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Laurent Expansion for Differential Complexes

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Abstract*: The powerful tool is Laurent series for the study of local behavior of functions around singularities. In the context of differential complexes, Laurent expansions provide a method to represent solutions of differential equations with constant coefficients as series.Therefor, this paper shade a light explores the foundational concepts of Laurent expansions in the study of differential complexes*

Keywords: Differential Operators, elliptic equation , Differential Complexes, cohomologies

Introduction

The 1950s saw a particularly intense study of the local properties of solutions to general elliptic differential equations, particularly for those with Hoelder continuous coefficients. These properties, zeros of finite or infinite order and unique continuation of the solutions, included smoothness of the solution, point singularities and removable singularities, expansions of the solutions in asymptotic or convergent series, and branch points of the solutions. Bers's survey provides a handy bibliography for these inquiries [1]. The usage of a parametrix with estimates for the coefficients of the major component of the differential equation at a unique point, or "freezing," was the fundamental tool in this case. The more knowledge on the local behavior of solutions to homogeneous differential equations with constant coefficients, the more successful advancement will be. You can think of elliptic operators (or, more broadly, differential operators of constant strength) as perturbations of operators whose coefficients are constants (see H¨ormander [3, Ch. 13])

The Theorem on Exponential Representation (cf. Palamodov [4, Ch.II]) provides a wealth of local information for solutions of a broad system of differential equations with constant coefficients. The Fourier transformation serves as the basis for both of these approaches, which are closely related to one another: the parametrix method and the exponential representation method. Two benefits of the parametrix technique are mentioned: Two main benefits are: 1) a broader range of applications (differential operators with variable coefficients) and 2) constructiveness.

Harvey and Polking [2] proposed the original idea of Laurent series for solutions of homogeneous elliptic differential equations with constant coefficients in Rn. They recommended selecting the coefficients so that the series' terms coincide with their Cauchy main values (i.e., the coefficients are uniquely defined).

Cohomology of Hilbert complexes

For a complex of a differential operator whose coefficients are constants, the cohomology spaces $\mathcal{H}^i\left(\mathcal{L}_0^1(\mathcal{E}^\cdot)\right)$ can be expressed in terms of algebra. To be more precise, let $\mathcal{P}$ represent the ring of polynomials of n complex variables (with complex coefficients)

$\mathcal{Z} = (\mathcal{Z}_1, \dots, \mathcal{Z}_n)\epsilon\mathbb{C}^n$. To each differential operator $\mathcal{P}_i = \mathcal{P}_i(\mathcal{D})$ there corresponds a $(\mathcal{K}_{i+1} \times \mathcal{K}_i)$-matrix of polynomials $\mathcal{P}_i(\mathcal{Z})$, which is obtained from $\mathcal{P}_i(\mathcal{D})$by the change of variables$\mathcal{D}\ to\ \mathcal{Z}$. Here, $\mathcal{K}_i$denotes the rank of $\mathcal{E}_i$.The matrix $\mathcal{P}_i(\mathcal{Z})$, is called the full symbol of the differential operator $\mathcal{P}_i$.

Then, the complex $\mathcal{E}^\cdot$ gives rise to the complex of free $\mathcal{P}$-modules and their $\mathcal{P}$-mappings

$$\mathcal{P}^{\mathcal{K}}: 0 \longrightarrow \mathcal{P}^{\mathcal{K}_0} \xrightarrow{\mathcal{P}_0(\mathcal{Z})} \mathcal{P}^{\mathcal{K}_1} \xrightarrow{\mathcal{P}_1(\mathcal{Z})} \dots\dots \xrightarrow{\mathcal{P}_{N-1}(\mathcal{Z})} \mathcal{P}^{\mathcal{K}_N} \longrightarrow 0 \qquad (2.1)$$

Laurent series for elliptic complexes

Suppose $\mathcal{E}^\cdot = \{\mathcal{E}^i, \mathcal{P}_i\}$ is an elliptic complex of finite length $\mathcal{N}$ of homogeneous differential operators with constant coefficients .For $j = 0, 1, \dots$, let $\mathcal{P}_i$ denote the vector subspace of $\mathcal{P}$ consisting of all homogeneous polynomials of degree j. Thus, $\mathcal{P}_i = \oplus_j \mathcal{P}_i$. Since each differential operator $\mathcal{P}_i$ is homogeneous, the mappings $\mathcal{P}_i(\mathcal{Z})$in the sequence (2.1) preserve the grading $\mathcal{P}_{\mathcal{K}i} = \oplus_j \mathcal{P}_{\mathcal{K}i}$ Consequently, the cohomology spaces of the complex (2.1) decompose into direct sums

$$\mathcal{H}^{i+1}(\mathcal{P}^{\mathcal{K}i}) = \oplus_j \frac{\mathcal{Z}^{i+1}(\mathcal{P}^{\mathcal{K}}) \cap \mathcal{P}_j^{\mathcal{K}_i}}{\mathcal{B}^{i+1}(\mathcal{P}^{\mathcal{K}i}) \cap \mathcal{P}_j^{\mathcal{K}_i}}$$

We also use similar method for the subspace's algebraic complement $\mathcal{Z}^{i+1}(\mathcal{P}^{\mathcal{K}})$ in $\mathcal{B}^{i+1}(\mathcal{P}^{\mathcal{K}i})$
It follows that the differential operators Pi's principal and full symbols overlap due to their homogeneity., i.e. $\sigma(\mathcal{P}_i)(\mathcal{L}) = \mathcal{P}_i(\mathcal{L})$.
Now, the ellipticity of the complex $\mathcal{E}$ means that the sequence of symbols

$$\mathcal{C}^{\mathcal{K}}: 0 \longrightarrow \mathcal{C}^{\mathcal{K}_0} \xrightarrow{\mathcal{P}_0(\mathcal{L})} \mathcal{C}^{\mathcal{K}_1} \xrightarrow{\mathcal{P}_1(\mathcal{L})} \ldots \ldots \xrightarrow{\mathcal{P}_{N-1}(\mathcal{L})} \mathcal{C}^{\mathcal{K}_N} \longrightarrow 0 \qquad 3.1$$

is exact for each real cotangent vector $\mathcal{L}$, lying on the unit sphere $S_{n-1} = \{\mathcal{L} \in R_n : |\mathcal{L}| = 1\}$.
However, a basic principle derived from linear algebra indicates that the complex (3.1) at the term $\mathcal{C}^{\mathcal{K}_i}$ is comparable to the matrix's non-singularity

$$\Delta_i(\mathcal{L}) = (\widetilde{\mathcal{P}}_i(\mathcal{L}))^*\mathcal{P}_i(\mathcal{L}) + \mathcal{P}_{i-1}(\mathcal{L})(\widetilde{\mathcal{P}_{i-1}}(\mathcal{L}))^*$$

Where $\widetilde{\mathcal{P}}_i(\mathcal{L}) = |\mathcal{L}|^{2(\mathcal{P}-\mathcal{P}_i)}$ and $\mathcal{P}$ is the largest of the numbers $\mathcal{P}_i$.
Lemma 3.1[6] For each $j = 0, 1, \ldots$, the sesquilinear form

$$(\mathcal{P}_1, \mathcal{P}_2) = \int_{s^{n-1}} ((\Delta_i(\mathcal{L}))^{-1} (\widetilde{\mathcal{P}}_i(\mathcal{L}))^*\mathcal{P}_1(\mathcal{L}), (\Delta_i(\mathcal{L}))^{-1} (\widetilde{\mathcal{P}}_i(\mathcal{L}))^*\mathcal{P}_2(\mathcal{L}))ds(\mathcal{L}) \qquad 3.2$$

yields a scalar product on $\mathcal{Z}^{i+1}(\mathcal{P}^{\mathcal{K}}) \cap \mathcal{P}_j^{\mathcal{K}_{i+1}}$

proof For the proof we have only to verify the assertion that $(\mathcal{P}, \mathcal{P}) = 0$, for $\mathcal{P} \in \mathcal{Z}^{i+1}(\mathcal{P}^{\mathcal{K}}) \cap \mathcal{P}_j^{\mathcal{K}_{i+1}}$ implies $\mathcal{P} = 0$.
Indeed, if $(\mathcal{P}, \mathcal{P}) = 0$, then $(\Delta_i(\mathcal{L}))^{-1} (\widetilde{\mathcal{P}}_i(\mathcal{L}))^*\mathcal{P}(\mathcal{L}) = 0$ for each $\mathcal{L} \in s_{n-1}$. Since also $\mathcal{P}_{i+1}(\mathcal{L})\mathcal{P}(\mathcal{L}) = 0$, we have

$$\begin{aligned}\mathcal{P}(\mathcal{L}) &= (\Delta_i(\mathcal{L}))^{-1}\Delta_{i+1}(\mathcal{L})\mathcal{P}(\mathcal{L}) \\ &= ((\Delta_{i+1}(\mathcal{L}))^{-1} (\widetilde{\mathcal{P}_{i+1}}(\mathcal{L}))^*\mathcal{P}_{i+1}(\mathcal{L}) + \mathcal{P}_i(\mathcal{L})(\Delta_i(\mathcal{L}))^{-1} (\widetilde{\mathcal{P}}_i(\mathcal{L}))^*)\mathcal{P}(\mathcal{L}) \\ &= 0,\end{aligned}$$

for each $\mathcal{L}$ on the sphere s^{n-1}.
However, in view of the homogeneity, Thus, $\mathcal{P} = 0$,as required. ■
Let $\mathcal{H}_j^{i+1}(\mathcal{P}^{\mathcal{K}})$ stand for the orthogonal complement of the vector subspace $\mathcal{B}^{i+1}(\mathcal{P}^{\mathcal{K}}) \cap \mathcal{P}_j^{\mathcal{K}_{i+1}}$ in $\mathcal{Z}^{i+1}(\mathcal{P}^{\mathcal{K}}) \cap \mathcal{P}_j^{\mathcal{K}_{i+1}}$ with respect to the scalar product 3.2 Since, for $j < \mathcal{P}_i$, it is clear that $\mathcal{B}^{i+1}(\mathcal{P}^{\mathcal{K}}) \cap \mathcal{P}_j^{\mathcal{K}_{i+1}} = \{0\}$,we have

$$\mathcal{H}_j^{i+1}(\mathcal{P}^{\mathcal{K}}) = \mathcal{Z}^{i+1}(\mathcal{P}^{\mathcal{K}}) \cap \mathcal{P}_j^{\mathcal{K}_{i+1}} \quad \text{for such j.}$$

Lemma 3.2[6] For $j \geq \mathcal{P}_i$, the space $\mathcal{H}_j^{i+1}(\mathcal{P}^{\mathcal{K}})$ consists in those and only those
$\mathcal{Z}^{i+1}(\mathcal{P}^{\mathcal{K}}) \cap \mathcal{P}_j^{\mathcal{K}_{i+1}}$ for which

$$= \int_{s^{n-1}} ((\Delta_i(\mathcal{L}))^{-1} (\widetilde{\mathcal{P}}_i(\mathcal{L}))^*\hbar(\mathcal{L}), \mathcal{P}(\mathcal{L}))ds(\mathcal{L}) = 0 \qquad 3.3$$

Proof By definition, $\mathcal{H}_j^{i+1}(\mathcal{P}^{\mathcal{K}})$ consists of those $\hbar \in \mathcal{Z}^{i+1}(\mathcal{P}^{\mathcal{K}}) \cap \mathcal{P}_j^{\mathcal{K}_{i+1}}$ for which $(\hbar, \mathcal{P}_2) = 0$ whenever $\mathcal{P}_2 \in \mathcal{B}^{i+1}(\mathcal{P}^{\mathcal{K}}) \cap \mathcal{P}_j^{\mathcal{K}_{i+1}}$.
Let us consider the sesquilinear form (3.2) for such $\mathcal{P}_2$, i.e. for $\mathcal{P}_2(\mathcal{L}) = \mathcal{P}_i(\mathcal{L})\mathcal{P}(\mathcal{L})$
, where $\wp \in \mathcal{P}_{j-p_i}^{\mathcal{K}_i}$, We have

$$\begin{aligned}(\hbar, \mathcal{P}_2) &= \int_{s^{n-1}} ((\Delta_i(\mathcal{L}))^{-1} (\widetilde{\mathcal{P}}_i(\mathcal{L}))^*\hbar(\mathcal{L}), (\Delta_i(\mathcal{L}))^{-1} (\widetilde{\mathcal{P}}_i(\mathcal{L}))^*\wp(\mathcal{L}))ds(\mathcal{L}) \\ &= \int_{s^{n-1}} ((\Delta_i(\mathcal{L}))^{-1} (\widetilde{\mathcal{P}}_i(\mathcal{L}))^*\hbar(\mathcal{L}), \wp(\mathcal{L}))ds(\mathcal{L}) \\ &- \int_{s^{n-1}} ((\Delta_i(\mathcal{L}))^{-1} (\widetilde{\mathcal{P}}_i(\mathcal{L}))^*\hbar(\mathcal{L}), (\Delta_i(\mathcal{L}))^{-1}\mathcal{P}_{i-1}(\mathcal{L}) (\widetilde{\mathcal{P}_{i-1}}(\mathcal{L}))^*\wp(\mathcal{L}))ds(\mathcal{L})\end{aligned}$$

Since $\mathcal{P}\Delta = \Delta\mathcal{P}$ and $\mathcal{P}^*\widetilde{\mathcal{P}^*} = 0$, the integral in the last term on the right side is identically equal to zero. From this the assertion of the lemma follows. ■

Now, if we set $\mathcal{H}_j^{i+1}(\mathcal{P}^{\mathcal{K}}) = \bigoplus j\, \mathcal{H}_j^{i+1}(\mathcal{P}^{\mathcal{K}})$, then we have the following:
Lemma 3.3[6] For any i = 0, 1, . . . ,N − 1, $\mathcal{B}^{i+1}(\mathcal{P}^{\mathcal{K}}) = \mathcal{H}_j^{i+1}(\mathcal{P}^{\mathcal{K}}) \oplus j\mathcal{Z}^{i+1}(\mathcal{P}^{\mathcal{K}})$.
Proof. Indeed, by the Rellich Theorem, for each j = 0, 1, . . ., the space $\mathcal{H}_j^{i+1}(\mathcal{P}^{\mathcal{K}})$ is the algebraic complement of $\mathcal{B}^{i+1}(\mathcal{P}^{\mathcal{K}}) \cap \mathcal{P}_j^{\mathcal{K}_{i+1}}$ in $\mathcal{Z}^{i+1}(\mathcal{P}^{\mathcal{K}}) \cap \mathcal{P}_j^{\mathcal{K}_{i+1}}$ Thus, in view of the homogeneity of $\mathcal{P}_i(\mathcal{L})$

$$\begin{aligned}\mathcal{Z}^{i+1}(\mathcal{P}^{\mathcal{K}}) &= \bigoplus j\mathcal{Z}^{i+1}(\mathcal{P}^{\mathcal{K}}) \cap \mathcal{P}_j^{\mathcal{K}_{i+1}} \\ &= \bigoplus j(\mathcal{H}_j^{i+1}(\mathcal{P}^{\mathcal{K}}) \oplus \mathcal{B}^{i+1}(\mathcal{P}^{\mathcal{K}}) \cap \mathcal{P}_j^{\mathcal{K}_{i+1}}\end{aligned}$$

$$= (\bigoplus j\, \mathcal{H}_j^{i+1}(\mathcal{P}^{\mathcal{K}}))\ \oplus (\mathcal{B}^{i+1}(\mathcal{P}^{\mathcal{K}}) \cap \mathcal{P}_j^{\mathcal{K}_{i+1}})$$
$$= \mathcal{H}^{i+1}(\mathcal{P}^{\mathcal{K}})\ \oplus \mathcal{B}^{i+1}(\mathcal{P}^{\mathcal{K}}),$$

as required. ■

Before formulating the definitive result, we make some remarks regarding the fundamental solution for $\mathcal{E}^{\cdot}$. As we have said, $\Delta_i = \Delta_i(\mathcal{D})$are elliptic differential operators of order $2p$ with constant coefficients. Denote by $e(x)$ a fundamental solution of convolution type for the (scalar) differential operator $\prod_{i=0}^{N} \Delta_i(\mathcal{D})$ and set

$$\Phi_i(x) = \widetilde{\mathcal{P}_{i-1}^{*}}(\mathcal{D})\big(adj\Delta_i(\mathcal{D})\big)\left(\prod_{\substack{0\le j\le N\\ i\ne j}} \det \Delta_j(\mathcal{D})\right) e(x) \qquad 3.3$$

for $i = 1, \dots, N$ and $\Phi_i(x) = 0$ for the remaining i.

In the following lemma we make more explicit the construction of a fundamental solution of convolution type for a non-degenerate complex of differential operators with constant coefficients (cf. Tarkhanov [7]).

Lemma 3.4 As defined by (3.3), $\Phi\ \ (x) = (\Phi_i(x))$is a fundamental solution of convolution type for the complex $\mathcal{E}^{\cdot}$.

Proof. Indeed, from the equality $\mathcal{P}_i\Delta_i = \mathcal{P}_{i+1}\Delta_i$ we obtain

$$\mathcal{P}_{i+1}\Delta_i\mathcal{P}_i(det\Delta_i) = (det\Delta_{i+1})\, \mathcal{P}_i(adj\Delta_i)$$

Whence

$$\mathcal{P}_{i-1}(\mathcal{D})\Phi_i(x) + \Phi_{i+1}(x)(x)\mathcal{P}_i(\mathcal{D}) =$$
$$\left(\mathcal{P}_{i-1}\widetilde{\mathcal{P}_{i-1}^{*}}(\mathcal{D})\ adj\Delta_i(\mathcal{D})\left(\prod_{\substack{0\le j\le N\\ i\ne j}} \det \Delta_j\right) + \widetilde{\mathcal{P}_{i-1}^{*}}(\mathcal{D})\ adj\Delta_i\mathcal{P}_i\left(\prod_{\substack{0\le j\le N\\ i\ne j}} \det \Delta_j\right)\right) e(x)$$
$$= \Delta_i adj\Delta_i\left(\prod_{\substack{0\le j\le N\\ i\ne j}} \det \Delta_j\right) e(x)$$
$$= \delta(x) I_{\mathcal{K}_i}$$
■

Theorem 3.5 (cf. [5]) Let $\mathcal{E}^{\cdot}$ be an elliptic complex of homogeneous differential operators with constant coefficients on X. Then, for each section $u \in \mathcal{Z}^i(\mathcal{D}^i\left(\mathcal{E}^i\big|_{X\setminus 0}\right)) \cap \mathcal{D}'(\mathcal{E}^i)$, there is a unique class $[\mathcal{U}_e] \in \frac{\mathcal{Z}^i(\mathcal{D}^i(\mathcal{E}^{\cdot})}{\mathcal{P}(\mathcal{Z}^{i-1}(\mathcal{D}'(\mathcal{E}^{\cdot}|_{X\setminus 0}))\cap\mathcal{D}'(\mathcal{E}^{i-1})}$ and column vectors $\hbar_j \in \mathcal{H}^{>+\infty}_{|}(\mathcal{P}\|), j = 0,1, \dots . J(\mathcal{U})$ such that

$$\mathcal{U} = \mathcal{U}_e + \sum_j \Phi * \big(\hbar_j(\mathcal{D})\delta\big) \qquad on \qquad X\setminus 0. \qquad 3.4$$

In conclusion, we emphasize that in (3.4) the sum is finite. If for a section $u \in \mathcal{Z}^i(\mathcal{D}^i\left(\mathcal{E}^i\big|_{X\setminus 0}\right))$, we assume only that it extends to the whole neighborhood in the class of hyperfuntions, then, as before, we are guaranteed the existence of an expansion (3.4), but the sum on the right-hand side of (3.4) may be infinite [5].

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