International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 8 Issue 8 August - 2024, Pages: 87-89

Hodge Theory For Elliptic Complexes

Azal J. Mera[1], Mohammed A. Akraa[2], Zahir.D.AL-Nafie[3],

University of Babylon, College of Education for pure Science, Iraq
[1]azal.mera@uobabylon.edu.iq , [2] mat.mohammed.akraa@uobabylon.edu.iq , [3]pure.zahir.dobeas@uobabylon.edu.iq

Abstract: *This paper discusses Hodge theory for elliptic complex of compact Manifold without boundary. The combination of algebraic geometry and differential geometry can be bridged by Hodge theory, which is a powerful tool in mathematics.*

Keywords: Hodge theory; Complex manifolds; Cohomology; Differential operators.

Introduction

Algebraic geometry and complex manifolds rely heavily on Hodge theory, which is a significant area in mathematics. The work of Georges de Rham on de Rham cohomology is the basis for W.V.D Hodge's development in the 1930s.

The study of cohomologies in a class of sections of a particular differential complex on a manifold constitutes the problem of global solvability in the broadest sense. Only the classical variation of this problem, involving elliptic complexes on compact manifolds, has been solved; the Hodge theory Wells provides the solution [4].
In this explication, we aim to state and demonstrate Hodge theory's central theorem. In the second section, we discuss harmonic spaces. In the third section, we discuss Hodge theory and its theorem, which leads to orthogonal decomposition.Hodge [2] proved the outcome of this study for the de Rham complex on X.

Harmonic spaces

Let E be an elliptic complex of (finite) length N on X, and suppose that X is an n-dimensional differentiable compact manifold.

We fix Hermitian metrics in the bundles $E^i$and appositive volume form $v \in \varepsilon(\Omega)$ on X. The scalar product $(,)_X = \int_X (,)_x\, v(x)$ then exists in each of the vector spaces $D(E^i)$ .As the completions of $D(E^i)$ in the norm $\|.\| = (,)_X^{1/2}$., define the Hilbert spaces $L^2(E^i)$.

Set $\mathcal{H}^i(E^\cdot) = \{f \in \mathcal{E}(E^i): Pf = P^*f = 0\}$, and impose this vector space with the topology induced from $\mathcal{E}(E^i)$. $f \in \mathcal{H}^i(E^\cdot)$ represent The sections are said Harmonic sections of E^i.

Preposition2.1 *The natural mappings i are continuous monomorphisms*

$$: \mathcal{H}^i(E^\cdot) \to H\left(\mathcal{E}(E^i)\right)$$

Proof: suppose that $f \in \mathcal{H}^i(E^\cdot)$ and $f = Pu$ for some section $u \in \mathcal{E}(E^{i-1})$. Then

$$(f,f)_X = (Pu,f)_X = (u,P^*f)_X = 0,$$

thus $f = 0$ follows . From the definitions of the induced and quotient topologies, The continuity of the monomorphism i follows

Hoge theory

As a matter of the fact i is a topological isomorphism, and the next assertion is to prove this fact.

Theorem 2.2 *Let $E^\cdot$ be an elliptic complex of finite length Non a compact manifold X.Then there are pseudo − differential operators $H: D'(E^\cdot) \to \mathcal{E}(E^\cdot)$of degree 0 and $\Pi: D'(E^\cdot) \to D'(E^\cdot)$ of degree (-1)such that:*

1) H_i is the L^2 − orthogonal projection on the finite − dimensional space $\mathcal{H}^i(E^\cdot)$ having the kernel

$$H_i(x,y) = \sum_j h_{ij}(x) \otimes\star\, h_{ij}(y) \in \mathcal{E}\left(E^i \boxtimes E^{i'}\right),$$

where$\{h_{ij}\}$is an orthonormal basis for $\mathcal{H}^i(E^\cdot)$, and $PH = Hp = 0$;

2) the operetor Π has a kernel $\Pi(x,y) \in \mathcal{E}\left(\left(E^i \boxtimes E^{i'}\right)\middle| X \times X\backslash\Delta\right)$ and for all $f \in D'(E^i)$

$$\Pi Pf + P\Pi f = f - Hf. \quad 2.1$$

Lemma 2.3 1) $For f \in \mathcal{E}(E^i), \Delta f = 0$ *if and only if* $f \in \mathcal{H}^i(E^\cdot)$,
2) $For\ f \in \mathcal{E}(E^i), \Delta^* f = 0$ *if and only if* $\tilde{P} f = P^* f = 0.$

Proof 1) the sufficiency is obvious, and we will prove the necessity. Assume that $f \in \mathcal{E}(E^i)$ and $\Delta f = 0$. Since $P_i P_{i-1} = 0$, we have

$$(\Delta f, P^* P f)_X = (QP^* Pf, P^* Pf)_X = 0.$$

hence it follows that $P^* P f = 0$ by condition (2.2),or $PF = 0$. Consequently $PQP^* f = 0$ and

$$(PQP^* f, f)_X = (QP^* f, P^* f)_X = 0.$$

Hence it again follows that$P^* f = 0$ by condition (2.2). Thus, $f \in \mathcal{H}^i(E^\cdot)$ as required.

The assertion 2) is verified in the same way.

Let us continue the proof of the theorem 2. 2 Consider for each $i$the differential operator $L_i = \Delta_i^* \Delta_i$ in $do_p(E^i \to E^i)$ which is obviously selfadjoint and elliptic.

If follows from part 1)of Lemma 2. 3 that $Lf = 0$ for a section $f \in \mathcal{E}(E^i)$ if and only if $f \in \mathcal{H}^i(E^\cdot)$.

Therefor, by using the fundamental Decomposition theorem for selfadjoint elliptic differential operators on compact manifolds ,we infer that there are selfadjoint linear mappings $G_i: \mathcal{E}(E^i) \to \mathcal{E}(E^i)$ which satisfy

$$G_i L_i = 1 - H_i \text{ on } \mathcal{E}(E^i). \quad 2.3$$

The operator H_i is defined in part 10 of theorem2.2. Notice that $GH = 0$ by construction.

Let now $\mathcal{G} \in \text{pseudo} - \text{differential operators} \mathit{of\ the\ typ}\ (E^i \to E^i)\ \mathit{for\ order}\ 4$ be a selfadjoint parametrix for L_i and selfadjointness L_i. We have

$$\mathcal{G}_i L_i = 1 - S_i \quad 2.4$$

for some $S_i \in \text{pdo}_{-\infty}(E^i \to E^i)$.

From(2. 3) and (2. 4)we get

$$(G_i - \mathcal{G}_i) L_i = -(H_i - S_i) \text{ on } \mathcal{E}(E^i).$$

Applying the operator $\mathcal{G}_i$ to the equality from the right, we get

$$G_i - \mathcal{G}_i = (G_i - \mathcal{G}_i) S_i^* - (H_i - S_i) \mathcal{G}_i \text{ on } \mathcal{E}(E^i). \quad 2.5$$

Hence, $G_i - \mathcal{G}_i \in \mathcal{L}_b(\mathcal{E}(E^i) \to \mathcal{E}(E^i))$ has a smooth kernel. Thus $G_i \in \text{pdo}_{-4p}(E^i \to E^i)$ has the same symbol as $\mathcal{G}_i$.

Set $\Pi_i = G_{i-1} \Delta_{i-1}^* \breve{P}_{i-1}^*$, so that $\Pi_i \in \text{pdo}_{-p_i - 1}(E^i \to E^{i-1})(i \in Z)$.

By equality (2. 3) we have for each section $f \in \mathcal{E}(E^i)$

$$G_i L_i f = G_i \Delta_i^* \big(\widetilde{P_i^*} P_i + P_{i-1} \widetilde{P_{i-1}^*}\big) f = \Pi_{i+1} P_i f + P_{i-1} \Pi_i f + \big((G_i \Delta_i^*) P_{i-1} - P_{i-1}(G_{i-1} \Delta_{i-1}^*)\big) \widetilde{P_{i-1}^*} f$$
$$= f - H_i f.$$

Set $T_i = (G_{i+1} \Delta_{i+1}^*) P_i - P_i (G_i \Delta_i^*)$ here,so that T is a pseudo-differential operator of degree 1 with respect to the grading $E^\cdot$. Then the proof of theorem 2.2 will be complete, if we show that $T_{i-1} \widetilde{P_{i-1}^*}: \mathcal{E}(E^i) \to \mathcal{E}(E^i)$ is the zero operator>

According to (2. 3) we get for $f \in \mathcal{E}(E^{i-1})$

$$G_{i-1} \Delta_{i-1}^* \Delta_{i-1} f = f - H_{i-1} f,$$
$$G_i \Delta_i^* \Delta_i P_{i-1} f = P_{i-1} f - H_i P_{i-1} f.$$

Applying the differential operator P_{i-1} to the first of these equalities, comparing the result with the second equality, and using the commutativity relationships $P_i\Delta_i = \Delta_{i+1}P_i$,we see that $T_{i-1}\Delta_{i-1}: \mathcal{E}(E^{i-1}) \to \mathcal{E}(E^i)$ is the zero operator.

But then $\Delta_{i-1}^* T_{i-1}^*: \mathcal{E}(E^i) \to \mathcal{E}(E^{i-1})$ is the zero operator too. By part 2) of Lemma 2.3 we get that $\widetilde{P_{i-1}} T_{i-1}^* = 0$, or $T_{i-1}\widetilde{P_{i-1}^*} = 0$ as required.

Thus we have proved formula (2.1) for section $f \in \mathcal{E}(E^i)$. But since the kernel Π is continuable, then by the continuity formula, (2.1) is also valid for sections $f \in D(E^i)$. Thus the theorem is proved.

Corollary *if $E^\cdot$ is an elliptic complex on a compact manifold X, then*

$$H^\cdot\big(\mathcal{E}(E^\cdot)\big) \cong \mathcal{H}^\cdot(E^\cdot)$$

In particular the cohomology spaces $H^\cdot(\mathcal{E}(E^\cdot))$ are finite – dimensional.

Proof. It follows immediately from formula (2.1) that the natural mapping $i: \mathcal{H}^\cdot(E^\cdot) \to H^\cdot(\mathcal{E}(E^\cdot))$ is an epimorphism. Hence by Proposition 2.1 it is sufficient to prove only the continuity of inverse mapping i^{-1}. But (2.1) implies that i^{-1} may be de composed as

$$i^{-1}: H^\cdot\big(\mathcal{E}(E^\cdot)\big) \xrightarrow{\varphi^{-1}} Z\big(\mathcal{E}(E^\cdot)\big) \xrightarrow{H} \mathcal{H}^\cdot(E^\cdot)$$

Where φ is the quotient mapping. Hence i^{-1} is continuous because φ is open and H is continuous.

corollary *If $E^\cdot$ is a compatibility complex for a sufficiently regular differential operator P_0 with injective symbol on X, then the Cohomology spaces $H^\cdot\big(\mathcal{E}(E^\cdot)\big)$ $(i \in Z)$ are finite − dimensional.*

Orthogonal decomposition

If the order of all the differential operators P_i in the complex $E^\cdot$ are the same, the theorem 2. 2 can be made more exact. The following statement.

corollary (**Hodge theorem**) *let $E^\cdot$ be an elliptic of differential operators of the same order p on a compact manifold X . Then there are selfadjoint pseudo − diferential operator $H: \mathcal{E}(E^\cdot) \to \mathcal{E}(E^\cdot)$ and $G: \mathcal{E}(E^\cdot) \to \mathcal{E}(E^\cdot)$ of degree 0 such that:*

1) *H is the L^2 − orthogonal projection onto the finite − dimentsional space $\mathcal{H}^\cdot(E^\cdot)$ and it has the kernel*

$$H(x,y) = \sum_{ij} h_{ij}(x) \otimes \star\, h_{ij}(y) \in \mathcal{E}((E^\cdot \boxtimes E^{\cdot\prime})^0),$$

where $\{h_{ij}\}$ is an orthonormal basis for $\mathcal{H}^\cdot(E^\cdot)$, and $PH = HP = 0$;

2) *the Green operator G has the kernal $G(x,y) \in \mathcal{E}((E^\cdot \boxtimes E^{\cdot\prime})^0|_{X\times X\setminus\triangle})$*

This yields the L^2 − orthogonal decomposition

$$f = Hf + PPGf + PPGf \text{ for all } f \in \mathcal{E}(E^{\,i}).$$

Proof. It is sufficient to take $Q_i = 1, L_i = \Delta_i$, and repeat the proof of theorem 2.2 with slight modification.

References:

[1] Georges de Rham. Vari_et_es di__erentiables. Formes, courants, formes harmoniques. Actualites Sci. Indust, no. 1222. Hermann, Paris, 1955.
[2] W. V. D. Hodge. The theory and application of harmonic integrals. Cambridge Univ. Press, New York, 1941
[3] K. Kodaira. On a di_erential-geometric method in the theory of analytic stacks. Proc. Nat. Acad. Sci., 39: 1268{1273, 1953.
[4] R.Wells. Di_erential analysis on complex manifolds. Prentice-Hall, Englewood Cli_s, N.J., 1973.
[5] Tarkhanov, N., Complexes of Di_erential Operators, Kluwer Academic Publishers, Dordrecht, NL, 1995.