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On Chromaticity of Circulant Graph

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Abstract: *Coloring graphs is a fundamental problem that arose during the attempt to resolve the four-color theorem. The main focus lies in finding the minimum number of colors required for a proper graph coloring. Additionally, there is an interest in determining the total count of distinct proper colorings achievable with a specific number of colors on a graph. These values can be computed using the Chromatic Polynomial, a specialized function linked to each graph. Graphs G and H are considered chromatically equivalent if they have the same chromatic polynomial. A graph G is chromatically unique if it is isomorphic to any graph H that is chromatically equivalent to G. The exploration of chromatically equivalent and chromatically unique problems is known as chromaticity. This paper explores the chromaticity of circulant graph, focusing on their chromatic equivalence and uniqueness.*

Keywords: Chromatic polynomials, chromatically equivalent, chromatically unique, circulant graph.

1. INTRODUCTION

A graph G is considered planar if it can be represented on a plane without any edges crossing each other. Koh, K.M. and K.L. Teo defined a λ-coloring of a graph G as a mapping $\Phi : V(G) \to \{1,2,3,\dots,\lambda\}$ such that: $\Phi(a) \neq \Phi(b)$ for every edge $ab \in E(G)$. The smallest value of λ for which G can be properly colored is known as the chromatic number, and G is then referred to as being λ-colorable. In their quest to prove the four-color problem (the conjecture that every planar graph is 4-colorable), mathematicians developed various useful techniques for addressing graph coloring problems. Birkhoff [3] proposed a method to tackle the four-color problem by introducing a function $P(M,\lambda)$, representing the count of proper λ-colorings of a map M. This function, $P(M,\lambda)$, corresponds to a polynomial known as the chromatic polynomial of M.

In 1932, Whitney [11] took the study of chromatic polynomials from maps to graphs to a new level, contributing significantly to its expansion. He also made significant strides in establishing fundamental results in this field. Then, in 1968, Read [9] sought to determine a necessary and sufficient condition for two graphs to be chromatically equivalent (χ-equivalent), meaning to have identical chromatic polynomials.
Chao and Whitehead Jr. [14] defined a graph as chromatically unique (χ-unique) if no other graphs share its chromatic polynomial, and raised another question: What is the necessary and sufficient condition for a graph to be chromatically unique?
The study of chromaticity delves into the aforementioned questions regarding chromatic equivalence and chromatic uniqueness.
Throughout the period when the Four-Color Problem remained unsolved for over a century, various approaches were introduced in pursuit of a solution to this renowned problem [7].
The order n graph is produced by linking new edges to each pair of vertices in the cycle C_n that have a distance of k. This is represented by $C_n(k)$, where n and k are integers and $2 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, $\lfloor x \rfloor$ *is* the biggest number that is equal to or less than x, the graph $C_n(2)$, is called a chorded cycle [10].The prism graph Y_n is defined as the Cartesian product of $C_n \times K_2$, where K_2 represents the complete graph with two vertices and C_n represents the cycle graph with n vertices [2]. The Mobius Ladder M_n is obtained by connecting the opposite end points of two copies of $P_n \times P_2$, forming a graph as described in reference [11].
A Turan graph, also known as a maximally saturated graph, is represented by $T(n;k)$ and is defined as the complete k-partite graph of order n with all parts having sizes approximately equal to $\lfloor n/k \rfloor$ *or* $\lceil n / k \rceil$ [1].

A circulant graph with n nodes and jumps $s_1, s_2, \cdots, s_k$ is denoted by $C_n^{s_1,s_2,\dots,s_k}$. This is the regular graph of $2k$ with n vertices labeled $\{0, 1, 2,\cdots, n-1\}$, such that each vertex i $(0 \leq i \leq n-1)$ is adjacent to $2k$ vertices $i \pm s_1, i \pm s_2,\cdots, i \pm s_k \mod n$. The simplest circulant graph is the n vertex cycle C_n^1 . The next simplest is the square of the cycle $C_n^{1,2}$ in which each vertex is connected to both of its neighbors and to the neighbors of its neighbor [6].

2. REQUIREMENTS:

This section introduces some established results that contribute to proving the main result.
Example 2.1. [7] For the complete graph K_n of order n, we have
$$P(K_n ; \lambda) = \lambda(\lambda - 1) \cdots (\lambda - n + 1)$$
Theorem 2.1. [7]. In a graph G, let u and v be two non-adjacent vertices. Then,
$$P(G; \lambda) = P(G + uv, \lambda) + P(G \circ uv, \lambda)$$

Where $G + uv$ is created by appending a new edge uv to G, where $u, v \in V(G), uv \notin E(G)$; $G \circ uv$ is created by contracting the two vertices that coincide with e and eliminating all but one of the many edges, if any.

Theorem 2.2. [14]. Assume that e is an edge in graph G, Then,

$$P(G, \lambda) = P(G - e; \lambda) - P(G \circ e; \lambda)$$

It is acknowledged to analyze a graph drawing to indicate its chromatic polynomial, where $G - e$ is the graph formed from G by eliminating e.

Example 2.2. [7]. Let C_n be a cycle graph of order n. Then

$$P(C_n, \lambda) = (\lambda - 1)n + (-1)n(\lambda - 1)$$

Theorem 2.3. (Zykov [15]). Let G_1 and G_2 be two graphs and $G_1 \epsilon \xi [G_1 U_r G_2]$,
Then

$$P(G; \lambda) = \frac{P(G_1, \lambda)P(G_1, \lambda)}{P(K_r, \lambda)}$$

Theorem 2.4. [10] If G and H are two graphs that are chromatically equivalent, then we get:

1. $|V(G)| = |V(H)|$
2. $|E(G)| = |E(H)|$
3. $\chi(G) = \chi(H)$
4. G is connected iff H is connected
5. G is 2-connected iff H is 2-connected
6. $g(G) = g(H)$
7. There are an equal number of shortest cycles between G and H.
8. G is bipartite iff H is bipartite.

Example 2.3. [5] Any graph that is empty O_n, complete K_n and cycle C_m, where $n \geq 1$ and $m \geq 3$, is χ-unique.

Theorem 2.5. [13] For every p; $q \geq 2$, the complete bipartite graph $K(p, q)$ is χ-unique.

Theorem 2.6. [4] If $|ni - nj| \leq 1$ for all $i, j = 1,2, \dots, t$., then the complete t-partite graph $K(n_1, n_2, \dots, n_t)$ is χ-unique for all $t \geq 2$

Theorem 2.7. [9] When p is less than 5, the complement of $\overline{C_p}$ is χ-unique.

Conjecture 2.1. [10] The chorded cycle $C_n(2)$ is χ-unique. for all $n \geq 4$,

Conjecture 2.2. [12] For each $n \geq 3$, the prism $C_n \times K_2$ is χ-unique.

Theorem 2.8. [4] The Turan graph $T(n, k)$ with $1 \leq k \leq n - 1$ is χ-unique.

Conjecture 2.3. [10] the Mobius Ladder M_n, n ≥ 3 is χ-unique.

3. RESULTS

Proposition 3.1. For all $n \geq 3$, the circulant graph C_n^1 is χ-unique.

Proof. Since C_n^1 is isomorphic to cycle C_n and by example 2.1, C_n is χ-unique. In that case, the circulant graph C_n^1 is χ-unique.

Proposition 3.2. The circulant graph $C_n^{1,2,3,\dots,\left\lfloor\frac{n}{2}\right\rfloor}$; $n \geq 3$ is χ-unique.

Proof. $C_n^{1,2,3,\dots,\left\lfloor\frac{n}{2}\right\rfloor}$ is isomorphic to complete K_n, with $n \geq 3$. then the proposition becomes true by example 2.1.

Proposition 3.3 If G is defined as $C_{2n}^{1,3,5,\dots,n-\frac{1+(-1)^n}{2}}$, then G is χ-unique with $n \geq 3$.

Proof. The complete bipartite graph $K(p, q)$ is isomorphic to G. Then, according to theorem 2.5, G is χ-unique.

Proposition 3.4. For $n \geq 2$, $C_{3n}^{1,2,4,5,7,8,\dots,\left\lfloor\frac{n}{2}\right\rfloor}$ is χ-unique.

Proof. The complete 3-partite graph $K(n, n, n)$ where $n \geq 2$ is isomorphic to $C_{3n}^{1,2,4,5,7,8,\dots,\left\lfloor\frac{n}{2}\right\rfloor}$. Since $K(n, n, n)$ is χ-unique according to theorem 2.6, so too is $C_{3n}^{1,2,4,5,7,8,\dots,\left\lfloor\frac{n}{2}\right\rfloor}$ χ-unique.

Proposition 3.5. If $G = C_{4n}^{1,2,3,5,6,7,\dots,\left\lfloor\frac{n}{2}\right\rfloor}$ then G with $n \geq 2$ is χ- unique.

Proof. According to theorem 2.6, the graph $K(n, n, n, n)$ is χ-unique with $n \geq 2$, since G ≅ $K(n, n, n, n)$. Proposition are real.

Proposition 3.6. The graph $C_{2n}^{1,n}$ with n ≥ 3 is χ-unique.

Proof. Let G = $C_{2n}^{1,n}$, G is isomorphic to Mobius ladders M_n, $n \geq 3$ then G is χ-unique by conjecture 2.3.

Proposition 3.7. $C_n^{1,2}$, n ≥ 4 is χ-unique.

Proof. The circulant graph $C_n^{1,2}$ is isomorphic to the graph $C_n(2)$, and the graph $C_n(2)$ is χ-unique by conjecture 2.1, then the Proposition is realized.

Proposition 3.8. $C_{2n}^{1,2,..,n-1}$ is χ-unique where $n \geq 1$.

Proof. The circulant graph $C_{2n}^{1,2,..,n-1}$ is isomorphic to turan graph $T(2n,n)$ and $T(2n,n)$ is χ-unique according to theorem 3.8, $C_{2n}^{1,2,..,n-1}$ is hence χ-unique.

Proposition 3.9. $C_{2n}^{1,2,3,..,n-2,n}$ is χ-unique where n is even and $n \geq 4$.

Proof. Let G = $C_{2n}^{1,2,3,..,n-2,n}$, and observe that G is isomorphic for every n is even and $n \geq 4$ to $\overline{C_{2n}}$ Next, where $n \geq 4$ we conclude that $\overline{C_{2n}}$ is χ-unique using Theorem 2.7, In the event that n is even and $n \geq 4$, G = $C_{2n}^{1,2,3,..,n-2,n}$ is likewise χ-unique.

Proposition 3.10. The circulant graph $C_{2n+1}^{1,2,3,..,n-1}$ is χ-unique where n ≥ 4.

Proof. Considering that $G = C_{2n+1}^{1,2,3,..,n-1}$, and that G is isomorphic to $\overline{C_{2n+1}}$ where $n \geq 4$, we can infer by theorem 2.7, that $\overline{C_{2n+1}}$ In the case where $n \geq 4$ is χ-unique graph, $G = C_{2n+1}^{1,2,3,..,n-1}$ is likewise a χ-unique.

Proposition 3.11. For $n \geq 3$, $C_{2n}^{1,1,n}$ is χ-unique.

Proof. Assuming G to be the circulant graph $C_{2n}^{1,1,n}$, $G \cong C_{2n+1}^{1,2}$ and χ-equivalent to $C_{2n+1}^{1,2}$ then G is χ-unique by proposition 3.7.

Proposition 3.12. when n is odd & $n \geq 3$, the circulant graph $C_{2n}^{2,n}$ is χ-unique.

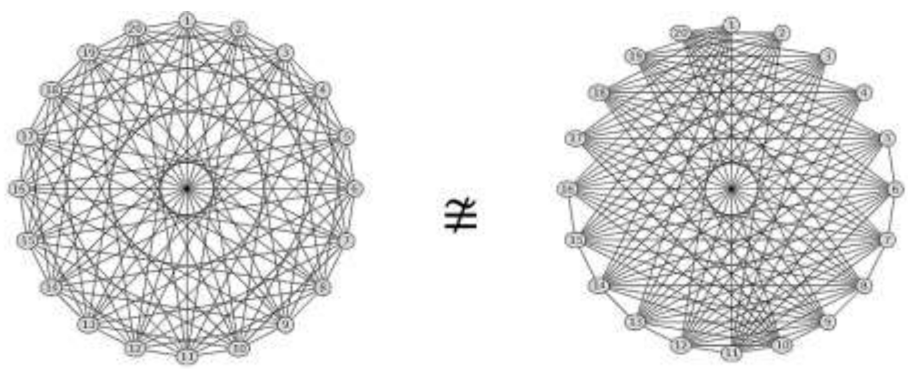
Proof. for odd n ≥ 3, the proposition is realized since $C_{2n}^{2,n}$ is isomorphic to the prism $C_n \times K_2$, which is thereafter χ-unique according to conjecture 2.2.

Problem 3.1. Is every circulant graph χ-unique?

No, as demonstrated by the example that follows.

Example 3.1. the circulant graph $C_{20}^{1,3,5,7,9,10}$, is not χ-unique.

Figure1, shows that, although H has the same chromatic polynomial as $C_{20}^{1,3,5,7,9,10}$, it is not isomorphic to it, because the circulant graph $C_{20}^{1,3,5,7,9,10}$ does not have a degree 10 vertex, but the graph H does. The outcome was shown using the Maple software.



$C_{20}^{1,3,5,7,9,10}$ $\ncong$ H

Figure1: $C_{20}^{1,3,5,7,9,10}$ is not isomorphic to H

$$P\left(C_{20}^{1,3,5,7,9,10},\lambda\right) = P(H,\lambda) = \lambda^{20} - 110\lambda^{19} + 5895\lambda^{18} - 203470\lambda^{17} + 5047210\lambda^{16} - 95204972\lambda^{15} + 1410827870\lambda^{14} - 16748216540\lambda^{13} + 161080904565\lambda^{12} - 1262004786854\lambda^{11} + 8060199694731\lambda^{10} - 41820067733590\lambda^{9} + 174896929272720\lambda^{8} - 582075172381440\lambda^{7} + 1512323170349408\lambda^{6} - 2982043806619200\lambda^{5} + 4276697472098560\lambda^{4} - 4166497012924800\lambda^{3} + 2435161460129792\lambda^{2} - 633586821259776\lambda$$

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