X. ⌂

__**Theorem 3.20.**__

If a cubic set $\Omega =<\tilde{\mu}_{\Omega}, \lambda_{\Omega}>$ is a cubic subalgebra of X, then the upper

$[s_1,s_2]$-Level and Lower t-Level of Ω are subalgebras of X.

Proof.

Let $x, y \in U(\tilde{\mu}_{\Omega} |[s_1,s_2])$, then $\tilde{\mu}_{\Omega}(x) \succcurlyeq [s_1,s_2]$ and $\tilde{\mu}_{\Omega}(y) \succcurlyeq [s_1,s_2]$. It follows that

$\tilde{\mu}_{\Omega}(x+y) \succcurlyeq \text{rmin}\{\tilde{\mu}_{\Omega}(x), \tilde{\mu}_{\Omega}(y)\} \succcurlyeq [s_1,s_2]$, so that $x+y \in U(\tilde{\mu}_{\Omega} |[s_1,s_2])$.

Similarly, $x-y \in U(\tilde{\mu}_{\Omega} |[s_1,s_2])$.

Hence $U(\tilde{\mu}_{\Omega} |[s_1,s_2])$ is SA-subalgebra of X.

Let $x, y \in L(\lambda_{\Omega}|t)$, then $\lambda_{\Omega}(x) \leq t$ and $\lambda_{\Omega}(y) \leq t$. It follows that

$\lambda_{\Omega}(x+y) \leq \max\{\lambda_{\Omega}(x), \lambda_{\Omega}(y)\}\leq t$, so that $x+y \in L(\lambda_{\Omega}|t)$.

Similarly, $x-y \in L(\lambda_{\Omega}|t)$

Hence $L(\lambda_{\Omega}|t)$ is SA-subalgebra of X. ⌂

__**Corollary 3.21.**__

Let $\Omega =<\tilde{\mu}_{\Omega}, \lambda_{\Omega}>$ be a cubic fuzzy SA-subalgebra of X, then

$\Omega([s_1,s_2];t) = U(\tilde{\mu}_{\Omega} |[s_1,s_2])\cap L(\lambda_{\Omega}|t)$

$=\{x\in X|\tilde{\mu}_{\Omega}(x)\succcurlyeq[s_1,s_2],\lambda_{\Omega}(x)\leq t \}$ is a cubic fuzzy SA-subalgebra of X

The following example shows that the converse of Corollary (3.21) is not valid

__**Example 3.22.**__

Let X ={0,a,b,c,d} be SA-algebra in example (3.8) and cubic set $\Omega=(\tilde{\mu}_\Omega,\lambda_\Omega)$ of X by

$$\tilde{\mu}_\Omega(x)=\begin{cases}[0.6,0.8], & \text{if } x=0,\\ [0.5,0.6], & \text{if } x\in\{a,b,c\},\\ [0.3,0.4], & \text{if } x\in\{d\},\end{cases} \quad \text{and} \quad v_\Omega(x)=\begin{cases}0.1, & \text{if } x=0,\\ 0.3, & \text{if } x\in\{a,b,d\},\\ 0.8, & \text{if } x\in\{c\},\end{cases}$$

We take $[s_1,s_2]=[0.41,0.48]$ and $t=0.4$, then

$\Omega([s_1,s_2];t)=U(\tilde{\mu}_\Omega|[s_1,s_2])\cap L(\lambda_\Omega|t)=\{x\in X|\tilde{\mu}_\Omega(x)\succcurlyeq[s_1,s_2],\lambda_\Omega(x)\leq t\}$

$=\{0,a,b,c\}\cap\{0,a,b,d\}=\{0,a,b\}$ is SA-subalgebra of X, but $\Omega=<\tilde{\mu}_\Omega,\lambda_\Omega>$ is not a cubic fuzzy SA-subalgebra since $\tilde{\mu}_\Omega(c*b)\not\succcurlyeq \text{rmin}\{\tilde{\mu}_\Omega(c),\tilde{\mu}_\Omega(b)\}$ and $\lambda_\Omega(c*d)\nleq\max\{\lambda_\Omega(c),\lambda_\Omega(b)\}$.

Theorem 3.23.

Let $\Omega=<\tilde{\mu}_\Omega,\lambda_\Omega>$ be a cubic fuzzy subset of X such that the sets $U(\tilde{\mu}_\Omega|[s_1,s_2])$ and $L(v_\Omega|t)$ are SA-subalgebras of X, for every $[s_1,s_2]\in D[0,1]$and $t\in[0,1]$, then $\Omega=<\tilde{\mu}_\Omega,\lambda_\Omega>$ is a cubic fuzzy SA-subalgebra of X.

Proof.

Let $U(\tilde{\mu}_\Omega|[s_1,s_2])$and $L(v_\Omega|t)$ are SA-subalgebras of X, for every $[s_1,s_2]\in D[0,1]$

and $t\in[0,1]$ on the contrary, let $x_0,y_0\in X$ be such that

$\tilde{\mu}_\Omega(x_0+y_0)\prec \text{rmin}\{\tilde{\mu}_\Omega(x_0),\tilde{\mu}_\Omega(y_0)\}$.

Let $\tilde{\mu}_\Omega(x_0)=[\theta_1,\theta_2]$ and $\tilde{\mu}_\Omega(y_0)=[\theta_3,\theta_4]$ and $\tilde{\mu}_\Omega(x_0+y_0)=[s_1,s_2]$.

Then $[s_1,s_2]\prec \text{rmin}\{[\theta_1,\theta_2],[\theta_3,\theta_4]\}=[\min\{\theta_1,\theta_3\},\min\{\theta_2,\theta_4\}]$.

So, $s_1<\min\{\theta_1,\theta_3\}$and $s_2<\min\{\theta_2,\theta_4\}$. Let us consider,

$[\rho_1,\rho_2]=\frac{1}{2}[\tilde{\mu}_\Omega(x_0+y_0)+\text{rmin}\{\tilde{\mu}_\Omega(x_0),\tilde{\mu}_\Omega(y_0)\}]$

$=\frac{1}{2}\left[[s_1,s_2]+[min\{\theta_1,\theta_3\},min\{\theta_2,\theta_4\}]\right]$

$=\left[\frac{1}{2}(s_1+\min\{\theta_1,\theta_3\}),\frac{1}{2}(s_2+\min\{\theta_2,\theta_4\})\right]$.

Therefore, $\min\{\theta_1,\theta_3\}>\rho_1=\frac{1}{2}(s_1+\min\{\theta_1,\theta_3\})>s_1$ and

$\min\{\theta_2,\theta_4\}>\rho_2=\frac{1}{2}(s_2+\min\{\theta_2,\theta_4\})>s_2$.

Hence $[\min\{\theta_1,\theta_3\},\min\{\theta_2,\theta_4\}]\succ[\rho_1,\rho_2]\succ[s_1,s_2]$, so that $(x_0+y_0)\notin U(\tilde{\mu}_\Omega|[s_1,s_2])$ which is a contradiction, since $\tilde{\mu}_\Omega(x_0)=[\theta_1,\theta_2]\succ[\min\{\theta_1,\theta_3\},\min\{\theta_2,\theta_4\}]\succ[\rho_1,\rho_2]$ and $\tilde{\mu}_\Omega(y_0)=[\theta_3,\theta_4]\succ[\min\{\theta_1,\theta_3\},\min\{\theta_2,\theta_4\}]\succ[\rho_1,\rho_2]$ this implies

$(x_0+y_0)\in U(\tilde{\mu}_\Omega|[s_1,s_2])$. Thus $\tilde{\mu}_\Omega(x+y)\succcurlyeq \text{rmin}\{\tilde{\mu}_\Omega(x),\tilde{\mu}_\Omega(y)\}$, for all $x,y\in X$.

Again, Let $x_0,y_0\in X$such that $\lambda_\Omega(x_0+y_0)>\max\{\lambda_\Omega(x_0),\lambda_\Omega(y_0)\}$.

Let $\lambda_\Omega(x_0)=\eta_1$, $\lambda_\Omega(y_0)=\eta_2$ and $\lambda_\Omega(x_0+y_0)=t$, then $t>\max\{\eta_1,\eta_2\}$.

Let us consider, $t_1=\frac{1}{2}[\lambda_\Omega(x_0 + y_0)+\max\{\lambda_\Omega(x_0), \lambda_\Omega(y_0)\}]$.

We get that $t_1=\frac{1}{2}(t+\max\{\eta_1,\eta_2\})$, therefore,

$\eta_1<t_1=\frac{1}{2}(t+\max\{\eta_1,\eta_2\})<t$ and $\eta_2<t_1=\frac{1}{2}(t+\max\{\eta_1,\eta_2\})<t$, hence,

$\max\{\eta_1,\eta_2\}<t_1<t = \lambda_\Omega(x_0 * y_0)$.

So that $x_0 + y_0 \notin L(\lambda_\Omega|t)$ which is a contradiction, since $\lambda_\Omega(x_0) = \eta_1 \leq \max\{\eta_1,\eta_2\}<t_1$ and $\lambda_\Omega(y_0)=\eta_2 \leq \max\{\eta_1,\eta_2\} < t_1$, this implies $x_0 * y_0 \in L(\lambda_\Omega|t)$ this implies

$\lambda_\Omega(x+y)\leq \max\{\lambda_\Omega(x), \lambda_\Omega(y)\}$, for all $x, y \in X$.

Therefore, $\tilde{\mu}_\Omega(x+y) \succcurlyeq \text{rmin}\{\tilde{\mu}_\Omega(x),\tilde{\mu}_\Omega(y)\}$ and $\lambda_\Omega(x+y) \leq \max\{\lambda_\Omega(x),\lambda_\Omega(y)\}$.

Similarly, $\tilde{\mu}_\Omega(x-y) \succcurlyeq \text{rmin}\{\tilde{\mu}_\Omega(x),\tilde{\mu}_\Omega(y)\}$ and $\lambda_\Omega(x-y) \leq \max\{\lambda_\Omega(x),\lambda_\Omega(y)\}$.

Hence, $\Omega =<\tilde{\mu}_\Omega, \lambda_\Omega>$ is a cubic fuzzy SA-subalgebra of X.⌂

Theorem 3.24.

Any SA-subalgebra of SA-algebra$(X;+,-,0)$ can be realized as both the upper $[s_1,s_2]$-Level and Lower t-Level of some cubic SA-subalgebra of X.

Proof.

Let P be a cubic fuzzy SA-subalgebra of X and Ω be cubic fuzzy subset on X defined by

Let P be a cubic fuzzy SA-subalgebra of X and Ω be cubic fuzzy subset on X defined by

$\tilde{\mu}_\Omega(x)=\begin{cases}[\alpha_1,\alpha_2], & \text{if } x \in P\\ [0,0], & \text{otherwise}\end{cases}$ and $v_\Omega(x)=\begin{cases}\beta, & \text{if } x \in P\\ 1, & \text{othrwise}\end{cases}$

For all$[\alpha_1,\alpha_2] \in D[0,1]and\beta \in [0,1]$, we consider the following cases

Case 1) If $x, y\in P$, then $\tilde{\mu}_\Omega(x)=[\alpha_1,\alpha_2]$, $\lambda_\Omega(x)=\beta$ and $\tilde{\mu}_\Omega(y)=[\alpha_1,\alpha_2]$,$\lambda_\Omega(y)=\beta$.

Thus, $\tilde{\mu}_\Omega(x+y)=[\alpha_1,\alpha_2]=\text{rmin}\{[\alpha_1,\alpha_2],[\alpha_1,\alpha_2]\}=\text{rmin}\{\tilde{\mu}_\Omega(x),\tilde{\mu}_\Omega(y)\}$ and

$\lambda_\Omega(x+y) =\beta=\max[\beta_1,\beta_2]=\max\{\lambda_\Omega(x), \lambda_\Omega(y)\}$.

Case 2) If $x \in P$and $y\notin P$,then $\tilde{\mu}_\Omega(x)=[\alpha_1,\alpha_2]$, $\lambda_\Omega(x)=\beta$ and $\tilde{\mu}_\Omega(y)=[0,0]$,$\lambda_\Omega(y)=1$.

Thus $\tilde{\mu}_\Omega(x+y)=[0,0] \succcurlyeq \text{rmin}\{[\alpha_1,\alpha_2],[0,0]\}=\text{rmin}\{\tilde{\mu}_\Omega(x),\tilde{\mu}_\Omega(y)\}$ and

$\lambda_\Omega(x+y) \leq 1 = \max[\beta, 1] = \max\{\lambda_\Omega(x), \lambda_\Omega(y)\}$.

Case 3) If $x\notin P$ and $y\in P$, then$\tilde{\mu}_\Omega(x) = [0,0]$, $\lambda_\Omega(x)=1$and $\tilde{\mu}_\Omega(y)=[\alpha_1,\alpha_2]$,$\lambda_\Omega(y)= \beta$

Thus,$\tilde{\mu}_\Omega(x+y)=[0,0] \succcurlyeq \text{rmin}\{[0,0],[\alpha_1,\alpha_2]\}= \text{rmin}\{\tilde{\mu}_\Omega(x),\tilde{\mu}_\Omega(y)\}$ and

$\lambda_\Omega(x+y) \leq 1 =\max[1,\beta]=\max\{\lambda_\Omega(x), \lambda_\Omega(y)\}$.

Case 4) If $x \notin P, y \notin P$ and y, then $\tilde{\mu}_{\Omega}(x) = [0,0], \lambda_{\Omega}(x)=1$ and $\tilde{\mu}_{\Omega}(y)=[0,0], \lambda_{\Omega}(y)=1$

Now, $\tilde{\mu}_{\Omega}(x+y)=[0,0]=rmin\{[0,0],[0,0]\}=rmin\{\tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\}$ and $\lambda_{\Omega}(x+y) \leq 1 = max[1,1]=max\{\lambda_{\Omega}(x),\lambda_{\Omega}(y)\}$.

Hence, $\tilde{\mu}_{\Omega}(x+y) \succcurlyeq rmin\{\tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\}$ and $\lambda_{\Omega}(x+y) \leq max\{\lambda_{\Omega}(x),\lambda_{\Omega}(y)\}$.

Similarly, $\tilde{\mu}_{\Omega}(x-y) \succcurlyeq rmin\{\tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\}$ and $\lambda_{\Omega}(x-y) \leq max\{\lambda_{\Omega}(x),\lambda_{\Omega}(y)\}$.

Therefore, Ω is a cubic fuzzy SA-subalgebra of X.⌂

4. Cubic Fuzzy SA-ideals of SA-algebra

In this section, we will introduce a new notion called cubic fuzzy SA-ideal of SA-algebras and study several properties of it.

Definition 4.1.

Let $(X;+,-,0)$ be an SA-algebra. A cubic fuzzy subset $\Omega = <\tilde{\mu}_{\Omega}, \lambda_{\Omega}>$ of X is called **cubic fuzzy SA-ideal of *X*** if, for all $x,y,z \in X$:

(ABI_1) $\tilde{\mu}_{\Omega}(0) \succcurlyeq \tilde{\mu}_{\Omega}(x)$, and $\lambda_{\Omega}(0) \leq \lambda_{\Omega}(x)$,

(ABI_2) $\tilde{\mu}_{\Omega}(x+y) \succcurlyeq rmin\{\tilde{\mu}_{\Omega}(x+z), \tilde{\mu}_{\Omega}(y-z)\}, and$

$$\lambda_{\Omega}(x+y) \leq max\{\lambda_{\Omega}(x+z), \lambda_{\Omega}(y-z)\}.$$

Example 4.2.

Let $X = \{0, a, b, c, d\}$ be a set with the following tables:

+	0	a	b	c	d
0	0	a	b	c	d
a	a	b	c	d	0
b	b	c	d	0	a
c	c	d	0	a	b
d	d	0	a	b	c

-	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	a
b	b	b	0	0	a
c	c	b	d	0	a
d	d	d	d	d	0

Then $(X;*,0)$ is an SA-algebra. It is easy to show that $I=\{0,c\}$ and $J=\{0,d\}$ are SA-subalgebras of X. We defined two cubic set $\Omega_1 = \{(x, \tilde{\mu}_{\Omega 1}(x), \lambda_{\Omega 1}(x)) \mid x \in X\}$ and $\Omega_2 = \{(x, \tilde{\mu}_{\Omega 2}(x), \lambda_{\Omega 2}(x)) \mid x \in X\}$ of X by :-

$$\tilde{\mu}_{\Omega 1}(x) = \begin{cases} [0.5,0.8], & \text{if } x \in \{0,c\}, \\ [0.4,0.7], & \text{if } x \in \{a,b\}, \\ [0.3,0.8], & \text{otherwise} \end{cases} \quad \lambda_{\Omega 1}(x) = \begin{cases} 0.2, & \text{if } x \in \{0,c\}, \\ 0.6, & \text{if } x \in \{a,b\}, \\ 0.4, & \text{otherwise} \end{cases}$$

$$\tilde{\mu}_{\Omega 2}(x) = \begin{cases} [0.4,0.9], & \text{if} x \in \{0,d\}, \\ [0.3,0.7], & \text{otherwise.} \end{cases} \quad \text{and} \quad \lambda_{\Omega 2}(x) = \begin{cases} 0.1, & \text{if} x \in \{0,d\}, \\ 0.4, & \text{otherwise.} \end{cases}$$

Then Ω_1 and Ω_2 are cubic fuzzy SA-ideal of X .

Proposition 4.3.

The R-intersection of any set of cubic fuzzy SA-ideal of X is also cubic fuzzy SA-ideal of X.

Proof.

Let $\{\Omega_i | i\in\Lambda\}$ be family of cubic fuzzy SA-ideals of X, then for any $x, y, z \in X$,

$(\cap \tilde{\mu}_{\Omega i})(0)$ =rinf $(\tilde{\mu}_{\Omega i}(0)) \succcurlyeq$ rinf $(\tilde{\mu}_{\Omega i}(x)) = (\cap \tilde{\mu}_{\Omega i})(x)$ and

$(\vee \lambda_{\Omega i})(0) = \sup \lambda_{\Omega i}(0) \le \sup \lambda_{\Omega i}(y) = (\vee \lambda_{\Omega i})(y).$

$(\cap \tilde{\mu}_{\Omega i}(x+y))$ = rinf $(\tilde{\mu}_{\Omega i}(x+y))$

$\succcurlyeq$ rinf (rmin $\{\tilde{\mu}_{\Omega i}(x+z), \tilde{\mu}_{\Omega i}(y-z)\}$)

= rmin { rinf $(\tilde{\mu}_{\Omega i}(x+z)$,rinf $(\tilde{\mu}_{\Omega i}(y-z))\}$

= rmin { $(\cap \tilde{\mu}_{\Omega i})(x+z), (\cap \tilde{\mu}_{\Omega i})(y-z)\}$

$(\vee \lambda_{\Omega i})(x+y) = \sup \lambda_{\Omega i}(x+y)$

$\le \sup\{\max\{\lambda_{\Omega i}(x+z), \lambda_{\Omega i}(y-z)\}\}$

$= \max\{\sup(\lambda_{\Omega i}(x+z)), \sup(\lambda_{\Omega i}(y-z))\}$

$= \max\{(\vee \lambda_{\Omega i})(x+z), (\vee \lambda_{\Omega i})(y-z)\}.$

Hence, R-intersection of Ω_i is a cubic fuzzy SA-ideal of X. ⌂

Remark 4.4.

The P-intresection of any sets of cubic fuzzy SA-ideal need not be a cubic fuzzy SA-ideal, for example:

Example 4.5.

Let X = {0, a, b, c, d} be a set with the following tables:

+	0	a	b	c	d
0	0	a	b	c	d
a	a	b	c	d	0
b	b	c	d	0	a
c	c	d	0	a	b
d	d	0	a	b	c

-	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	a
b	b	b	0	0	a
c	c	b	d	0	a
d	d	d	d	d	0

Then $(X; *, 0)$ is an SA-algebra. It is easy to show that I ={0, c} and J ={0, b} are SA-ideals of X. We defined two cubic set $\Omega_1 = <\tilde{\mu}_{\Omega 1}, \lambda_{\Omega 1}>$ and $\Omega_2 = <\tilde{\mu}_{\Omega 2}, \lambda_{\Omega 2}>$ of X by :-

$$\tilde{\mu}_{\Omega 1}(x) = \begin{cases} [0.6, 0.7], & \text{f } x \in \{0, c\}, \\ [0.1, 0.2], & \text{if } x \in \{d\}, \\ [0.3, 0.4], & \text{otherwise} \end{cases} \qquad \lambda_{\Omega 1}(x) = \begin{cases} 0.1, & \text{if } x \in \{0, c\}, \\ 0.6, & \text{if } x \in \{d\}, \\ 0.2, & \text{otherwise} \end{cases}$$

$$\tilde{\mu}_{\Omega 2}(x) = \begin{cases} [0.8, 0.9], & \text{if} x \in \{a, d\}, \\ [0.3, 0.4], & \text{otherwise,} \end{cases} \quad \text{and} \quad \lambda_{\Omega 2}(x) = \begin{cases} 0.1, & \text{if} x \in \{0, b\}, \\ 0.4, & \text{otherwise.} \end{cases}$$

Then Ω_1 and Ω_2 are cubic fuzzy SA-ideal of X , but P-intersection of Ω_1 and Ω_2 are not cubic fuzzy SA-ideals of X . Since

$(\cap \tilde{\mu}_{\Omega i})(b) = [0.3,0.4] \nsucceq [0.5,0.6] = \text{rmin}\{(\cap \tilde{\mu}_{\Omega i})(d * a), (\cap \tilde{\mu}_{\Omega i})(a)\}$
and $(\wedge v_{\Omega i})(d * b) = 0.4 \nleq 0.2 = \min\{(\wedge v_{\Omega i})((d * a) * b), (\wedge v_{\Omega i})(a)\}$.

Proposition 4.6.

Let $\Omega_i = <\tilde{\mu}_{\Omega i}, \lambda_{\Omega i}>$ be a cubic fuzzy SA-ideal of SA-algebra $(X ; +, -, 0)$, where $i \in \Lambda$, $\text{rsup}\{\text{rmin}\{\tilde{\mu}_{\Omega i}(x), \tilde{\mu}_{\Omega i}(y)\}\} = \text{rmin}\{\text{rsup}\,\tilde{\mu}_{\Omega i}(x), \text{rsup}\,\tilde{\mu}_{\Omega i}(y)\}$, for all $x \in X$, then the P-intresection of Ω_i is also a cubic fuzzy SA-ideal of X.

Proof. Let $\Omega_i = \{\langle x, \tilde{\mu}_{\Omega i}, (x), \lambda_{\Omega i}(x)\rangle | x \in X\}$ where $i \in \Lambda$**,** be a set of cubic ideal of X , for all $x \in X$, then for $x , y \in X$,

$(\cap \tilde{\mu}_{\Omega i})(0) = \text{rinf}\,(\tilde{\mu}_{\Omega i}(0)) \succcurlyeq \text{rinf}\,(\tilde{\mu}_{\Omega i}(x)) = (\cap \tilde{\mu}_{\Omega i})(x)$ and

$(\wedge \lambda_{\Omega i})(0) = \inf \lambda_{\Omega i}(0) \le \inf \lambda_{\Omega i}(y) = (\wedge \lambda_{\Omega i})(y)$.

$(\cap \tilde{\mu}_{\Omega i})(x + y) = \text{rinf}\,\tilde{\mu}_{\Omega i}(x + y)$

$\succcurlyeq \text{rsup}\{\text{rmin}\{\tilde{\mu}_{\Omega i}(x + z), \tilde{\mu}_{\Omega i}(y - z)\}\}$

$= \text{rmin}\{\text{rsup}\,\tilde{\mu}_{\Omega i}(x + z), \text{rsup}\,\tilde{\mu}_{\Omega i}(y - z)\}$

$= \text{rmin}\{(\cap \tilde{\mu}_{\Omega i})(x + z), (\cap \tilde{\mu}_{\Omega i})(y - z)\}$

and $(\wedge \lambda_{\Omega i})(x + y) = \inf\ v_{\Omega i}(x + y)$

$\le \inf\{\max\{\lambda_{\Omega i}(x + z), \lambda_{\Omega i}(y - z)\}\}$

$= \max\{\inf \lambda_{\Omega i}(x + z), \inf \lambda_{\Omega i}(y - z)\}$

$= \max\{(\wedge \lambda_{\Omega i})(x + z), (\wedge \lambda_{\Omega i})(y - z)\}$.

$\le \min\{(\wedge \lambda_{\Omega i})(x + z), (\wedge \lambda_{\Omega i})(y - z)\}$.

Hence, P-intersection of Ω_i is a cubic fuzzy SA-ideal of X. ⌂

Proposition 4.8.

The P-union of any set of cubic fuzzy SA-ideal of X is also cubic fuzzy SA-ideal of X.

Proof.

Let $\Omega_i = \{\langle x, \tilde{\mu}_{\Omega i}(x), \lambda_{\Omega i}(x)\rangle | x \in X\}$ where $i \in \Lambda$**,** be a set of cubic fuzzy SA-ideal of X and $x, y \in X$, then $(\cup \tilde{\mu}_{\Omega i})(0) = \text{rinf}\,(\tilde{\mu}_{\Omega i}(0)) \succcurlyeq \text{rinf}\,(\tilde{\mu}_{\Omega i}(x)) = (\cup \tilde{\mu}_{\Omega i})(x)$ and

$(\vee \lambda_{\Omega i})(0) = \sup \lambda_{\Omega i}(0) \le \sup \lambda_{\Omega i}(y) = (\vee \lambda_{\Omega i})(y)$.

$(\cup \tilde{\mu}_{\Omega i})(x + y) = \text{rsup}\ \tilde{\mu}_{\Omega i}(x + y)$

$\succcurlyeq \text{rsup}\{\text{rmax}\{\tilde{\mu}_{\Omega i}(x + z), \tilde{\mu}_{\Omega i}(y - z)\}\}$

$\succcurlyeq \text{rmax}\{\text{rsup}\,\tilde{\mu}_{\Omega i}(x + z), \text{rsup}\,\tilde{\mu}_{\Omega i}(y - z)\}$

$= \text{rmax}\,\{(\cup \tilde{\mu}_{\Omega i})(x + z), (\cup \tilde{\mu}_{\Omega i})(y - z)\}$.

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$(\vee \mathrm{v}_{\Omega\mathrm{i}})(x+y) = \sup \mathrm{v}_{\Omega\mathrm{i}}(x+y)$

$\leq \sup\{\max\{\mathrm{v}_{\Omega\mathrm{i}}(x+z), \mathrm{v}_{\Omega\mathrm{i}}(y-z)\}\}$

$= \max\{\sup \mathrm{v}_{\Omega\mathrm{i}}(x+z), \sup \mathrm{v}_{\Omega\mathrm{i}}(y-z)\} = \max\{(\vee \mathrm{v}_{\Omega\mathrm{i}})(x+z), (\vee \mathrm{v}_{\Omega\mathrm{i}})(y-z)\}$,Hence, P-union of Ω_{i} is a cubic fuzzy SA-ideal of X.

Remark 4.8.

The R-union of any sets of cubic fuzzy SA-ideal need not be a cubic fuzzy SA-ideal, for example:

Example 4.9.

Let X = {0, a, b, c, d} be a set with the following tables:

+	0	a	b	c	d
0	0	a	b	c	d
a	a	b	c	d	0
b	b	c	d	0	a
c	c	d	0	a	b
d	d	0	a	b	c

-	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	a
b	b	b	0	0	a
c	c	b	d	0	a
d	d	d	d	d	0

Then $(X; +, -, 0)$ is an SA-algebra. It is easy to show that I ={0, c} and J ={0, d} are ideals of X.We defined two cubic set $\Omega_1 = <\tilde{\mu}_{\Omega 1}, \lambda_{\Omega 1}>$ and $\Omega_2 = <\tilde{\mu}_{\Omega 2}, \lambda_{\Omega 2}>$ of X by :-

$\tilde{\mu}_{\Omega 1}(\mathrm{x}) = \begin{cases} [0.6,0.7], & \text{if} \mathrm{x} \in \{0, \mathrm{c}\}, \\ [0.1,0.2], & \text{otherwise}, \end{cases}$ $\lambda_{\Omega 1}(\mathrm{x}) = \begin{cases} 0.2, & \text{if} \mathrm{x} \in \{0, \mathrm{c}\}, \\ 0.6, & \text{otherwise}, \end{cases}$

$\tilde{\mu}_{\Omega 2}(\mathrm{x}) = \begin{cases} [0.8,0.9], & \text{if} \mathrm{x} \in \{0, \mathrm{d}\}, \\ [0.3,0.4], & \text{otherwise}, \end{cases}$ and $\lambda_{\Omega 2}(\mathrm{x}) = \begin{cases} 0.1, & \text{if} \mathrm{x} \in \{0, \mathrm{d}\}, \\ 0.4, & \text{otherwise}. \end{cases}$

Then Ω_1 and Ω_2 are cubic fuzzy SA-ideals of X , but P–union of $\Omega_1$and Ω_2 are not cubic fuzzy SA-ideal of X .Since

$(\cup \tilde{\mu}_{\Omega\mathrm{i}})(\mathrm{d}) = [0.3,0.4] \not\succcurlyeq [0.6,0.7] = \mathrm{rmin}\{(\cup \tilde{\mu}_{\Omega\mathrm{i}})(\mathrm{c} * \mathrm{d}), (\cup \mu_{\Omega\mathrm{i}})(\mathrm{c})\}$
and $(\wedge \lambda_{\Omega\mathrm{i}})(\mathrm{d}) = 0.4 \not\leq 0.2 = \max\{(\wedge \lambda_{\Omega\mathrm{i}})(\mathrm{c} * \mathrm{d}), (\wedge \lambda_{\Omega\mathrm{i}})(\mathrm{c})\}$.

Proposition 4.11.

Let $\Omega_{\mathrm{i}} = < \tilde{\mu}_{\Omega\mathrm{i}}, \lambda_{\Omega\mathrm{i}} >$ be a cubic ideal of SA-algebra $(X; +, -, 0)$, where $\mathrm{i} \in \Lambda$ $\mathrm{rsup}\{\mathrm{rmin}\{\tilde{\mu}_{\Omega\mathrm{i}}(\mathrm{x}), \tilde{\mu}_{\Omega\mathrm{i}}(\mathrm{y})\}\} = \mathrm{rmin}\{\mathrm{rsup}\, \tilde{\mu}_{\Omega\mathrm{i}}(\mathrm{x}), \mathrm{rsup}\, \tilde{\mu}_{\Omega\mathrm{i}}(\mathrm{y})\}$, for all $\mathrm{x} \in \mathrm{X}$, then the R-union of Ω_{i} is also a cubic fuzzy SA-ideal of X.

Proof.

Let $\Omega_{\mathrm{i}} = \{\langle \mathrm{x}, \tilde{\mu}_{\Omega\mathrm{i}}, (\mathrm{x}), \lambda_{\Omega i}(\mathrm{x})\rangle | \mathrm{x} \in \mathrm{X}\}$where $\mathrm{i} \in \Lambda$, be a set of cubic ideal of X , then for $x, y \in X$,

$(\cup \tilde{\mu}_{\Omega\mathrm{i}})(0) = \mathrm{rinf}\, (\tilde{\mu}_{\Omega\mathrm{i}}(0)) \succcurlyeq \mathrm{rinf}\, (\tilde{\mu}_{\Omega\mathrm{i}}(\mathrm{x})) = (\cap \tilde{\mu}_{\Omega\mathrm{i}})(\mathrm{x})$ and

$(\wedge \lambda_{\Omega\mathrm{i}})(0) = \inf \lambda_{\Omega\mathrm{i}}(0) \leq \inf \lambda_{\Omega\mathrm{i}}(\mathrm{y}) = (\wedge \lambda_{\Omega\mathrm{i}})(\mathrm{y})$.

$(\cup \tilde{\mu}_{\Omega\mathrm{i}})(x+y) = \mathrm{rinf}\, \tilde{\mu}_{\Omega\mathrm{i}}(x+y)$

$\succcurlyeq \mathrm{rsup}\{\mathrm{rmin}\{\tilde{\mu}_{\Omega\mathrm{i}}(x+z), \tilde{\mu}_{\Omega\mathrm{i}}(y-z)\}\}$

$= \text{rmin}\{\text{rsup}\ \tilde{\mu}_{\Omega i}(x+z), \text{rsup}\ \tilde{\mu}_{\Omega i}(y-z)\}$

$= \text{rmin}\{(\cup \tilde{\mu}_{\Omega i})(x+z), (\cup \tilde{\mu}_{\Omega i})(y-z)\}$

and $(\wedge \lambda_{\Omega i})(x+y) = \inf\ v_{\Omega i}(x+y)$

$\leq \inf\{\max\{\lambda_{\Omega i}(x+z), \lambda_{\Omega i}(y-z)\}\}$

$= \max\{\inf \lambda_{\Omega i}(x+z), \inf \lambda_{\Omega i}(y-z)\}$

$= \max\{(\wedge \lambda_{\Omega i})(x+z), (\wedge \lambda_{\Omega i})(y-z)\}$.

Hence, R-union of Ω_i is a cubic fuzzy SA-ideal of X. ⌂

Proposition 4.11.

Let $(X;+,-,0)$ be an SA-algebra. If a cubic subset $\Omega = <\tilde{\mu}_{\Omega}, \lambda_{\Omega}>$ of X, then Ω is a cubic fuzzy SA-ideal of X, then for all $\tilde{t} \in$ D[0, 1] and s ∈ [0, 1], the set $\tilde{U}(\Omega; \tilde{t}, s)$ is an SA-ideal of X.

Proof.

Assume that $\Omega = <\tilde{\mu}_{\Omega}, \lambda_{\Omega}>$ is a cubic fuzzy SA-ideal of X and let $\tilde{t} \in$ D[0, 1] and

s ∈ [0, 1], be such that $\tilde{U}(\Omega; \tilde{t}, s) \neq \emptyset$, and let $x, y, z \in X$ such that

$x+z, y-z \in \tilde{U}(\Omega; \tilde{t}, s)$, then $\tilde{\mu}_{\Omega}(x+z) \succcurlyeq \tilde{t}$, $\tilde{\mu}_{\Omega}(y-z) \succcurlyeq \tilde{t}$ and $\lambda_{\Omega}(x+z) \leq$ s, $\lambda_{\Omega}(y-z) \leq$ s. Since Ω is a cubic fuzzy SA-ideal of X, we get

$\tilde{\mu}_{\Omega}(x+y) \succcurlyeq \text{rmin}\{\tilde{\mu}_{\Omega}(x+z), \tilde{\mu}_{\Omega}(y-z)\} \succcurlyeq \tilde{t}$ and

$\lambda_{\Omega}(x+y) \leq \max\{\lambda_{\Omega}(x+z), \lambda_{\Omega}(y-z)\} \leq$ s.

Hence the set $\tilde{U}(\Omega; \tilde{t}, s)$ is an SA-ideal of X.⌂

Proposition 4.12.

Let $(X;+,-,0)$ be an SA-algebra. If for all $\tilde{t} \in$ D[0, 1] and s ∈ [0, 1], the set

$\tilde{U}(\Omega; \tilde{t}, s)$ is an SA-ideal of X, then $\Omega = <\tilde{\mu}_{\Omega}, \lambda_{\Omega}>$ is a cubic fuzzy SA-ideal of X.

Proof.

Suppose that $\tilde{U}(\Omega; \tilde{t}, s)$ is an SA-ideal of X and let $x, y, z \in X$ be such that

$\tilde{\mu}_{\Omega}(x+y) \prec \text{rmin}\{\tilde{\mu}_{\Omega}(x+z), \tilde{\mu}_{\Omega}(y-z)\}$, and $\lambda_{\Omega}(x+y) > \max\{\lambda_{\Omega}(x+z), \lambda_{\Omega}(y-z)\}$.

Consider $\tilde{\beta} = 1/2\ \{\tilde{\mu}_{\Omega}(x+y) + \text{rmin}\{\tilde{\mu}_{\Omega}(x+z), \tilde{\mu}_{\Omega}(y-z)\}\}$ and

$\beta = 1/2\ \{\lambda_{\Omega}(x+y) + \max\{\lambda_{\Omega}(x+z), \lambda_{\Omega}(y-z)\}\}$.

We have $\tilde{\beta} \in$ D[0, 1] and β∈ [0, 1], and $\tilde{\mu}_{\Omega}(x+y) \prec \tilde{\beta} \prec \text{rmin}\{\tilde{\mu}_{\Omega}(x+z), \tilde{\mu}_{\Omega}(y-z)\}$, and $\lambda_{\Omega}(x+y) > \beta > \max\{\lambda_{\Omega}(x+z), \lambda_{\Omega}(y-z)\}$.

It follows that $(x+z),(y-z) \in \widetilde{U}\,(\Omega;\tilde{t},s)$, and $(x+y) \notin \widetilde{U}\,(\Omega;\tilde{t},s)$. This is a contradiction and therefore $\Omega =<\tilde{\mu}_\Omega , \lambda_\Omega>$ is a cubic fuzzy SA-ideal of . ⌂

Theorem 4.13.

Cubic set $\Omega =<\tilde{\mu}_\Omega , \lambda_\Omega>$ is a cubic ideal of X if and only if, μ^-_Ω and μ^+_Ω be fuzzy SA-ideals of X and λ_Ω be anti-fuzzy SA-ideals of .

Proof.

Let μ^-_Ω and μ^+_Ω be fuzzy SA-ideals of X and λ_Ω be anti-fuzzy SA-ideals of X and $x, y, z \in X$, then

$\mu^-_\Omega(0) \geq \mu^-_\Omega(x)$, $\mu^+_\Omega(0) \geq \mu^+_\Omega(x)$ and $v_\Omega(0) \leq v_\Omega(x)$.

$\mu^-_\Omega(x+y) \geq \min\{\mu^-_\Omega(x+z),\mu^-_\Omega(y-z)\}$, $\mu^+_\Omega(x+y) \geq \min\{\mu^+_\Omega(x+z),\mu^+_\Omega(y-z)\}$ and $\lambda_\Omega(x+y) \leq \max\{\lambda_\Omega(x+z),\lambda_\Omega(y-z)\}$. Now,

$\tilde{\mu}_\Omega(x+y) = [\mu^-_\Omega(x+y),\mu^+_\Omega(x+y)]$

$\succcurlyeq [\min\{\mu^-_\Omega(x+z),\mu^-_\Omega(y-z)\}, \min\{\mu^+_\Omega(x+z),\mu^+_\Omega(y-z)\}]$

$= \text{rmin}\{[\mu^-_\Omega(x+z),\mu^+_\Omega(x+z)],[\mu^-_\Omega(y-z),\mu^+_\Omega(y-z)]\}$

$= \text{rmin}\{\tilde{\mu}_\Omega(x+z),\tilde{\mu}_\Omega(y-z)\}$, therefore

$\tilde{\mu}_\Omega(x+y) \succcurlyeq \text{rmin}\{\tilde{\mu}_\Omega(x+z),\tilde{\mu}_\Omega(y-z)\}$. And

$\lambda_\Omega(x+y) \leq \max\{\lambda_\Omega(x+z),\lambda_\Omega(y-z)\}$.

Hence Ω is a cubic fuzzy SA-ideal of X.

Conversely, assume that Ω is a cubic fuzzy SA-ideal of X, for any $x, y, z \in X$,

$\tilde{\mu}_\Omega(0) \succcurlyeq \tilde{\mu}_\Omega(x)$ and $\lambda_\Omega(0) \leq \lambda_\Omega(x)$, and

$[\mu^-_\Omega(x+y),\mu^+_\Omega(x+y)]=\tilde{\mu}_\Omega(y) \succcurlyeq \text{rmin}\{\tilde{\mu}_\Omega(x+z),\tilde{\mu}_\Omega(y-z)\}$

$= \text{rmin}\{[\mu^-_\Omega(x+z),\mu^+_\Omega(x+z)],[\mu^-_\Omega(y-z),\mu^+_\Omega(y-z)]\}$

$= [\min\{\mu^-_\Omega(x+z),\mu^-_\Omega(y-z),\{\mu^+_\Omega(x+z),\mu^+_\Omega(y-z)\}]$.

Thus $\mu^-_\Omega(x+y) \geq \{\mu^-_\Omega(x+z),\mu^-_\Omega(y-z)\}$,

$\mu^+_\Omega(x+y) \geq \{\mu^+_\Omega(x+z),\mu^+_\Omega(y-z)\}$ and

$\lambda_\Omega(x+y) \leq \max\{\lambda_\Omega(x+z),\lambda_\Omega(y-z)\}$,

Therefore μ^-_Ω and μ^+_Ω be fuzzy SA-ideals of X and λ_Ω be anti-fuzzy SA-ideal of X. ⌂

Theorem 4.14.

Every cubic fuzzy SA-ideal of SA-algebra $(X; +,-,0)$ is a cubic fuzzy SA-subalgebra of X.

Proof:

Let $(X;+,-,0)$ be an SA-algebra and $\Omega =< \tilde{\mu}_{\Omega}(x), \lambda_{\Omega}(x) >$ is a cubic fuzzy SA-ideal of X. Since Ω is an cubic fuzzy SA-ideal of X, then by Proposition (4.11) , for every $\zeta \in (0,1]$, $\tilde{t} \in$ D[0, 1] and $s \in [0,1]$, $\tilde{U}(\Omega;\tilde{t},s) = \{ x \in X \mid \tilde{\mu}_{\Omega}(x) \succcurlyeq \tilde{t}, \lambda_{\Omega}(x) \leq s\}$, is ideal of X . By Proposition (2.9), for every $\zeta \in (0,1]$, $\tilde{t} \in$ D[0, 1] and $s \in [0,1]$, $\tilde{U}(\Omega;\tilde{t},s)$ is SA-subgalgebra of X . Hence μ is a cubic fuzzy SA-subalgebra of X by Proposition (3.19). ⌂

Remark 4.15.

The converse of Theorem (4.14) is not true as the following example:

Example 4.16.

Let X={0,1,2, 3} in which (+, −) be a defined by the following tables:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

-	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

Then $(X;+,-,0)$ is an SA-algebra. Define a cubic set $\Omega =<\tilde{\mu}_{\Omega}, \lambda_{\Omega}>$ of X is fuzzy subset $\mu: X \rightarrow [0,1]$ by: $\tilde{\mu}_{\Omega}(x) = \begin{cases} [0.3,0.9] & if x = \{0,1,2\} \\ [0.1,0.6] & otherwise \end{cases}$ and $\lambda_{\Omega} = \begin{cases} 0.1 & if x = \{0,1,2\} \\ 0.6 & otherwise \end{cases}$.

The cubic set $\Omega =< \tilde{\mu}_{\Omega}(x), \lambda_{\Omega}(x) >$ is not a cubic fuzzy SA-subalgebra of X.

Note that λ_{Ω} is not an anti-fuzzy SA-ideal of X since

$\lambda_{\Omega}(4) = 0.24 > 0.04 = \max\{\lambda_{\Omega}(1*4), \lambda_{\Omega}(1)\}$

$= \max\{\lambda_{\Omega}(1), \lambda_{\Omega}(1)\} = \lambda_{\Omega}(1)$. Hence Ω is not cubic fuzzy SA-ideal of X.

Theorem 4.17.

Let B a nonempty subset of X and $\Omega =<\tilde{\mu}_{\Omega}, \lambda_{\Omega}>$ be a cubic set of X defined by $\tilde{\mu}_{\Omega}(x) = \begin{cases} [\alpha_1, \alpha_2], & \text{if } x \in B \\ [\beta_1, \beta_2], & \text{otherwise} \end{cases}$ and $v_{\Omega}(x) = \begin{cases} \gamma, & \text{if } x \in B \\ \delta, & \text{otherwise} \end{cases}$

for all $[\alpha_1, \alpha_2], [\beta_1, \beta_2] \in D[0,1]$ and $\gamma, \delta \in [0,1]$ with $[\alpha_1, \alpha_2] \geq [\beta_1, \beta_2]$ and $\gamma \leq \delta$. Then Ω is a cubic fuzzy SA-ideal of X if and only if, B is an SA-ideal of X.

Proof.

Let Ω be a cubic fuzzy SA-ideal of X and x, y,z ∈B, then

$\tilde{\mu}_{\Omega}(x+y) \succcurlyeq \text{rmin}\{\tilde{\mu}_{\Omega}(x+z), \tilde{\mu}_{\Omega}(y-z)\} = \text{rmin}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2]$ and

$\lambda_\Omega(x+y) \leq \max\{\lambda_\Omega(x+z),\lambda_\Omega(y-z)\}=\{\gamma,\gamma\}=\gamma$.

So $y-z \in$ B. Hence B is an SA-ideal of X.

Conversely, suppose that B is an SA-ideal of X and let $x, y, z \in X$. Consider two cases.

Case 1) If $(x+z),(y-z) \in$B then $x+y \in$B, thus $\tilde{\mu}_\Omega(x+y)=[\alpha_1,\alpha_2]$=rmin$\{\tilde{\mu}_\Omega(x+z),\tilde{\mu}_\Omega(y-z)\}$ and $\lambda_\Omega(x+y)=\gamma$=max$\{\lambda_\Omega(x+z),\lambda_\Omega(y-z)\}$=max$\{\gamma,\gamma\}$.

Case 2) If $(x+z) \in$B and $(y-z) \notin$B , thus $\tilde{\mu}_\Omega(x+y) \geq [\beta_1,\beta_2]$ =rmin$\{\tilde{\mu}_\Omega(x+z),\tilde{\mu}_\Omega(y-z)\}$ and $\lambda_\Omega(x+y) \leq \delta$ =max$\{\lambda_\Omega(x+z),\lambda_\Omega(y-z)\}$.

Case 3) If $(x+z) \notin$B and $(y-z) \in$B , thus $\tilde{\mu}_\Omega(x+y) \geq [\beta_1,\beta_2]$ =rmin$\{\tilde{\mu}_\Omega(x+z),\tilde{\mu}_\Omega(y-z)\}$ and $\lambda_\Omega(x+y) \leq \delta$ =max$\{\lambda_\Omega(x+z),\lambda_\Omega(y-z)\}$.

Case 4) if $(y-z) \notin$B or $(x+y) \notin$B, then $\tilde{\mu}_\Omega(x+y) \geq [\beta_1,\beta_2]$ =rmin$\{\tilde{\mu}_\Omega(x+z),\tilde{\mu}_\Omega(y-z)\}$ and $\lambda_\Omega(x+y) \leq \delta$ =max$\{\lambda_\Omega(x+z),\lambda_\Omega(y-z)\}$.

Hence, Ω is cubic fuzzy SA-ideal of X. ⌂

Theorem 4.18.

If a cubic fuzzy subset $\Omega = <\tilde{\mu}_\Omega, \lambda_\Omega>$ is a cubic fuzzy SA-ideal of X, then the upper

$[s_1,s_2]$-Level and Lower t-Level of Ω are SA-ideals of X.

Proof.

Let $(x+z),(y-z) \in U(\tilde{\mu}_\Omega|[s_1,s_2])$, then $\tilde{\mu}_\Omega(x+z) \succcurlyeq [s_1,s_2]$ and $\tilde{\mu}_\Omega(y-z) \succcurlyeq [s_1,s_2]$. It follows that

$\tilde{\mu}_\Omega(x+y) \succcurlyeq$ rmin$\{\tilde{\mu}_\Omega(x+z), \tilde{\mu}_\Omega(y-z)\} \succcurlyeq [s_1,s_2]$, so that $(x+y) \in U(\tilde{\mu}_\Omega|[s_1,s_2])$.

Hence $U(\tilde{\mu}_\Omega|[s_1,s_2])$ is an SA-ideal of X.

Let $(x+z),(y-z) \in L(v_\Omega|t)$, then $\lambda_\Omega(x+z) \leq t$ and $\lambda_\Omega(y-z) \leq t$. It follows that

$\lambda_\Omega(x+y) \leq \max\{\lambda_\Omega(x+z), \lambda_\Omega(y-z)\} \leq t$, so that $(x+y) \in L(\lambda_\Omega|t)$.

Hence $L(\lambda_\Omega|t)$ is an SA-ideal of X. ⌂

Corollary 4.19.

Let $\Omega = <\tilde{\mu}_\Omega, \lambda_\Omega>$ be a cubic fuzzy SA-ideal of X, then

$U(\tilde{\mu}_\Omega|[s_1,s_2]) \cap L(\lambda_\Omega|t)=\{x \in X | \tilde{\mu}_\Omega(x) \geq [s_1,s_2], \lambda_\Omega(x) \leq t\}$ is an SA-ideal of X

The following example shows that the converse of Corollary (4.19) is not valid

Example 4.20.

Let X ={0,a,b,c,d} be SA-algebra in example (3.8) and cubic set $\Omega=(\tilde{\mu}_\Omega,\lambda_\Omega)$ of X by

$\tilde{\mu}_{\Omega}(x) = \begin{cases} [0.6,0.8], & \text{if } x = 0, \\ [0.5,0.6], & \text{if } x \in \{a,b,c\}, \\ [0.3,0.4], & \text{if } x \in \{d\}, \end{cases}$ and $\lambda_{\Omega}(x) = \begin{cases} 0.1, & \text{if } x = 0, \\ 0.3, & \text{if } x \in \{a,b,c\}, \\ 0.8, & \text{if } x \in \{d\}, \end{cases}$

We take $[s_1,s_2]=[0.41,0.48]$ and $t=0.4$, then

$U(\tilde{\mu}_{\Omega} \| [s_1,s_2]) \cap L(\lambda_{\Omega}|t)=\{x\in X|\tilde{\mu}_{\Omega}(x) \succcurlyeq [s_1,s_2], \lambda_{\Omega}(x)\le t\}$

$=\{0, a, b, c\}\cap\{0, a, b, d\}=\{0, a, b\}$ is subalgebra of X, but $\Omega = <\tilde{\mu}_{\Omega}, \lambda_{\Omega}>$ is not a cubic subalgebra since $\tilde{\mu}_{\Omega}(c*b) \not\succcurlyeq rmin\{\tilde{\mu}_{\Omega}(c), \tilde{\mu}_{\Omega}(b)\}$ and $\lambda_{\Omega}(c*b) \nleq \max\{\lambda_{\Omega}(c), \lambda_{\Omega}(b)\}$.

Theorem 4.21.

Let $\Omega=(\tilde{\mu}_{\Omega},\lambda_{\Omega})$ be a cubic fuzzy subset of X such that the sets $U(\tilde{\mu}_{\Omega} \| [s_1,s_2])$ and $L(\lambda_{\Omega}|t)$ are SA-ideals of X, for every $[s_1,s_2]\in D[0,1]$ and $t \in[0,1]$, then $\Omega = <\tilde{\mu}_{\Omega}, \lambda_{\Omega}>$ is a cubic fuzzy SA-ideal of X.

Proof.

Let $U(\tilde{\mu}_{\Omega} \| [s_1,s_2])$ and $L(\lambda_{\Omega}|t)$ are ideals of X, for every $[s_1,s_2]\in D[0,1]$

and $t \in[0,1]$ on the contrary, let $x_0, y_0, z_0 \in X$ be such that

$\tilde{\mu}_{\Omega}(x_0 + y_0) \prec rmin\{\tilde{\mu}_{\Omega}(x_0 + z_0), \tilde{\mu}_{\Omega}(y_0 - z_0)\}$.

Let $\tilde{\mu}_{\Omega}(x_0 + z_0) =[\theta_1,\theta_2]$ and $\tilde{\mu}_{\Omega}(y_0 - z_0) =[\theta_3,\theta_4]$ and $\tilde{\mu}_{\Omega}(x_0 + y_0)=[s_1,s_2]$.

Then $[s_1,s_2] \prec rmin\{[\theta_1,\theta_2],[\theta_3,\theta_4]\}=[\min\{\theta_1,\theta_2\},\min\{\theta_3,\theta_4\}]$.

So, $s_1< \min\{\theta_1,\theta_3\}$ and $s_2< \min\{\theta_2,\theta_4\}$. Let us consider,

$[\rho_1,\rho_2] = \frac{1}{2}[\tilde{\mu}_{\Omega}(x_0 + y_0)+ rmin\{\tilde{\mu}_{\Omega}(x_0 + z_0), \tilde{\mu}_{\Omega}(y_0 - z_0)\}]$

$= \frac{1}{2}\left[[s_1, s_2] + [min\{\theta_1, \theta_3\}, min\{\theta_2, \theta_4\}]\right]$

$=\left[\frac{1}{2}(s_1 + \min\{\theta_1, \theta_3\}), \frac{1}{2}(s_2 + \min\{\theta_2, \theta_4\})\right]$.

Therefore, $\min\{\theta_1, \theta_3\}>\rho_1=\frac{1}{2}(s_1 + \min\{\theta_1, \theta_3\})>s_1$ and

$\min\{\theta_2, \theta_4\}>\rho_2=\frac{1}{2}(s_2 + \min\{\theta_2, \theta_4\})>s_2$.

Hence $[\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] \succ[\rho_1,\rho_2] \succ [s_1,s_2]$, so that $(x_0 + y_0) \notin U(\tilde{\mu}_{\Omega} \| [s_1,s_2])$ which is a contradiction, since $\tilde{\mu}_{\Omega}(x_0 + z_0)=[\theta_1, \theta_2] \succ [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] \succ [\rho_1,\rho_2]$ and $\tilde{\mu}_{\Omega}(y_0 - z_0)=[\theta_3, \theta_4] \succ [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] \succ [\rho_1,\rho_2]$ this implies

$(x_0 + y_0) \in U(\tilde{\mu}_{\Omega} \| [s_1,s_2])$. Thus $\tilde{\mu}_{\Omega}(x + y) \succcurlyeq rmin\{\tilde{\mu}_{\Omega}(x + z),\tilde{\mu}_{\Omega}(y - z)\}$, for all $x, y, z \in X$. Again, Let $x_0, y_0, z_0 \in X$ such that $\lambda_{\Omega}(x_0 + y_0)> \max\{\lambda_{\Omega}(x_0 + z_0), \lambda_{\Omega}(y_0 - z_0)\}$.

Let $\lambda_{\Omega}(x_0 + z_0)=\eta_1$, $\lambda_{\Omega}(x_0 + y_0)=\eta_2$ and $\lambda_{\Omega}(x_0 + z_0) = t$, then $t > \max\{\eta_1,\eta_2\}$.

Let us consider, $t_1= \frac{1}{2}[\lambda_{\Omega}(x_0 + y_0)+ \max\{\lambda_{\Omega}(x_0 + z_0), \lambda_{\Omega}(y_0 - z_0)\}]$.

We get that $t_1=\frac{1}{2}(t+\max\{\eta_1,\eta_2\})$, therefore,

$\eta_1<t_1=\frac{1}{2}(t+\max\{\eta_1,\eta_2\})<t$ and $\eta_2<t_1=\frac{1}{2}(t+\max\{\eta_1,\eta_2\})<t$, hence,

$\max\{\eta_1,\eta_2\}<t_1<t=\lambda_\Omega(x_0+y_0)$. So that $x_0 * z_0 \notin L(\lambda_\Omega|t)$ which is a contradiction, since $\lambda_\Omega(x_0 * y_0)=\eta_1\leq\max\{\eta_1,\eta_2\}<t_1$ and $\lambda_\Omega(y_0-z_0)=\eta_2\leq\max\{\eta_1,\eta_2\}<t_1$, this implies $(x_0+y_0)\in L(\lambda_\Omega|t)$ this implies $\lambda_\Omega(x+y)\leq\max\{\lambda_\Omega(x+z),\lambda_\Omega(y-z)\}$, for all $x,y,z\in X$.

Hence, $\Omega=<\tilde{\mu}_\Omega,\lambda_\Omega>$ is a cubic fuzzy SA-ideal of X.⌂

5. Homomorphism of Cubic on SA-algebra

In this section, we will present some results on images and preimages of cubic fuzzy SA-subalgebra and cubic fuzzy SA-ideals of SA-algebras.

Theorem 5.1.

A homomorphic preimage of cubic fuzzy SA-subalgebra is also cubic fuzzy SA-subalgebra.

Proof.

Let $f:(X;+,-,0)\rightarrow(Y;+',-',0')$be homomorphism from an SA -algebra X into an SA-algebra Y. If $\beta=<\tilde{\mu}_\beta,\lambda_\beta>$is cubic fuzzy SA-subalgebra of Y and $\Omega=<\tilde{\mu}_\Omega,\lambda_\Omega>$ the preimage of β under f, then $\tilde{\mu}_{f^{-1}(\beta)}(x)=\tilde{\mu}_\Omega(f(x))$, $\lambda_{f^{-1}(\beta)}(x)=\lambda_\Omega(f(x))$, for all $x\in X$. Let $x\in X$, then $(\tilde{\mu}_{f^{-1}(\beta)}0)=\tilde{\mu}_\Omega(f(0))\succcurlyeq\tilde{\mu}_\Omega(f(x))=\tilde{\mu}_{f^{-1}(\beta)}(x)$, and

$(\lambda_{f^{-1}(\beta)})(0)=\lambda_\Omega(f(0))\leq\lambda_\Omega(f(x))=\lambda_{f^{-1}(\beta)}(x)$. Now, let $x, y\in X$, then

$\tilde{\mu}_{f^{-1}(\beta)}(x+y)=\tilde{\mu}_\Omega(f(x+y))=\tilde{\mu}_\Omega(f(x)*'f(y))$

$\succcurlyeq \text{rmin}\{\tilde{\mu}_\Omega(f(x),\tilde{\mu}_\Omega(f(y))\}$

$=\text{rmin}\{\tilde{\mu}_{f^{-1}(\beta)}(x),\tilde{\mu}_{f^{-1}(\beta)}(y)\}$ and

$\lambda_{f^{-1}(\beta)}(x+y)=\lambda_\Omega(f(x+y))=\lambda_\Omega(f(x)*'f(y))$

$\leq\max\{\lambda_\Omega(f(x),\lambda_\Omega(f(y))\}=\max\{\lambda_{f^{-1}(\beta)}(x),\lambda_{f^{-1}(\beta)}(y)\}$.

Similarly, $\tilde{\mu}_{f^{-1}(\beta)}(x-y)\succcurlyeq\text{rmin}\{\tilde{\mu}_{f^{-1}(\beta)}(x),\tilde{\mu}_{f^{-1}(\beta)}(y)\}$ and

$\lambda_{f^{-1}(\beta)}(x-y)\leq\max\{\lambda_{f^{-1}(\beta)}(x),\lambda_{f^{-1}(\beta)}(y)\}$. ⌂

Theorem 5.2.

Let $f:(X;+,-,0)\rightarrow(Y;+',-',0')$ be an epimorphism from an SA-algebra X into an SA-algebra Y. For every cubic fuzzy SA-subalgebra $\Omega=<\tilde{\mu}_\Omega,\lambda_\Omega>$ of X with **sup and inf properties**, then $f(\Omega)$ is a cubic fuzzy SA-subalgebra of Y.

Proof.

Since $f(\tilde{\mu}_{\Omega})(y') = \underset{x\in f^{-1}(y')}{rsup}\ \tilde{\mu}_{\Omega}(x)$ and

$f(\lambda_{\Omega})(y') = \underset{x\in f^{-1}(y')}{inf}\ \lambda_{\Omega}(x)$ for all y' ∈ Y and

$rsup(\emptyset) = [0,0]$ and inf $(\emptyset) = 1$. We have prove that

$f(\tilde{\mu}_{\Omega})$ (x'+y') ≽rmin {$f(\tilde{\mu}_{\Omega})$(x'), $\tilde{\mu}_{\Omega}$ (y')}, and

$f(\lambda_{\Omega})$ (x'+y')≤ rmax{$f(\lambda_{\Omega})$(x') , $f(\lambda_{\Omega})$ (y')}, for all x', y' ∈ Y.

$f(\tilde{\mu}_{\Omega})(x' +' y') = \underset{t\in f^{-1}(x'*y')}{rsup}\ \tilde{\mu}_{\Omega}(t) = \tilde{\mu}_{\Omega}(x_0 + y_0)$

≽ rmin {$\tilde{\mu}_{\Omega}$ (x_0), $\tilde{\mu}_{\Omega}$ (y_0)}= rmin { $\underset{t\in f^{-1}(x')}{rsup}\ \tilde{\mu}_{\Omega}(t)$, $\underset{t\in f^{-1}(y')}{rsup}\ \tilde{\mu}_{\Omega}(t)$ }

= rmin { $f(\tilde{\mu}_{\Omega})(x')$,$f(\tilde{\mu}_{\Omega})(y')$ } and

$f(\lambda_{\Omega})(x' +' y') = \underset{t\in f^{-1}(x'*y')}{inf}\ \lambda_{\Omega}(t) = \lambda_{\Omega}(x_0 + y_0)$

≤ max {$\lambda_{\Omega}(x_0)$, $\lambda_{\Omega}(y_0)$} = max { $\underset{t\in f^{-1}(x')}{inf}\ \lambda_{\Omega}(t)$, $\underset{t\in f^{-1}(y')}{inf}\ \lambda_{\Omega}(t)$ }

= max { $f(\lambda_{\Omega})(x')$,$f(\lambda_{\Omega})(y')$ } .

Similarly, $f(\tilde{\mu}_{\Omega})$(x$-y$) ≽ rmin{ $f(\tilde{\mu}_{\Omega})$(x),$f(\tilde{\mu}_{\Omega})$ (y)} and

$f(\lambda_{\Omega})$ (x-y) ≤ max {$f(\lambda_{\Omega})$ (x), $f(\lambda_{\Omega})$ (y)}.

Hence, $f(\Omega)$ is a cubic fuzzy SA-subalgebra of Y.⌂

<u>Theorem 5.3.</u>

A homomorphic pre-image of cubic fuzzy SA-ideal is also cubic fuzzy SA-ideal.

Proof.

Let $f:(X;+,-,0) \to (Y;+',-',0')$be homomorphism from an SA-algebra X into an SA-algebra Y. If β =<$\tilde{\mu}_{\beta}$, λ_{β}>is a cubic ideal of Y and Ω =<$\tilde{\mu}_{\Omega}$, λ_{Ω}>the pre-image of β under f , then $\tilde{\mu}_{\Omega}$ (x) = $\tilde{\mu}_{\beta}$ (f (x)) , λ_{Ω} (x) = λ_{β} (f (x)), for all $x \in X$. Let x ∈X , then

($\tilde{\mu}_{\Omega}$)(0) = $\tilde{\mu}_{\beta}$ (f (0))≽$\tilde{\mu}_{\beta}$ (f (x)) = $\tilde{\mu}_{\Omega}$ (x), and (λ_{Ω})(0) = λ_{β} (f (0))≤λ_{β} (f (x)) = λ_{Ω} (x).

Now, let $x, y, z \in X$, then $\tilde{\mu}_{\Omega}$ $(x + y)$ = $\tilde{\mu}_{\beta}$ (f $(x + y)$)

≽ rmin {$\tilde{\mu}_{\beta}$ (f $(x + z)$,$\tilde{\mu}_{\beta}$ (f $(y - z)$) }

= rmin $\{\tilde{\mu}_\Omega(x+z), \tilde{\mu}_\Omega(y-z)\}$, and

$\lambda_\Omega(x+y) = \lambda_\beta(f(x+y))$

$\leq \max\{\lambda_\beta(f(x+z), \lambda_\beta(f(y-z))\} = \max\{\lambda_\Omega(x+z), \lambda_\Omega(y-z)\}$. ⌂

Theorem 5.4.

Let $f:(X;+,-,0) \to (Y;+',-',0')$ be an epimorphism from an SA-algebra X into an SA-algebra Y. For every cubic fuzzy SA-ideal $\Omega = <\tilde{\mu}_\Omega, \lambda_\Omega>$ of X with **sup and inf properties**, then $f(\Omega)$ is a cubic fuzzy SA-ideal of Y.

Proof.

Since $\tilde{\mu}_\beta(x'+'z') = f(\tilde{\mu}_\Omega)(x'+'z') = \underset{x+z\in f^{-1}(x'+'z')}{rsup} \tilde{\mu}_\Omega(x+z)$ &

$\lambda_\beta(x'+'z') = f(\lambda_\Omega)(x'+'z') = \underset{x+z\in f^{-1}(x'+'z')}{inf} \lambda_\Omega(x+z)$ and

$\tilde{\mu}_\beta(y'-'z') = f(\tilde{\mu}_\Omega)(y'-'z') = \underset{y-z\in f^{-1}(y'-'z')}{rsup} \tilde{\mu}_\Omega(y-z)$ &

$\lambda_\beta(y'-'z') = f(\lambda_\Omega)(y'-'z') = \underset{y-z\in f^{-1}(y'-'z')}{inf} \lambda_\Omega(y-z)$ for all x', y', z'∈ Y and

rsup(∅) = [0, 0] and inf (∅) = 0 . We have prove that

$\tilde{\mu}_\beta$ (x' +y') ≽rmin {$\tilde{\mu}_\beta$ (x' +z'), $\tilde{\mu}_\beta$ (y'- z')}, and

λ_β (x' +y')≤ max{λ_β (x' +z'), λ_β (z' -z')}, for all x', y' , z' ∈ Y .

Let $f:(X;+,-,0) \to (Y;+',-',0')$ be epimorphism of SA-algebras, $\Omega = <\tilde{\mu}_\Omega, \lambda_\Omega>$ is a cubic fuzzy SA-ideal of X has sup and inf properties and $\beta = <\tilde{\mu}_\beta, \lambda_\beta>$the image of$\Omega = <\tilde{\mu}_\Omega, \lambda_\beta>$under f . Since $\Omega = <\tilde{\mu}_\Omega, \lambda_\Omega>$ is a cubic fuzzy SA-ideal of X, we have $(\tilde{\mu}_\Omega)(0) \succcurlyeq \tilde{\mu}_\Omega(x)$ and $(\lambda_\Omega)(0) \leq \lambda_\Omega(x)$, for all x ∈X .

Note that, 0 ∈ f^{-1}(0') where 0,0' are the zero of X and Y, respectively. Thus

$\tilde{\mu}_\beta(0') = \underset{t\in f^{-1}(0')}{r\sup} \tilde{\mu}_\Omega(t) = \tilde{\mu}_\Omega(0) \succcurlyeq \tilde{\mu}_\Omega(x) = \underset{t\in f^{-1}(x')}{r\sup} \tilde{\mu}_\Omega(t) = \tilde{\mu}_\beta(x')$,

$\lambda_\beta(0') = \underset{t\in f^{-1}(0')}{\inf} \lambda_\Omega(t) = \lambda_\Omega(0) \leq \lambda_\Omega(x) = \underset{t\in f^{-1}(x')}{\inf} \lambda_\Omega(t) = \lambda_\beta(x')$,for all x ∈X , which implies that $\tilde{\mu}_\beta$ (0') ≽ $\tilde{\mu}_\beta(x')$, and $\lambda_\beta(0') \leq \lambda_\beta(x')$, for all x' ∈Y .

For any x' , y' , z' ∈ Y, let $x_0 \in f^{-1}$(x') , $y_0 \in f^{-1}$(y') and $z_0 \in f^{-1}$(z') be such that

$\tilde{\mu}_\Omega(x_0+z_0) = \underset{t\in f^{-1}(x'+'z')}{rsup} \tilde{\mu}_\Omega(t)$, $\tilde{\mu}_\Omega(y_0-z_0) = \underset{t\in f^{-1}(y'-'z')}{rsup} \tilde{\mu}_\Omega(t)$ and

$\tilde{\mu}_\Omega(x_0+y_0) = \tilde{\mu}_\beta(x'+'y') = \underset{t\in f^{-1}(x'+'y')}{rsup}$

and $\lambda_\Omega(x_0+z_0) = \underset{t\in f^{-1}(x'+'z')}{inf} \lambda_\Omega(t)$, $\lambda_\Omega(y_0-z_0) = \underset{t\in f^{-1}(y'-'z')}{inf} \lambda_\Omega(t)$

And $\lambda_\Omega(x_0+y_0)=\inf_{t\in f^{-1}(x'+y')}\lambda_\Omega(t)$

Also , $\tilde{\mu}_\beta(x'*y')=\underset{t\in f^{-1}(x'*y')}{rsup}\tilde{\mu}_\Omega(t)=\tilde{\mu}_\Omega(x_0+y_0)$

$\geqslant$ rmin $\{\tilde{\mu}_\Omega(x_0+z_0),\tilde{\mu}_\Omega(y_0-z_0)\}$,

$=$ rmin$\{\underset{t\in f^{-1}(x'+z')}{rsup}\tilde{\mu}_\Omega(t),\underset{t\in f^{-1}(y'-z')}{rsup}\tilde{\mu}_\Omega(t)\}$

= rmin $\{\tilde{\mu}_\beta(x'+z'),\tilde{\mu}_\beta(y'-z')\}$and

$\lambda_\Omega(x'*y')=\inf_{(y_0)\in f^{-1}(y')}\lambda_\Omega(y_0)\leq$ max $\{\lambda_\Omega(x_0+z_0),\lambda_\Omega(y_0-z_0)\}$ $=$ max$\{\inf_{t\in f^{-1}(x'+z')}\lambda_\Omega(t),\inf_{t\in f^{-1}(y'-z')}\lambda_\Omega(t)\}$ = max $\{\lambda_\Omega(x'+z'),\lambda_\Omega(y'-z')\}$.

Hence, β is a cubic fuzzy SA-ideal of . ⌂

6. Cartesian product on Cubic Fuzzy of SA -algebra

In this section we introduce the notions of Cartesian product of cubic fuzzy SA -subalgebras and cubic fuzzy ideals in a SA -algebra.

Definition 6.1.

Let $\Omega_1=<\tilde{\mu}_{\Omega_1},\lambda_{\Omega_1}>$ be a cubic fuzzy subset of X and $\Omega_2=<\tilde{\mu}_{\Omega_2},\lambda_{\Omega_2}>$ be a cubic fuzzy subset of Y. The Cartesian product of Ω_1 and Ω_2 is defined as $\Omega_1\times\Omega_2=$ (X×Y, $\tilde{\mu}_{\Omega_1}\times\tilde{\mu}_{\Omega_2}$, $\lambda_{\Omega_1}\times\lambda_{\Omega_2}$) where , $\tilde{\mu}_{\Omega_1}\times\tilde{\mu}_{\Omega_2}$: X×Y→[0,1] and $\lambda_{\Omega_1}\times\lambda_{\Omega_2}$: X×Y→[0,1] ∀ x ∈ X, y ∈ Y, such that (, $\tilde{\mu}_{\Omega_1}\times\tilde{\mu}_{\Omega_2}$)(x,y) = rmin$\{\tilde{\mu}_{\Omega_1}(x),\tilde{\mu}_{\Omega_2}(y)\}$ and $(\lambda_{\Omega_1}\times\lambda_{\Omega_2})$(x,y) = max$\{\lambda_{\Omega_1}(x),\lambda_{\Omega_2}(y)\}$

Theorem 6.2.

Let $\Omega_1=<\tilde{\mu}_{\Omega_1},\lambda_{\Omega_1}>$ be a cubic fuzzy SA -subalgebra of X and $\Omega_2=<\tilde{\mu}_{\Omega_2},\lambda_{\Omega_2}>$be a cubic fuzzy SA-subalgebra of Y, then $\Omega_1\times\Omega_2$ is a cubic fuzzy SA-subalgebras of X×Y.

Proof:

Let $(x_1,y_1)\in X\times Y$and $(x_2,y_2)\in X\times Y$, then

$(\tilde{\mu}_{\Omega_1}\times\tilde{\mu}_{\Omega_2})((x_1,y_1)+(x_2,y_2))=(\tilde{\mu}_{\Omega_1}\times\tilde{\mu}_{\Omega_2})((x_1+x_2),(y_1+y_2))$

$=\min\{\tilde{\mu}_{\Omega_1}(x_1+x_2),\tilde{\mu}_{\Omega_2}(y_1+y_2)\}$

$\geqslant\min\{\text{rmin}(\tilde{\mu}_{\Omega_1}(x_1),\tilde{\mu}_{\Omega_1}(x_2)),\text{rmin}(\tilde{\mu}_{\Omega_2}(y_1),\tilde{\mu}_{\Omega_2}(y_2))\}$

$=\text{rmin}\{\min\{\tilde{\mu}_{\Omega_1}(x_1),\tilde{\mu}_{\Omega_2}(y_1)\},\min\{\tilde{\mu}_{\Omega_1}(x_2),\tilde{\mu}_{\Omega_2}(y_2)\}\}$

$=\text{rmin}\{(\tilde{\mu}_{\Omega_1}\times\tilde{\mu}_{\Omega_2})(x_1,y_1),(\tilde{\mu}_{\Omega_1}\times\tilde{\mu}_{\Omega_2})(x_2,y_2)\}$,

Similarly, $(\tilde{\mu}_{\Omega_1}\times\tilde{\mu}_{\Omega_2})((x_1,y_1)-(x_2,y_2))\geqslant rmin\{(\tilde{\mu}_{\Omega_1}\times\tilde{\mu}_{\Omega_2})(x_1,y_1),(\tilde{\mu}_{\Omega_1}\times\tilde{\mu}_{\Omega_2})(x_2,y_2)\}$. And $(\lambda_{\Omega_1}\times\lambda_{\Omega_2})((x_1,y_1)+(x_2,y_2))=(\lambda_{\Omega_1}\times\lambda_{\Omega_2})((x_1+x_2),(y_1+y_2))$

$= \max\{\lambda_{\Omega_1}(x_1+x_2), \nu_B(y_1+y_2)\}$

$\leq \max\{\max\{\lambda_{\Omega_1}(x_1), \lambda_{\Omega_1}(x_2)\}, \max\{\lambda_{\Omega_2}(y_1), \lambda_{\Omega_2}(y_2)\}\}$

$= \max\{\max\{\lambda_{\Omega_1}(x_1), \lambda_{\Omega_2}(y_1)\}, \max\{\lambda_{\Omega_1}(x_2), \lambda_{\Omega_2}(y_2)\}\}$

$= \max\{(\lambda_{\Omega_1} \times \lambda_{\Omega_2})(x_1, y_1), (\lambda_{\Omega_1} \times \lambda_{\Omega_2})(x_2, y_2)\},$

Similarly, $(\lambda_{\Omega_1} \times \lambda_{\Omega_2})((x_1, y_1) - (x_2, y_2)) \leq \max\{(\lambda_{\Omega_1} \times \lambda_{\Omega_2})(x_1, y_1), (\lambda_{\Omega_1} \times \lambda_{\Omega_2})(x_2, y_2)\}$.

Hence $\Omega_1 \times \Omega_2$ is a cubic fuzzy SA -subalgebra of X×Y. ⌂

Theorem 6.3.

Let Ω_1 =<$\tilde{\mu}_{\Omega_1}$, λ_{Ω_1}> be a cubic fuzzy SA -subset of X and Ω_2 =<$\tilde{\mu}_{\Omega_2}$, λ_{Ω_2}> be a cubic fuzzy SA -subset of Y. If $\Omega_1 \times \Omega_2$is a cubic fuzzy SA -subalgebra of X×Y , then Ω_1 is a cubic fuzzy SA -subalgebra of X and Ω_2 is a cubic fuzzy SA -subalgebra of Y.

Proof:

Assume that $\Omega_1 \times \Omega_2$is a cubic fuzzy SA -subalgebra of X×Y, then

$((\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2}))((x_1, y_1)+(x_2, y_2)) \succcurlyeq \text{rmin}\{(\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})(x_1, y_1), (\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})(x_2, y_2)\}$ ………….(1)

Putting $x_1 = x_2 = 0$ in (1) we get,

$(\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})((0, y_1)+(0, y_2)) \succcurlyeq \text{rmin}\{(\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})(0, y_1), (\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})(0, y_2)\}$,

$(\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})(0, y_1 + y_2) \succcurlyeq \text{rmin}\{(\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})(0, y_1), (\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})(0, y_2)\}$, then we have

$\tilde{\mu}_{\Omega_2}(y_1 + y_2) \succcurlyeq \min\{\tilde{\mu}_{\Omega_2}(y_1), \tilde{\mu}_{\Omega_2}(y_2)\}$, and

$(\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})((0, y_1) - (0, y_2)) \succcurlyeq \text{rmin}\{(\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})(0, y_1), (\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})(0, y_2)\}$,

$(\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})(0, y_1 - y_2) \succcurlyeq \text{rmin}\{(\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})(0, y_1), (\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})(0, y_2)\}$, then we have

$\tilde{\mu}_{\Omega_2}(y_1 - y_2) \succcurlyeq \min\{\tilde{\mu}_{\Omega_2}(y_1), \tilde{\mu}_{\Omega_2}(y_2)\}$.

Hence $\tilde{\mu}_{\Omega_2}$ is a fuzzy SA -subalgebra of Y. Also,

$(\lambda_{\Omega_1} \times \lambda_{\Omega_2})((x_1, y_1)+(x_2, y_2)) \leq \max\{(\lambda_{\Omega_1} \times \lambda_{\Omega_2})(x_1, y_1), (\lambda_{\Omega_1} \times \lambda_{\Omega_2})(x_2, y_2)\}$ ………….(2)

Putting $x_1 = x_2 = 0$ in (2) we get,

$(\lambda_{\Omega_1} \times \lambda_{\Omega_2})((0, y_1)+(0, y_2)) \leq \max\{(\lambda_{\Omega_1} \times \lambda_{\Omega_2})(0, y_1), (\lambda_{\Omega_1} \times \lambda_{\Omega_2})(0, y_2)\}$,

$(\lambda_{\Omega_1} \times \lambda_{\Omega_2})(0, y_1 + y_2) \leq \max\{(\lambda_{\Omega_1} \times \lambda_{\Omega_2})(0, y_1), (\lambda_{\Omega_1} \times \lambda_{\Omega_2})(0, y_2)\}$, then we have

$\lambda_{\Omega_2}(y_1 + y_2) \leq \max\{\lambda_{\Omega_2}(y_1), \lambda_{\Omega_2}(y_2)\}$,

$(\lambda_{\Omega_1} \times \lambda_{\Omega_2})((0, y_1) - (0, y_2)) \leq \max\{(\lambda_{\Omega_1} \times \lambda_{\Omega_2})(0, y_1), (\lambda_{\Omega_1} \times \lambda_{\Omega_2})(0, y_2)\}$,

$(\lambda_{\Omega_1} \times \lambda_{\Omega_2})(0, y_1 - y_2) \leq \max\{(\lambda_{\Omega_1} \times \lambda_{\Omega_2})(0, y_1), (\lambda_{\Omega_1} \times \lambda_{\Omega_2})(0, y_2)\}$, then we have

$\lambda_{\Omega_2}(y_1 - y_2) \leq \max\{\lambda_{\Omega_2}(y_1), \lambda_{\Omega_2}(y_2)\}$. Then Hence λ_{Ω_2} is anti-fuzzy SA -subalgebra of Y.

Hence Ω_2 is a cubic fuzzy SA -subalgebra of Y.

SimilarLy, Ω_1 is a cubic fuzzy SA -subalgebra of X. ⌂

Theorem 6.4.

Let Ω_1 =<$\tilde{\mu}_{\Omega_1}$, λ_{Ω_1}> be a cubic fuzzy SA -subset of X and Ω_2 =<$\tilde{\mu}_{\Omega_2}$, λ_{Ω_2}> be a cubic fuzzy SA -subset of Y. If $\Omega_1 \times \Omega_2$is a cubic fuzzy SA -ideal of X×Y , then Ω_1 is a cubic fuzzy SA -ideal of X and Ω_2 is a cubic fuzzy SA -ideal of Y.

Proof:

For any x = (x_1, x_2) ∈ X×X, we have

$(\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})(0) = (\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})(0,0) = \text{rmin}(\tilde{\mu}_{\Omega_1}(0), \tilde{\mu}_{\Omega_2}(0)) \succcurlyeq \text{rmin}(\tilde{\mu}_{\Omega_1}(x_1), \tilde{\mu}_{\Omega_2}(x_2))$

$= (\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})(x_1, x_2) = (\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})(x)$.

$(\lambda_{\Omega_1} \times \lambda_{\Omega_2})(0) = (\lambda_{\Omega_1} \times \lambda_{\Omega_2})(0,0) = \max(\lambda_{\Omega_1}(0), \lambda_{\Omega_2}(0)) \leq \max(\lambda_{\Omega_1}(x_1), \lambda_{\Omega_2}(x_2))$

$= (\lambda_{\Omega_1} \times \lambda_{\Omega_2})(x_1, x_2) = (\lambda_{\Omega_1} \times \lambda_{\Omega_2})(x)$.

Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$

$(\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})(x + y = (\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})((x_1, x_2) + (y_1, y_2))$

$= (\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})((x_1 + y_1), (x_2 + y_2))$

$= \text{rmin}\{\tilde{\mu}_{\Omega_1}(x_1 + y_1), \tilde{\mu}_{\Omega_2}(x_2 + y_2)\}$

$\succcurlyeq \text{rmin}\{\text{rmin}\{\tilde{\mu}_{\Omega_1}(x_1 + z_1)\ \tilde{\mu}_{\Omega_1}(y_1 - z_1)\}, \text{rmin}\{\tilde{\mu}_{\Omega_2}(x_2 + z_2), \tilde{\mu}_{\Omega_2}(y_2 - z_2)\}\}$

$= \text{rmin}\{\text{rmin}\{\tilde{\mu}_{\Omega_1}(x_1 + z_1), \tilde{\mu}_{\Omega_2}(x_2 + z_2)\}, \text{rmin}\{\tilde{\mu}_{\Omega_1}(y_1 - z_1), \tilde{\mu}_{\Omega_2}(y_2 - z_2)\}\}$

$= \text{rmin}\{(\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})(x_1, x_2) + (z_1, z_2)), (\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})((y_1, y_2) - (z_1, z_2))\}$

$= \text{rmin}\{(\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})(x + z), (\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})(y - z)\}$,

$(\lambda_{\Omega_1} \times \lambda_{\Omega_2})(x + y) = (\lambda_{\Omega_1} \times \lambda_{\Omega_2})((x_1, x_2) + (y_1, y_2))$

$= (\lambda_{\Omega_1} \times \lambda_{\Omega_2})((x_1 + y_1), (x_2 + y_2))$

$= \max\{\lambda_{\Omega_1}(x_1 + y_1), \lambda_{\Omega_2}(x_2 + y_2)\}$

$\leq \max\{\max\{\lambda_{\Omega_1}(x_1 + z_1), \lambda_{\Omega_1}(y_1 - z_1)\}, \max\{\lambda_{\Omega_2}(x_2 + z_2), \lambda_{\Omega_2}(y_2 - z_2)\}\}$

$= \max\{\max\{\lambda_{\Omega_1}(x_1 + z_1), \lambda_{\Omega_2}(x_2 + z_2)\}, \max\{\lambda_{\Omega_1}(y_1 - z_1), \lambda_{\Omega_2}(y_2 - z_2)\}\}$

$= \max\{(\lambda_{\Omega_1} \times \lambda_{\Omega_2})(x_1, x_2) + (z_1, z_2)), (\lambda_{\Omega_1} \times \lambda_{\Omega_2})(y_1, y_2) - (z_1, z_2))\}$

$= \max\{(\lambda_{\Omega_1} \times \lambda_{\Omega_2})(x + z), (\lambda_{\Omega_1} \times \lambda_{\Omega_2})(y - z)\}$.

Hence $\Omega_1 \times \Omega_2$ is a cubic fuzzy SA -ideal of X×Y. ⌂

Theorem 6.5.

Let $\Omega_1 =<\tilde{\mu}_{\Omega_1}, \lambda_{\Omega_1}>$ be a cubic fuzzy SA -subset of X and $\Omega_2 =<\tilde{\mu}_{\Omega_2}, \lambda_{\Omega_2}>$ be a cubic fuzzy SA -subset of Y. If $\Omega_1 \times \Omega_2$ is a cubic fuzzy SA -ideal of X×Y, then Ω_1 is a cubic fuzzy SA-ideal of X and Ω_2 is a cubic fuzzy SA -ideal of Y.

Proof:

Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2) \in$ X×X

$(\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})((0,0),(0,0)) = \text{rmin}((\tilde{\mu}_{\Omega_1})(0,0), (\tilde{\mu}_{\Omega_2})(0,0))$

$\succcurlyeq \text{rmin}\{(\tilde{\mu}_{\Omega_1})(x_1, y_1), (\tilde{\mu}_{\Omega_2})(x_2, y_2)\}$

Putting $y = (y_1, y_2) = (0,0)$, we have $\tilde{\mu}_{\Omega_1}(0,0) \succcurlyeq \tilde{\mu}_{\Omega_1}(x_1, x_2)$,

$(\lambda_{\Omega_1} \times \lambda_{\Omega_2})((0,0),(0,0)) = \max\{(\lambda_{\Omega_1} \times \lambda_{\Omega_2})(0,0), (\lambda_{\Omega_1} \times \lambda_{\Omega_2})(0,0)\}$

$\leq \max\{(\lambda_{\Omega_1})(x_1, y_1), (\lambda_{\Omega_2})(x_2, y_2)\}$

Putting $y = (y_1, y_2) = (0,0)$, we have $\lambda_{\Omega_1}(0,0) \leq \lambda_{\Omega_2}(x_1, x_2)$.

Assume that $\Omega_1 \times \Omega_2$ is a cubic fuzzy SA-ideal of X×Y, then

$(\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})((x_1, x_2) + (y_1, y_2)) \succcurlyeq \text{rmin}\{(\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})((x_1, x_2) + (z_1, z_2)), (\tilde{\mu}_{\Omega_1} \times \tilde{\mu}_{\Omega_2})$

$((y_1, y_2) - (z_1, z_2))\}$………………….. (1)

Putting $x_2 = y_2 = z_2 = 0$, then we have

$(\tilde{\mu}_{\Omega_1})((x_1, 0) + (y_1, 0)) \succcurlyeq \text{rmin}\{(\tilde{\mu}_{\Omega_1})((x_1, 0) + (z_1, 0)), (\tilde{\mu}_{\Omega_1})((y_1, 0) - (z_1, 0))\}$, thus

$\tilde{\mu}_{\Omega_1}(x_1 + y_1) \geq \text{rmin}\{\tilde{\mu}_{\Omega_1}((x_1 + z_1), \tilde{\mu}_{\Omega_1}(y_1 - z_1)\}$. And

$(\lambda_{\Omega_1} \times \lambda_{\Omega_2})((x_1, x_2) + (y_1, y_2)) \leq \max\{(\lambda_{\Omega_1} \times \lambda_{\Omega_2})((x_1, x_2) + (z_1, z_2)), (\lambda_{\Omega_1} \times \lambda_{\Omega_2})$

$((y_1, y_2) - (z_1, z_2))\}$………………….. (2)

Putting $x_2 = y_2 = z_2 = 0$, then we have

$(\lambda_{\Omega_1})((x_1, 0) + (y_1, 0)) \leq \max\{(\lambda_{\Omega_1})((x_1, 0) + (z_1, 0)), (\lambda_{\Omega_1})((y_1, 0) - (z_1, 0))\}$, thus

$\lambda_{\Omega_1}(x_1 + y_1) \leq \max\{\lambda_{\Omega_1}((x_1 + z_1), \lambda_{\Omega_1}(y_1 - z_1)\}$.

Hence Ω_1 is a cubic fuzzy SA -ideal of X. Similarly, Ω_2 is a cubic fuzzy SA -ideal of Y.⌂

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