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Solve Nonlinear Equations Numerically

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Abstract: *This research aims to calculate the approximate values of one- and two-dimensional nonlinear equations using a numerical method (iterative fixed point method). Then we improved the mentioned method and mentioned its applications in several fields, including engineering, physics, electricity, and others.*

Keywords: Nonlinear Equations, numerical method, Fixed-point method.

1. Introduction:

Nonlinear equations are essential for comprehending and representing real-world phenomena that linear equations are unable to correctly depict [1]. These apps are used in a wide range of scientific and engineering fields, making them essential tools for researchers and professionals. The investigation of nonlinear equations frequently necessitates the utilization of advanced mathematical and numerical techniques to comprehend the intricacies they pose, resulting in a deeper understanding of systems that are intrinsically unpredictable and difficult [3,4,5]. In this research we will use the fixed point method to calculate nonlinear equations as the method gave high accuracy in the results as shown in the examples.

2.Locating the position of roots (programming method)[5]:

To locate the position of roots of the function (equation) $f(x)=0$ by using ***programming method,*** we use f(x) be continuous function on the interval [a,b]. We divide the interval [a,b] into n subintervals $a=x_0<x_1<\ldots<x_{n-1}<x_n=b$ where $x_i=a+ih,\ i=0,\ 1,\ \ldots,n;$ $h=\frac{b-a}{n}$. If $f(x_i)\times f(x_{i+1})<0$ for any $0\le i\le n$, then there exists c, $a<c<b$ for which $f(c)=0$.

Example 1:

To find the approximate location of the function $f(x)=x^3-4x+1=0$ on the interval [-1,4] with n=5.

Let n=5, $h=\frac{b-a}{n}=\frac{4-(-1)}{5}=1$

x	-1	0	1	2	3	4
f(x)	+	+	-	+	+	+

There is a root between (0,1) and (1,2).

3.Fixed-point method (iteration method) for solving a nonlinear equation with one variable

In this method, the equation $f(x)=0$ is written in the form $x=g(x)$ and it is said to solve such equation the **Fixed-point** of the function $g(x)$,That is, the Fixed-point of the function g is the root of the equation $f(x)=0$,To find this root let x_0 approximate value then $x_1=g(x_0)$ and $x_2=g(x_1)$ Similarly $x_3=g(x_2)$,….in general

$$x_{n+1}=g(x_n)\ ,\ \ n=1,2,3,\cdots \qquad \ldots(1)$$

and The enough condition for convergence of the formula above $|g'(x_0)|<1$ .and Stop condition is $|x_{n+1}-x_n|\le\in$ for any n.

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Example 2: The function g(x)=x, $0 \le x \le 1$ has a fixed point at each x in [0,1].

Example 3: The function g(x)=x-sin(πx) has exactly two fixed points in [0,1], x=0 and x=1.

4. The following theorem gives sufficient conditions for the existence and uniqueness of a fixed-point.

Theorem 1:

If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has a fixed point in [a,b]. Further ,suppose g'(x) exists on [a,b] and $|g'(x)| \le k < 1$ for all $x \in (a,b)$. Then g has a unique fixed point r in [a,b].

Proof: If g(a)=a or g(b)=b, the existence of a fixed point is obvious. Suppose not, then it must be true that g(a)>a and g(b)<b. Define D(x)=g(x)-x, D is continuous on [a,b] and moreover D(a)=g(x)-a>0, D(b)=g(b)-b<0.

The intermediate value theorem implies that there exists $r \in (a,b)$ for which D(r)=o. Thus g(r)-r=0 and r is a fixed point of g.

Suppose that u and v are both fixed points in [a,b] with $u \neq v$. By mean value theorem a number c exists between u and v, and hence in [a,b], with $|u-v| = |g(u)-g(v)| = |g'(c)||u-v| \le k|u-v| < |u-v|$

This is a contradiction. This contradiction must come from the only supposition $u \neq v$. Thus u=v and the fixed point in [a,b] is unique.

Theorem 2:

Let $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$. Further, let g'(x) exists on [a,b] with $|g'(x)| \le k < 1$ for all $x \in (a,b)$. If P_0 is any number in [a,b], then the sequence defined by $P_n = g(P_{n-1})$, $n \ge 1$ will converge to the unique fixed point P in [a,b] i.e. $\lim_{n\to\infty} P_n = P$.

Proof: By Theorem 1, a unique fixed point exists in [a,b]. Since g maps [a,b] into itself, the sequence $\{P_n\}_{n=0}^{\infty}$ is defined for all $n \ge 0$ and $P_n \in [a,b]$ for all n. Using $|g'(x)| \le k < 1$ and the mean value theorem

$$|P_n - P| = |g(P_{n-1}) - g(P)| = |g'(c)||P_n - P| \le k|P_n - P| \text{ where } c \in (a,b).$$

$$|P_1 - P| = |g(P_0) - g(P)| \le k|P_0 - P|$$

$$|P_2 - P| = |g(P_1) - g(P)| \le k|P_1 - P| \le k^2 |P_0 - P|$$

$$\vdots$$

$$|P_n - P| = |g(P_{n-1}) - g(P)| \le k^n |P_0 - P|.$$

Since k<1, then $\lim_{n\to\infty} |P_n - P| \le \lim_{n\to\infty} k^n |P_0 - P| = 0$ and $\{P_n\}_{n=0}^{\infty}$ converges to P.

5. Algorithm: Numerical Fixed-Point Iteration Method

Input:

- A continuous function g(x) derived from the equation f(x)=0 such that x=g(x).
- An initial guess x_0
- A tolerance level ϵ (a small positive number to determine when to stop).

- A maximum number of iterations N_{max} (to prevent infinite loops).

Output:

- Approximate solution x^* such that $|x_{n+1} - x_n| < \epsilon$.
- Number of iterations required.

Steps:

1. **Initialization**:

- Set n=0 (iteration counter).
- Set $x_n = x_0$ (initial guess).

2. **Iteration**:

- **While** $n < N_{max}$:
- ❖ Compute the next approximation: $x_{n+1} = g(x_n)$
- ❖ Check for convergence:
  - **If** $|x_{n+1} - x_n| < \epsilon$, **then**:
  - Set $x^* = x_{n+1}$.
  - **Output** x^* and the number of iterations n+1.
  - **Stop** the algorithm.
- ❖ Update $x_n = x_{n+1}$ (set the current guess to the new guess).
- ❖ Increment n=n+1.

3. **Non-Convergence Check**:

- **If** $n = N_{max}$ and convergence has not been reached:
- **Output** a message indicating that the method did not converge within the maximum number of iterations.
- **Output** the best approximation x_n found.

Example 4: The equation $x^3+4x^2-10=0$ has a unique root in [1,2]. There are many ways to change the equation to the form x=g(x) as follows:

(a) $x = g_1(x) = x - x^3 - 4x^2 + 10$

(b) $x = g_2(x) = \sqrt{\frac{10}{x} - 4x}$

(c) $x = g_3(x) = \frac{1}{2}\sqrt{10 - x^3}$

(d) $x = g_4(x) = \sqrt{\frac{10}{4 + x}}$

(e) $x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$ ……………..

To approximate the fixed point of a function g, we choose an initial approximation x_0 and generate the sequence $\{x_n\}_{n=0}^{\infty}$ by letting $x_n = g(x_{n-1})$ for each $n \geq 1$. Let $x_0 = 1.5$

(a) $|g_1'(1.5)| = |-8.75| > 1$ (diverge)

(b) $|g_2'(1.5)| = |5.17| > 1$ (diverge)

(c) $|g_3'(1.5)| = |0.6556| < 1$ (converge) ………………

If we use $g_3(x)$ to find a fixed point:

$$x_1 = g_3(x_0) = 1.2870$$

$x_2 = g_3(x_1) = 1.4025$

$x_3 = g_3(x_2) = 1.3455$

$\vdots$

$x_{20} = g_3(x_{19}) = 1.3652$ with error=6.5782×10^{-7}.

Example 5 :- Find an approximate solution of $f(x) = x^2 - x - 1 = 0$ in interval $[1,2]$ by using *Fixed-point method* where $\in = 0.0009$.

Solution:-

$f(1) = (1)^2 - (1) - 1 = -1$, $f(2) = (2)^2 - (2) - 1 = 1$

$x_{n+1} = g(x_n)$, $n = 1,2,3,\cdots$, $x_0 = 1.5$

$x = x^2 - 1 \Rightarrow g'(x) = 2x \Rightarrow |g'(x_0)| = |2x_0| = |2(1.5)| = 3 > 1$

So the formula $x = x^2 - 1$ (diverge)

$x^2 = x + 1 \Rightarrow x = \sqrt{x+1}$, $g'(x) = \dfrac{1}{2\sqrt{x+1}} \Rightarrow |g'(x_0)| = \left|\dfrac{1}{2\sqrt{x_0+1}}\right| = 0.3162 = 0.3 < 1$

The formula $x = \sqrt{x+1}$ gives a solution

$x_1 = \sqrt{x_0+1} = \sqrt{1.5+1} = 1.5811$, $|x_1 - x_0| = |1.5811 - 1.5| = 0.0811 > \in$

$x_2 = \sqrt{x_1+1} = \sqrt{1.5811+1} = 1.6066$, $|x_2 - x_1| = |1.6066 - 1.5811| = 0.0255 > \in$

$x_3 = \sqrt{x_2+1} = \sqrt{1.6066+1} = 1.6145$, $|x_3 - x_2| = |1.6145 - 1.6066| = 0.0079 > \in$

$x_4 = \sqrt{x_3+1} = \sqrt{1.6145+1} = 1.6169$, $|x_4 - x_3| = |1.6169 - 1.6145| = 0.0024 > \in$

$x_5 = \sqrt{x_4+1} = \sqrt{1.6169+1} = 1.6177$, $|x_5 - x_4| = |1.6177 - 1.6169| = 0.0008 < \in$

Then $x_5 = 1.6177$ be root required.

6.Fixed Point Method (Iteration Method) for Solving a System of Two Nonlinear Equations in Two Variables

We will discuss solving two equations in two unknowns

$$\left.\begin{matrix} f(x,y) = 0 \\ g(x,y) = 0 \end{matrix}\right\} \qquad \ldots(2)$$

As a first step in applying fixed-point method , we rewrite these equations in the following form

$$\left.\begin{aligned} x &= F(x,y) \\ y &= G(x,y) \end{aligned}\right\} \qquad \ldots(3)$$

So that any solution of (2) is a solution of (1), Let (λ,μ) be The real solution of (1)

i.e. $\left.\begin{aligned} f(\lambda,\mu)=0 \\ g(\lambda,\mu)=0 \end{aligned}\right\}$, Let (x_0,y_0) be an approximation to (λ,μ).

Starting with the approximate value (x_0,y_0) we can generate a sequence of approximate solutions $(x_i,y_i),\ i=0,1,2,\ldots$ of the two formulas:-

$$\left.\begin{aligned} x_{i+1} &= F(x_i,y_i) \\ y_{i+1} &= G(x_i,y_i) \end{aligned}\right\}, \quad i=0,1,2,\ldots \qquad \ldots(4)$$

the two conditions of convergence are:-

$\left|\frac{\partial F}{\partial x}\right| + \left|\frac{\partial G}{\partial x}\right| < 1 \ \wedge \left|\frac{\partial F}{\partial y}\right| + \left|\frac{\partial G}{\partial y}\right| < 1$ at the point (x_0,y_0)

And two stop conditions $|x_{i+1}-x_i| < \in \quad \vee \quad |y_{i+1}-y_i| < \in$ for any i.

Example 6:

Solve the following system:

$$\begin{aligned} 0.1x^2+0.1y^2-x+0.8=0 \\ 0.1x+0.1xy^2-y+0.8=0 \end{aligned}$$, where $x_0=y_0=0.5$, $\in=0.03$

The exact solution of this system is $x=y=1$.

Solution:

Rewrite above system as follows:

$$\begin{aligned} x &= F(x,y) = 0.1x^2+0.1y^2+0.8 \\ y &= G(x,y) = 0.1x+0.1xy^2+0.8 \end{aligned}$$

$$\frac{\partial F}{\partial x}=0.2x \ , \ \frac{\partial F}{\partial y}=0.2y \ , \ \frac{\partial G}{\partial x}=(0.1+0.1y^2) \ , \ \frac{\partial G}{\partial y}=0.2xy$$

$$|F_x|_{(0.5,\,0.5)} + |G_x|_{(0.5,\,0.5)} = |0.2(0.5)| + \left|0.1+0.1(0.5)^2\right| = 0.225 < 1$$

$$|F_y|_{(0.5,\,0.5)} + |G_y|_{(0.5,\,0.5)} = |0.2(0.5)| + |0.2(0.5*0.5)| = 0.15 < 1$$

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$$x_{i+1} = F(x_i, y_i) = 0.1x_i^2 + 0.1y_i^2 + 0.8$$
$$y_{i+1} = G(x_i, y_i) = 0.1x_i + 0.1x_i y_i^2 + 0.8$$

$$x_1 = 0.1(0.5)^2 + 0.1(0.5)^2 + 0.8 \Rightarrow x_1 = 0.85$$
$$y_1 = 0.1(0.5) + 0.1(0.5)(0.5)^2 + 0.8 \Rightarrow y_1 = 0.8625$$
$$|x_1 - x_0| = |0.85 - 0.5| = 0.35 > \in \wedge \ |y_1 - y_0| = |0.8625 - 0.5| = 0.3625 > \in$$

$$x_2 = 0.1(0.85)^2 + 0.1(0.8625)^2 + 0.8 \Rightarrow x_2 = 0.9466$$
$$y_2 = 0.1(0.85) + 0.1(0.85)(0.8625)^2 + 0.8 \Rightarrow y_1 = 0.9482$$
$$|x_1 - x_0| = |0.9466 - 0.85| = 0.097 \quad > \in \wedge \ |y_1 - y_0| = |0.9482 - 0.8625| = 0.086 > \in$$

Table (1)		
i	value x	value y
0	0.5	0.5
1	0.85	0.8625
2	0.9466	0.9482
3	0.9795	0.9798
4	0.9918	0.9920

Note:- To improve the results in the Fixed-point method, we can calculate y_i using x_{i+1} the obtained in the previous step by applying both $\left.\begin{aligned} x_{i+1} &= F(x_i, y_i) \\ y_{i+1} &= G(x_{i+1}, y_i) \end{aligned}\right\}, \quad i = 0, 1, 2, \ldots$

We will recalculate the table in the previous example using the note

$$x_{i+1} = F(x_i, y_i) = 0.1x_i^2 + 0.1y_i^2 + 0.8$$
$$y_{i+1} = G(x_{i+1}, y_i) = 0.1x_{i+1} + 0.1x_{i+1} y_i^2 + 0.8$$

Table (2)		
i	value x	value y
0	0.5	0.5
1	0.85	0.9063
2	0.9544	0.9738
3	0.9859	0.9921

7.Applications of the numerical fixed point method

1. Engineering

- Structural Analysis: The method is used to find equilibrium points in structures by solving nonlinear equations related to deformations and stresses.
- Control Systems: Fixed-point iteration is applied in the analysis and design of control systems, particularly in tuning controllers by solving nonlinear control equations.
- Thermal Systems: In heat transfer and thermodynamics, fixed-point methods can solve equations related to temperature distribution and heat flow.

2. Physics

- Quantum Mechanics: The method is applied in solving the Schrödinger equation under certain approximations, where the equation is expressed in a form suitable for iteration.
- Nonlinear Dynamics: Fixed-point iteration helps in analyzing stability and finding fixed points of nonlinear dynamical systems, such as chaotic systems.

3.Economics

- Equilibrium Models: Fixed-point iteration is used to find equilibrium prices and quantities in economic models, such as supply and demand models.
- Game Theory: The method is employed to find Nash equilibria in games, where each player's strategy depends on the strategies of others.

4. Computer Science

- Root-Finding Algorithms: Fixed-point iteration is a foundational method in numerical analysis for finding roots of nonlinear equations.
- Iterative Solvers: The method is used in iterative algorithms to solve large systems of linear or nonlinear equations, common in simulations and optimizations.
- Image Processing: Fixed-point methods can be applied in certain algorithms for image denoising and enhancement, where iterative schemes are designed to converge to a fixed point representing the desired image.

5.Fluid Mechanics

Flow Equations: In computational fluid dynamics, fixed-point iteration is used to solve the Navier-Stokes equations in their discretized forms, particularly when dealing with nonlinear terms.

These applications demonstrate the wide-ranging utility of the Numerical Fixed-Point Method across various scientific and engineering disciplines. The method's simplicity and effectiveness in handling nonlinear equations make it a valuable tool for researchers and practitioners.

8.Conclusion

The Numerical Fixed-Point Method is a useful and essential method in numerical analysis, widely applicable in several fields of study. Although the method is straightforward, its ability to converge and be effective relies on the characteristics of the function and the beginning conditions. Continual research and development efforts persist in improving its practicality and resolving its constraints.
This study offers a thorough comprehension of the Fixed-Point Method, encompassing its theoretical underpinnings, actual execution, applications, and potential future developments. The method is an essential element of numerical analysis and continues to discover fresh applications as computer techniques progress.

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