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The translation bifuzzy ψ-subalgebra of ψ-algebra

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Abstract*: we discuss the translation ψ-algebra and we get some relations, theorems, propositions and give examples of translation bifuzzy ψ-subalgebra. We show the notion of translation bifuzzy ψ-subalgebra of an ψ-algebra and investigate some of their properties.It follows, let $(X;+,-,0)$ denote an ψ-algebra, and for any fuzzy subset μ of X, we denote $T = 1 - sup\{\mu(x) \mid x \in X\}$ and $K = inf\{v(x) \mid x \in X\}$.*

Keywords: ψ-algebra, bifuzzy ψ-subalgebra, ψ-subalgebra, fuzzy ψ-subalgebra, translation bifuzzy ψ-subalgebra.

1. Introduction

In 1965, L.A. Zadeh introduced the notion of fuzzy subset, [12]. In 1980, Dubois and Prade [5] they studied fuzzy sets and systems. In 1983, Atanassov [1] defined the intuitionistic fuzzy sets and studied the intuitionistic fuzzy set theory. In 1994, Atanassov [2], introduced new operations defned over the intuitionistic fuzzy sets. In 1995, Gerstenkorn and Manko [6] renamed the intuitionistic fuzzy set as bifuzzy sets. Bifuzzy sets take the advantage of fuzzy sets to handle information with various facts of uncertainty such as fuzziness and randomness. The bifuzzy set has become a formal and useful tool for computer science to deal with bifuzzy information and uncertain information. In 2015, A.T. Hameed[7] introduced the idea of SA-algebras. In 2023 some concepts related to it such as SA-subalgebra, SA-ideal, fuzzy SA-subalgebra and fuzzy SA-ideal of SA-algebra. In 2023, N.H. Jaber and A.T. Hameed [8] introduced the notion of ψ-algebra, In 2023 N.H. Jaber and A.T. Hameed [9] introduced the notion of ψ-subalgebra and they introduced the concept of homomorphisms on ψ -algebra and fuzzy homomorphisms on ψ-algebra.

2. Preliminaries

In this section, we give some basic definitions and preliminaries proprieties of ψ-subalgebras and fuzzy ψ-subalgebras of ψ-algebra such that we include some elementary aspects that are necessary for this paper.

Definition 2.1.[8]. Let $(X;+,-,0)$ be an algebra with two operations $(+)$ and $(-)$ and constant (0). X is called an ***ψ*-algebra** if it satisfies the following properties: for all $x, y, z \in X$,

(ψ_1) $x - x = 0$,

(ψ_2) $(0 - x) + x = 0$,

(ψ_3) $(x - y) - z = x - (z + y)$,

(ψ_4) $(y + x) - (x - z) = y + z$.

In , we can define a binary relation ($\leq$) by : $x \leq y$ if and only if $x + y = 0$ and $x - y = 0$, $x, y \in X$.

Definition 2.2. [9].

Let $(X;+,-,0)$ be a ψ-algebra and let S be a nonempty set of X. S is called **a *ψ*-subalgebra of *X*** if $x + y \in S$ and $x - y \in S$, whenever $x, y \in S$.

Definition 2.3.[12].

Let X be a nonempty set, a fuzzy subset μ of X is a mapping $\mu: X \to [0,1]$.

Definition 2.4.[11].

For any $t \in [0,1]$ and a fuzzy subset μ in a nonempty set X, the set

$U(\mu, t) = \{x \in X \mid \mu(x) \geq t\}$ is called **an upper t-level cut of μ,** and the set $L(\mu, t) = \{x \in X \mid \mu(x) \leq t\}$ is called **a lower t-level cut of μ**.

Definition 2.5.[10].

Let $(X; +, -, 0)$ be a ψ-algebra, a fuzzy subset μ of X is called **a fuzzy ψ-subalgebra of X** if for all $x, y \in X$,

1- $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$ and
2- $\mu(x - y) \geq min\{\mu(x), \mu(y)\}$.

Proposition 2.6.[6].

1- Let μ be a fuzzy subset of ψ-algebra $(X; +, -, 0)$. If μ is a fuzzy ψ-subalgebra of X, for any t ∈ [0,1], μ_t is a ψ-subalgebra of X .

2- Let μ be a fuzzy subset of ψ-algebra $(X; +, -, 0)$. If for all t ∈ [0,1], μ_t is a ψ-subalgebra of X , then μ is a fuzzy ψ-subalgebra of X .

3- Let μ be a fuzzy ideal of ψ-algebra $(X; +, -, 0)$. If μ is a fuzzy ψ-ideal of X, then for any t ∈ [0,1], μ_t is an ψ-ideal of X .

4- Let μ be a fuzzy ideal of ψ-algebra $(X; +, -, 0)$. If for all t ∈ [0,1], μ_t is an ψ-ideal of X , then μ is a fuzzy ψ-ideal of X .

Now, we will recall the concept of anti-fuzzy subsets.

Definition 2.7. [5].

Let $(X; +, -, 0)$ be an ψ-algebra, a fuzzy subset μ of X is called **an anti-fuzzy ψ-subalgebra of X** if for all $x, y \in X$,

AFψS$_1$) $\mu(x + y) \leq max\{\mu(x), \mu(y)\}$,

AFψS$_2$) $\mu(x - y) \leq max\{\mu(x), \mu(y)\}$.

Proposition 2.8. [3].

Let μ be an anti-fuzzy subset of an ψ-algebra $(X; +, -, 0)$.

1- If μ is an anti-fuzzy ψ-subalgebra of , then it satisfies for any t∈[0, 1], $L(\mu, t) \neq \emptyset$ implies $L(\mu, t)$ is a ψ-subalgebra of X.

2- If $L(\mu, t)$ is a ψ-subalgebra of X, for all t∈[0, 1], $L(\mu, t) \neq \emptyset$, then μ is an anti-fuzzy ψ-subalgebra of X .

Definition 2.9. [4].

Let $f: (X; +, -, 0) \to (Y; +', -', 0')$be a mapping nonempty ψ-algebras X and Y respectively. If μ is anti-fuzzy subset of X, then the anti-fuzzy subset β of Y defined by:

$$f(\mu)(y) = \begin{cases} inf\{\mu(x): x \in f^{-1}(y)\} & if\ f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ 1 & otherwise \end{cases}$$

is said to be the image of μ under f.

Similarly if β is anti-fuzzy subset of , then the fuzzy subset $\mu = (\beta \circ f)$ of X (i.e the anti-fuzzy subset defined by $\mu(x) = \beta(f(x))$, for all

$x \in X$) is called the pre-image of β under f .

Now, we will recall the concept of bifuzzy subsets.

Definition 2.10. [2].

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be a bifuzzy subset of a ψ-algebra X. A is said to be **an bifuzzy ψ -subalgebra of X** if : for all x, y $\in$X,

(IFS1) $\mu_A(x+y) \geq min\{\mu_A(x), \mu_A(y)\}$ and

$\mu_A(x-y) \geq min\{\mu_A(x), \mu_A(y)\}$.

(IFS2) $\nu_A(x+y) \leq max\{\nu_A(x), \nu_A(y)\}$ and

$\nu_A(x-y) \leq max\{\nu_A(x), \nu_A(y)\}$.

i.e., μ_A is fuzzy ψ-subalgebra of ψ-algebra and ν_A is anti-fuzzy ψ-subalgebra of ψ-algebra.

3.The translations bifuzzy ψ-subalgebra of ψ-algebra

In this section, we discuss translation of ψ-algebra and we get some of relations, theorems, propositions and give examples of α-translation of bifuzzy ψ-subalgebra. We show the notion of translation bifuzzy ψ-subalgebra of an ψ-algebra and investigate some of their properties.

It follows, let $(X; +, -, 0)$ denote an ψ-algebra, and for any fuzzy subset μ of X, we denote $T = 1 - \sup\{\mu(x) \mid x \in X\}$ and $K = \inf\{v(x) \mid x \in X\}$.

Definition 3.1.

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be a bifuzzy subset of an ψ-algebra $(X; +, -, 0)$, μ_A be a fuzzy subset of X such that $\alpha \in [0, T]$ and ν_A be a fuzzy subset of X such that $\varepsilon \in [0, K]$. A mapping $(\mu_A)^{\mathrm{T}}_{\alpha} : X \to [0,1]$ and $(\nu_A)^{\mathrm{K}}_{\varepsilon} : X \to [0,1]$,

$A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^{\mathrm{T}}_{\alpha}(x), (\nu_A)^{\mathrm{K}}_{\varepsilon}(x)) \mid x \in X\}$ is called **a translation of *A*** if it satisfies: $\mu^{\mathrm{T}}_{\alpha}(x) = \mu_A(x) + \alpha$ and $(\nu_A)^{\mathrm{K}}_{\varepsilon}(x) = \nu_A(x) - \varepsilon$, for all $x \in X$.

Definition 3.2.

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be a bifuzzy subset of an ψ-algebra and $\alpha \in [0, T]$, $\varepsilon \in [0, K]$ of $(X; +, -, 0)$, then $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^{\mathrm{T}}_{\alpha}(x), (\nu_A)^{\mathrm{K}}_{\varepsilon}(x)) \mid x \in X\}$ is called **a translation bifuzzy ψ-subalgebra of *X***, if for all $x, y \in X$,

1− $(\mu_A)^{\mathrm{T}}_{\alpha}(x+y) \geq \min\{(\mu_A)^{\mathrm{T}}_{\alpha}(x), (\mu_A)^{\mathrm{T}}_{\alpha}(y)\}$,

2 − $(\mu_A)^{\mathrm{T}}_{\alpha}(x-y) \geq \min\{(\mu_A)^{\mathrm{T}}_{\alpha}(x), (\mu_A)^{\mathrm{T}}_{\alpha}(y)\}$ and

3 − $(\nu_A)^{\mathrm{K}}_{\varepsilon}(x+y) \leq \max\{(\nu_A)^{\mathrm{K}}_{\varepsilon}(x), (\nu_A)^{\mathrm{K}}_{\varepsilon}(y)\}$,

4 − $(\nu_A)^{\mathrm{K}}_{\varepsilon}(x-y) \leq \max\{(\nu_A)^{\mathrm{K}}_{\varepsilon}(x), (\nu_A)^{\mathrm{K}}_{\varepsilon}(y)\}$. i.e.,

1 − $\mu_A(x+y)+\alpha \geq \min\{\mu_A(x)+\alpha, \mu_A(y)+\alpha\}$

$= \min\{\mu_A(x), \mu_A(y)\} + \alpha$

2 − $\mu_A(x-y)+\alpha \geq \min\{\mu_A(x)+\alpha, \mu_A(y)+\alpha\}$

$= \min\{\mu_A(x), \mu_A(y)\} + \alpha,$

3 − $\nu_A(x+y)-\varepsilon \leq \max\{\nu_A(x)-\varepsilon, \nu_A(y)-\varepsilon\}$

$= \max\{\nu_A(x), \nu_A(y)\} - \varepsilon,$

4 − $\nu_A(x-y)-\varepsilon \leq \max\{\nu_A(x)-\varepsilon, \nu_A(y)-\varepsilon\}$

$= \max\{\nu_A(x), \nu_A(y)\} - \varepsilon.$

Example 3.3.

Let X ={0,1,2, 3} in which $(+, -)$ be a defined by the following table:

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	0	1	2

-	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

Then $(X; +, -, 0)$ is an ψ-algebra. It is easy to show that S_1 ={0,1} and S_2 ={0,2} are ψ-subalgebras of X. Define a fuzzy subset

$\mu_A: X \rightarrow [0,1]$ such that $\mu_A(0) = 0.7$, $\mu_A(1) = 0.6$, $\mu_A(2) = 0.4$, $\mu_A(3) = 0.3$, $\alpha = 0.15 \in [0,0.3]$.

$\nu_A: X \rightarrow [0,1]$ such that $\nu_A(0) = 0.3$, $\nu_A(1) = 0.4$, $\nu_A(2) = 0.6$, $\varepsilon = 0.25 \in [0,0.3]$.

Routine calculation gives that $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ is a bifuzzy ψ-subalgebra of X.

Also, gives that $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^{\mathrm{T}}_{\alpha}(x), (\nu_A)^{\mathrm{K}}_{\varepsilon}(x)) \mid x \in X\}$ is a translation bifuzzy ψ-subalgebra of X.

Theorem 3.4.

Let $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^{\mathrm{T}}_{\alpha}(x), (\nu_A)^{\mathrm{K}}_{\varepsilon}(x)) \mid x \in X\}$ be a translation bifuzzy ψ-subalgebra of an ψ-algebra $(X; +, -, 0)$ and $\alpha \in [0, T]$, $\varepsilon \in [0, K]$, then μ_A is a fuzzy ψ-subalgebra of X and ν_A is an anti-fuzzy ψ-subalgebra of X .

Proof:

Assume that A is a translation bifuzzy ψ-subalgebra of X, and $\alpha \in [0,\mathrm{T}]$, $\varepsilon \in [0, K]$. Let $x, y \in X$, then

$(\mu_A)^{\mathrm{T}}_{\alpha}(x+y) \geq \min\{(\mu_A)^{\mathrm{T}}_{\alpha}(x), (\mu_A)^{\mathrm{T}}_{\alpha}(y)\}$, that mean

$\mu_A(x+y)+\alpha \geq \min\{\mu_A(x)+\alpha, \mu_A(y)+\alpha\}$

$= \min\{\mu_A(x), \mu_A(y)\} + \alpha$, implies that $\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y)\}$.

Similarly, $\mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y)\}$.

$(\nu_A)_\varepsilon^{\mathrm{K}}(x+y) \leq \max\{(\nu_A)_\varepsilon^{\mathrm{K}}(x), (\nu_A)_\varepsilon^{\mathrm{K}}(y)\}$, that mean

$\nu_A(x+y) - \varepsilon \leq \max\{\nu_A(x) - \varepsilon, \nu_A(y) - \varepsilon\}$

$= \max\{\nu_A(x), \nu_A(y)\} - \varepsilon$, implies that

$\nu_A(x+y) \leq \max\{\nu_A(x), \nu_A(y)\}$.

Similarly, $\nu_A(x-y) \leq \max\{\nu_A(x), \nu_A(y)\}$.

Hence μ_A is a fuzzy ψ-subalgebra of X and ν_A is an anti-fuzzy ψ-subalgebra of X. ⌂

<u>Proposition 3.5.</u>

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be a bifuzzy subset of an ψ-algebra $(X; +, -, 0)$ and $\alpha \in [0, T]$, $\varepsilon \in [0, K]$ such that μ_A be a fuzzy ψ-subalgebra of X and ν_A be an anti-fuzzy ψ-subalgebra of X, then $(\mu_A)_\alpha^{\mathrm{T}}$ is a translation fuzzy ψ-subalgebra of X and $(\nu_A)_\varepsilon^{\mathrm{K}}$ is a translation anti-fuzzy ψ-subalgebra of X.

Proof:

Assume that μ_A is a fuzzy ψ-subalgebra of X and ν_A is an anti-fuzzy ψ-subalgebra of X, then for any $x, y \in X$ and $\alpha \in [0, T]$, $\varepsilon \in [0, K]$,

$\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and

$\mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y)\}$ implies that

$\mu_A(x+y) + \alpha \geq \min\{\mu_A(x), \mu_A(y)\} + \alpha = \min\{\mu_A(x) + \alpha, \mu_A(y) + \alpha\}$ and

$\mu_A(x-y) + \alpha \geq \min\{\mu_A(x), \mu_A(y)\} + \alpha = \min\{\mu_A(x) + \alpha, \mu_A(y) + \alpha\}$.

That mean $(\mu_A)_\alpha^{\mathrm{T}}(x+y) \geq \min\{(\mu_A)_\alpha^{\mathrm{T}}(x), (\mu_A)_\alpha^{\mathrm{T}}(y)\}$, and

$(\mu_A)_\alpha^{\mathrm{T}}(x-y) \geq \min\{(\mu_A)_\alpha^{\mathrm{T}}(x), (\mu_A)_\alpha^{\mathrm{T}}(y)\}$.

$\nu_A(x+y) \leq \max\{\nu_A(x), \nu_A(y)\}$ and

$\nu_A(x-y) \leq \max\{\nu_A(x), \nu_A(y)\}$ implies that

$\nu_A(x+y) - \varepsilon \leq \max\{\nu_A(x), \nu_A(y)\} - \varepsilon$

$= \max\{\nu_A(x) - \varepsilon, \nu_A(y) - \varepsilon\}$,

$\nu_A(x-y) - \varepsilon \leq \max\{\nu_A(x), \nu_A(y)\} - \varepsilon$

$= \max\{\nu_A(x) - \varepsilon, \nu_A(y) - \varepsilon\}$,

that mean $(\nu_A)_\varepsilon^{\mathrm{K}}(x+y) \leq \max\{(\nu_A)_\varepsilon^{K}(x), (\nu_A)_\varepsilon^{K}(y)\}$ and

$(\nu_A)_\varepsilon^{\mathrm{K}}(x-y) \leq \max\{(\nu_A)_\varepsilon^{\mathrm{K}}(x), (\nu_A)_\varepsilon^{\mathrm{K}}(y)\}$,

Hence μ_α^{T} is a translation fuzzy ψ-subalgebra of X and $\mathrm{v}_\varepsilon^{\mathrm{K}}$ is a translation anti-fuzzy ψ-subalgebra of X. ⌂

Definition 3.6.

For a fuzzy subset μ of an ψ-algebra $(X;+,-,0)$, $\alpha \in [0,T]$ and $t \in Im(\mu)$ with $t \geq \alpha$, let $U_\alpha(\mu;\ t) = \{x \in X \mid \mu(x) \geq t - \alpha\}$ and a fuzzy subset v of an ψ-algebra X, $\varepsilon \in [0,K]$ and $s \in Im(v)$ with $s \leq \varepsilon$, $L_\varepsilon(v;\ s) = \{x \in X \mid v(x) \leq s - \varepsilon\}$.

Remark 3.7.

1- If $(\mu_A)^T_\alpha$ is a translation fuzzy ψ-subalgebra of X, then it is that $U_\alpha(\mu_A; t)$ is a ψ-subalgebra of X, for any $t \in Im(\mu_A)$ with $t \geq \alpha$. Let $x, y \in U_\alpha(\mu_A; t)$, then $\mu_A(x) \geq t - \alpha$, and $\mu_A(y) \geq t - \alpha$, then $\min\{\mu_A(x), \mu_A(y)\} \geq t - \alpha$, since $(\mu_A)^T_\alpha$ is a translation fuzzy

ψ-subalgebra, then $\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y)\} \geq t - \alpha$, therefore $x + y \in U_\alpha(\mu_A; t)$ and

$\mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y)\} \geq t - \alpha$, therefore $x - y \in U_\alpha(\mu_A; t)$.

2- If $(\nu_A)^K_\varepsilon$ is a translation anti-fuzzy ψ-subalgebra of X, then it is that $L_\varepsilon(\nu_A;\ s)$ is a ψ-subalgebra of X, for any $s \in Im(\nu_A)$ with $s \leq \varepsilon$. Let $, y \in L_\varepsilon(v;\ s)$, then $\nu_A(x) \leq s - \varepsilon$, and $\nu_A(y) \leq s - \varepsilon$, then $\max\{\nu_A(x), \nu_A(y)\} \leq s - \varepsilon$, since $(\nu_A)^K_\varepsilon$ is a translation anti-fuzzy ψ-subalgebra, then $\nu_A(x+y) \leq \max\{\nu_A(x), \nu_A(y)\} \leq s - \varepsilon$, therefore

$x + y \in L_\varepsilon(\nu_A;\ s)$ and $\nu_A(x-y) \leq \max\{\nu_A(x), \nu_A(y)\} \leq s - \varepsilon$, therefore $x - y \in L_\varepsilon(\nu_A;\ s)$.

3- But if we do not give a condition that $(\mu_A)^T_\alpha$ is a translation fuzzy ψ-subalgebra of X, then $U_\alpha(\mu_A; t)$ is not a ψ-subalgebra of X or $(\nu_A)^K_\varepsilon$ is anti-fuzzy ψ-subalgebra of X, then $L_\varepsilon(\nu_A;\ s)$ is not a ψ-subalgebra of X as seen in the following example.

Example 3.8.

Consider $X = \{0, 1, 2, 3\}$ is a ψ-algebra which is given in Example (3.3). Define a fuzzy subset μ_A of X:

X	0	1	2	3
μ_A	0.7	0.6	0.4	0.3

Then $(\mu_A)^T_\alpha$ is not a translation fuzzy ψ-subalgebra of X.

Since $\mu_A(1+2) = 0.3 < 0.4 = \min\{\mu_A(1), \mu_A(2)\}$. For $\alpha = 0.1$ and $t = 0.5$, we obtain $U_\alpha(\mu_A; t) = \{0, 1, 2\}$ which is not an ψ-subalgebra of X since $1+2 = 3 \notin U_\alpha(\mu_A; t)$.

Proposition 3.9.

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be an bifuzzy subset of an ψ-algebra $(X;+,-,0)$ and $\alpha \in [0,T]$, $\varepsilon \in [0,K]$ such that $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^T_\alpha(x), (\nu_A)^K_\varepsilon(x)) \mid x \in X\}$ is a translation bifuzzy ψ-subalgebra of X, then $U_\alpha(\mu_A;\ t)$ and $L_\varepsilon(\nu_A;\ s)$ are ψ-subalgebras of X, for any $t \in Im(\mu_A)$, $s \in Im(\nu_A)$ with $t \geq \alpha$ and $s \leq \varepsilon$.

Proof:

Assume that $A^{(T,K)}_{(\alpha,\varepsilon)}$ is a translation bifuzzy ψ-subalgebra, then by Theorem (3.4), μ_A is a fuzzy ψ-subalgebra of X and ν_A is an anti-fuzzy ψ-subalgebra of X, then by Proposition (3.5), $(\mu_A)^T_\alpha$ is a translation fuzzy ψ-subalgebra and $(\nu_A)^K_\varepsilon$ is a translation anti-fuzzy ψ-subalgebra of X, therefore by Remark (3.8), $U_\alpha(\mu_A;\ t)$ and $L_\varepsilon(\nu_A;\ s)$ are ψ-subalgebras of X, for any $t \in Im(\mu_A)$, $s \in Im(\nu_A)$ with $t \geq \alpha$ and $s \leq \varepsilon$. ⌂

Theorem 3.10.

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be a bifuzzy subset of an ψ-algebra $(X; +, -, 0)$ and $\alpha \in [0,T]$, $\varepsilon \in [0,K]$ such that $U_\alpha(\mu_A; t)$ and $L_\varepsilon(\nu_A; s)$ are ψ-subalgebras of X, for all $t \in Im(\mu_A)$, $s \in Im(\nu)$ with $t \geq \alpha$ and $s \leq \varepsilon$, then $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^T_\alpha(x), (\nu_A)^K_\varepsilon(x)) \mid x \in X\}$ is a translation bifuzzy ψ-subalgebra of X.

Proof:

Assume that $x, y \in U_\alpha(\mu_A; t)$ and $(\mu_A)^T_\alpha$ of μ is not a fuzzy ψ-subalgebra of X, therefore

$(\mu_A)^T_\alpha(x+y) < t \leq \min\{(\mu_A)^T_\alpha(x), (\mu_A)^T_\alpha(y)\}$, then

$(\mu_A)(x) \geq t - \alpha$ and $(\mu_A)(y) \geq t - \alpha$, but $(\mu_A)(x+y) < t - \alpha$.

This shows that $x + y \notin U_\alpha(\mu_A; t)$. This is a contradiction,

and so $(\mu_A)^T_\alpha(x+y) \geq \min\{(\mu_A)^T_\alpha(x), (\mu_A)^T_\alpha(y)\}$, for all $x, y \in X$.

Similarly, $(\mu_A)^T_\alpha(x-y) \geq \min\{(\mu_A)^T_\alpha(x), (\mu_A)^T_\alpha(y)\}$.

Hence $(\mu_A)^T_\alpha$ is a translation fuzzy ψ-subalgebra of X.

$(\nu_A)^K_\varepsilon(x+y) > s \geq \max\{(\nu_A)^K_\varepsilon(x), (\nu_A)^K_\varepsilon(y)\}$, then

$(\nu_A)(x) \leq s - \varepsilon$ and $(\nu_A)(y) \leq s - \varepsilon$, but $(\nu_A)(x+y) > s - \varepsilon$.

This shows that $x + y \notin L_\varepsilon(\nu_A; s)$. This is a contradiction,

and so $(\nu_A)^K_\varepsilon(x+y) \leq \max\{(\nu_A)^K_\varepsilon(x), (\nu_A)^K_\varepsilon(y)\}$, for all $x, y \in X$.

Similarly, $(\nu_A)^K_\varepsilon(x-y) \leq \max\{(\nu_A)^K_\varepsilon(x), (\nu_A)^K_\varepsilon(y)\}$.

Therefore, $(\nu_A)^K_\varepsilon$ is a translation anti-fuzzy ψ-subalgebra of X.

Hence $A^{(T,K)}_{(\alpha,\varepsilon)}$ is a translation bifuzzy ψ-subalgebra of X.

Definition 3.11.

Let $(X; +, -, 0)$ be an ψ-algebra, μ_1 and μ_2 be fuzzy subsets of X, then μ_2 is called **a fuzzy extension** of μ_1. If $\mu_2(x) \geq \mu_1(x)$, for all $x \in X$.

Proposition 3.12.

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be a bifuzzy subset of an ψ-algebra $(X; +, -, 0)$, then the translation bifuzzy subset $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^T_\alpha(x), (\nu_A)^K_\varepsilon(x)) \mid x \in X\}$ of X is a fuzzy extension of A.

Proof:

Since $(\mu_A)^T_\alpha(x) = (\mu_A)(x) + \alpha \geq (\mu_A)(x)$, then $(\mu_A)^T_\alpha(x)$ is a fuzzy extension of $(\mu_A)(x)$, for all $x \in X$ and $(\nu_A)^K_\varepsilon(x) = (\nu_A)(x) - \varepsilon \leq (\nu_A)(x)$, then $(\nu_A)(x)$ is a fuzzy extension of $(\nu_A)^K_\varepsilon(x)$, for all $x \in X$. ⌂

Proposition 3.13.

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be a bifuzzy ψ-subalgebra of an ψ-algebra $(X; +, -, 0)$, then $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^{\mathrm{T}}_{\alpha}(x), (\nu_A)^{\mathrm{K}}_{\varepsilon}(x)) \mid x \in X\}$ is a translation bifuzzy ψ-subalgebra of X.

Proof:

Since A is a bifuzzy ψ-subalgebra of X, then $(\mu_A)^{\mathrm{T}}_{\alpha}$ is a translation fuzzy ψ-subalgebra of X and $(\nu_A)^{\mathrm{K}}_{\varepsilon}$ is a translation anti-fuzzy ψ-subalgebra of X by Proposition (5.1.5).

then $A^{(T,K)}_{(\alpha,\varepsilon)}$ of X is a translation bifuzzy ψ-subalgebra of X by Definition (5.1.2). ⌂

Remark 3.14.

The converse of Proposition (5.1.13) is not true as seen in the following example.

Example 3.15.

Let $(X; +, -, 0)$ be a ψ-algebra which is given in Example (5.1.3).

Define a fuzzy ψ-subalgebras (μ_A) and (ν_A) of X by:

X	0	1	2	3
μ_A	0.8	0.5	0.7	0.5
ν_A	0.3	0.4	0.5	0.3

Then μ_A is a fuzzy ψ-subalgebra of X and ν_A is anti-fuzzy ψ-subalgebra of X. Let $(\mu_A)^{\mathrm{T}}_{\alpha}$ be a fuzzy subsets of X where $\alpha = 0.1$ and $(\nu_A)^{\mathrm{K}}_{\varepsilon}$ be fuzzy subsets of X where $\varepsilon = 0.2$, μ_A is a fuzzy ψ-subalgebra of X given by :

X	0	1	2	3
$(\mu_A)^{\mathrm{T}}_{\alpha}$	0.9	0.6	0.8	0.6
$(\nu_A)^{\mathrm{K}}_{\varepsilon}$	0.1	0.2	0.3	0.1

Then $(\mu_A)^{\mathrm{T}}_{\alpha}$ is a fuzzy extension of μ_A, but the μ_A is not a fuzzy extension of $(\mu_A)^{\mathrm{T}}_{\alpha}$ and ν_A is a fuzzy extension of $(\nu_A)^{\mathrm{K}}_{\varepsilon}$, but the $(\nu_A)^{\mathrm{K}}_{\varepsilon}$ is not a fuzzy extension of ν_A

Proposition 3.16.

The intersection of translation bifuzzy ψ-subalgebras of an ψ-algebra $(X; +, -, 0)$ is a translation bifuzzy ψ-subalgebra of X.

Proof:

Let $\left\{(\mu_{A_i})^{\mathrm{T}}_{\alpha} \middle| i \in \Lambda \right\}$ be a family of translation bifuzzy ψ-subalgebra of μ_{A_i} of ψ-algebra X, then for any $x, y \in X, i \in \Lambda$,

$(\bigcap_{i\in\Lambda}(\mu_{A_i})_\alpha^T)(x+y) = \inf((\mu_{A_i})_\alpha^T(x+y))$

$= \inf(\mu_{A_i}(x+y)) + \alpha$

$\geq \inf\{\min\{\mu_{A_i}(x), \mu_{A_i}(y)\}\} + \alpha$

$= \min\{\inf \mu_{A_i}(x), \inf \mu_{A_i}(y)\} + \alpha$

$= \min\{\inf \mu_{A_i}(x) + \alpha, \inf \mu_{A_i}(y) + \alpha\}$

$= \min\{(\bigcap_{i\in\Lambda}(\mu_{A_i})_\alpha^T)(x), (\bigcap_{i\in\Lambda}(\mu_{A_i})_\alpha^T)(y)\}$.

Similarly, $(\bigcap_{i\in\Lambda}(\mu_{A_i})_\alpha^T)(x-y)$

$\geq \min\{(\bigcap_{i\in\Lambda}(\mu_{A_i})_\alpha^T)(x), (\bigcap_{i\in\Lambda}(\mu_{A_i})_\alpha^T)(y)\}$.

Let $\{(\nu_{A_i})_\varepsilon^K | i\in\Lambda\}$ be a family of translation bifuzzy ψ-subalgebra

of ν_{A_i}, then for any $x, y \in X$, $i \in \Lambda$,

$(\bigcup_{i\in\Lambda}(\nu_{A_i})_\varepsilon^K)(x+y) = \sup((\nu_{A_i})_\varepsilon^K(x+y))$

$= \sup(\nu_{A_i}(x+y)) - \varepsilon$

$\leq \sup\{\max\{\nu_{A_i}(x), \nu_{A_i}(y)\}\} - \varepsilon$

$= \max\{\sup(\nu_{A_i}(x)), \sup(\nu_{A_i}(y))\} - \varepsilon$

$= \max\{\sup(\nu_{A_i}(x)) - \varepsilon, \sup(\nu_{A_i}(y)) - \varepsilon\}$

$= \max\{(\nu_{A_i})_\varepsilon^K(x), \bigcup_{i\in\Lambda}(\nu_{A_i})_\varepsilon^K(y)\}$.

Similarity, $(\bigcup_{i\in\Lambda}(\nu_{A_i})_\varepsilon^K)(x-y)$

$\leq \max(\bigcup_{i\in\Lambda}(\nu_{A_i})_\varepsilon^K)(x), (\bigcup_{i\in\Lambda}(\nu_{A_i})_\varepsilon^K)(y)\}$. ⌂

Remark 3.17.

The union of a translation bifuzzy ψ-subalgebras of ψ-algebra $(X; +, -, 0)$, is not a translation bifuzzy ψ-subalgebra of X as seen in the following example.

Example 3.18.

Let $X = \{0, a, b, c, d\}$ be a set with the following table:

+	0	a	b	c	d
0	0	a	b	c	d
a	a	b	c	d	0
b	b	c	d	0	a
c	c	d	0	0	b
d	d	0	a	b	0

-	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	a
b	b	b	0	0	a
c	c	b	d	0	a
d	d	d	d	d	0

Then $(X;\ +,-,0)$ is an ψ-algebra. It is easy to show that $\mathrm{I}=\{0,\mathrm{c}\}$ and $\mathrm{J}=\{0,\mathrm{d}\}$ are ψ-subalgebras of X. We defined two sets

$(A_1)_{(\alpha,\varepsilon)}^{(T,K)} = \{(x,(\mu_{A_1})_\alpha^{\mathrm{T}}(x),(v_{A_1})_\varepsilon^{\mathrm{K}}(\,(x)) \mid x\in X\}$ and

$(A_2)_{(\alpha,\varepsilon)}^{(T,K)} = \{(x,(\mu_{A_2})_\alpha^{\mathrm{T}}(x),(v_{A_2})_\varepsilon^{\mathrm{K}}(x)) \mid x\ \in X\}$ of X by:

$$(\mu_{A_1})_\alpha^{\mathrm{T}}(x)=\begin{cases}0.8\ , & \text{if } x\in\{0,\mathrm{c}\},\\ 0.4, & \text{if } x\in\{\mathrm{a},\mathrm{b}\},\\ 0.5, & \text{otherwise}\end{cases}\qquad (v_{A_1})_\varepsilon^{\mathrm{K}}(x)=\begin{cases}0.2\ , & \text{if } x\in\{0,\mathrm{c}\},\\ 0.6, & \text{if } x\in\{\mathrm{a},\mathrm{b}\},\\ 0.5, & \text{otherwise}\end{cases}$$

$$(\mu_{A_2})_\alpha^{\mathrm{T}}(x)=\begin{cases}0.7\ , & \text{if } x\in\{0,\mathrm{d}\},\\ 0.5, & \text{otherwise.}\end{cases}\qquad (v_{A_2})_\varepsilon^{\mathrm{K}}(x)=\begin{cases}0.3, & \text{if } x\in\{0,\mathrm{d}\},\\ 0.5, & \text{otherwise.}\end{cases}$$

Then $(A_i)_{(\alpha,\varepsilon)}^{(T,K)}$ are translation bifuzzy ψ-subalgebras of , but the union of them are not translation bifuzzy ψ-subalgebras of X. Since $\left(\cup_{i\in\Lambda}(\mu_{A_i})_\alpha^{\mathrm{T}}\right)(\mathrm{c}+\mathrm{d})=\max\{0.4,0.5\}=0.5\ngeq 0.7$

$=\min\{\cup_{i\in\Lambda}(\mu_{A_i})_\alpha^{\mathrm{T}}(\mathrm{c}),\cup_{i\in\Lambda}(\mu_{A_i})_\alpha^{\mathrm{T}}(\mathrm{d})\}=\min\{\max\{0.8,0.5\},\max\{0.7,0.5\}\}$ and $\cup_{i\in\Lambda} v_{A_i}(\mathrm{c}+\mathrm{d})=\max\{0.6,0.5\}=0.6\nleq 0.5$

$$=\max\{\cup_{i\in\Lambda}(v_{A_i})_\varepsilon^{\mathrm{K}}(\mathrm{c}),\cup_{i\in\Lambda}(v_{A_i})_\varepsilon^{\mathrm{K}}(\mathrm{d})\}=\max\{\max\{0.2,0.5\},\max\{0.5,0.3\}\}$$

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