International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 8 Issue 8 August - 2024, Pages: 152-157

Some Properties of Zero Divisor Graph in the Ring $\boldsymbol{Z_n}$ for Some $\boldsymbol{n}$

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Abstract. *In this work, the zero divisor graph of the ring Z_n for some n are been formed as $\Gamma(Z_{p^2q^2})$ and $\Gamma(Z_{pqr})$. Furthermore, the cardinal of vertex set to these graphs are been determined. Also, the clique, maximum degree, and the pendants vertices are been founded. Finally, the domination number of the above graphs are been calculated.*

Keywords: zero divisor graph, domination number, pendants vertices, clique, and maximum degree.

INTRODUCTION

Graph theory is one of the most important branches of mathematics in studying the mathematical construction of most life problems in an appropriate way. So researchers began to study other branches of mathematics through the concepts of this branch, and thus began to deal with sub-branches such as algebraic graph [12], topological graph [14-15], fuzzy graph [28-31],[17], and [25], labeled graph [6], and others.

The graph theory also deals with many concepts, in which the concept of domination comes in the first place in terms of addressing many life applications, so it attracted many researchers to study this concept and in different forms according to the type of problem. The first to use the concept of domination was C. Berge in [7], followed by Babb Ore [16] in1962. The main definition of dominating set in graph theory is represented by the following: the subset D of the vertex set V represents the dominating set if it and its neighbors include all the vertex set of the graph. The minimum cardinality of all dominating set is called the domination number [8]. Many papers appeared recently deal with the concept of domination number with variant mode as in [1-5], [7], [14-21], and [26-27].There is a very strong relationship between graph theory and ring theory in terms of studying and analyzing most of the results by merging these two branches and obtaining a sub-branch which is algebraic graph.

The weighted graph is a rich branch of graph theory where it takes some numerical values to the edges or vertices. The concept of a zero divisor graph of a commutative ring was presented by I. Beck in [8]. Let R be a ring, and let G (R) be the graph such that the set of vertices is R and two vertices say that u and v are adjacent if $uv = 0$. This graph is denoted by Γ (R) and is called a zero divisor graph . The later there are many papers that deal with this concept [22-26]. For more details, the reader can be see [10-12].

During this work, the concept of the zero divisor graph of the commutative ring Z_n are been discussed for some integer number n as p^2q^2 and pqr, when p, q, and r prime numbers different from each other is discussed. Moreover, the cardinality of the vertex set of the graph whose vertices are the zero divisor of rings $Z_{p^2q^2}$ and Z_{pqr} are been determined. Also, the clique, maximum degree, and the domination number of the graphs mentioned above are been determined with some illustrate examples.

Theorem 1. [17]. $\left|V\left(\Gamma(Z_{pqr})\right)\right| = pq + pr + rq - (p + q + r)$ where $p, q, and\ r$ are primes numbers and $p < q < r$.

MAIN RESULTS

Theorem 2. $\left|V\left(\Gamma(Z_{p^2q^2})\right)\right| = pq^2 + qp^2 - 2 - \left\lfloor\frac{p^2q^2-q}{pq}\right\rfloor$ where p and q are primes numbers and $p < q$.

Proof. The vertices of the graph $\Gamma(Z_{p^2q^2}) = \{p, 2p, \ldots, p(pq^2-1), q, 2q, \ldots, q(p^2q-1)\}$. These vertices can be disjoint to two sets depend on the multiple of prime numbers p and q as follows:

$S_1 = \{p, 2p, \ldots., p(pq^2-1)\}$ and $S_2 = \{q, 2q, \ldots., q(p^2q-1)\}$. One can easily conclude that $|S_1| = pq^2 - 1$ and $|S_2| = p^2q - 1$. Now, to determine the vertex set of the graph $\Gamma(Z_{p^2q^2})$, firstly, we must find the intersection of two sets S_1 and S_2. Each vertex in the intersection set must multiple of prime number p and multiple of prime q, the intersection set contain all vertices that multiple of the number pq. Thus, the number of these vertices is equal in the sets S_1 and S_2. The number of multiple of prime number in the set S_1 is equal to $\left\lfloor \frac{p^2q^2-p}{pq} \right\rfloor$ and in the set S_2 is $\left\lfloor \frac{p^2q^2-q}{pq} \right\rfloor$. Therefore, the order of the graph $\Gamma(Z_{p^2q^2})$ is equal to $|S_1| + |S_2| - \left\lfloor \frac{p^2q^2-q}{pq} \right\rfloor$ and the result is obtained.

Example 3. In the graph $\Gamma(Z_{36}) \equiv \Gamma(Z_{2^23^2})$ the vertices of the set $S_1 = \{2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,32,34\}$ and the vertices of the set $S_2 = \{3,6,9,12,15,18,21,24,27,30,33\}$. It is easy to see that $S_1 \cap S_2 = \{6,12,18,24,30\}$. Thus, $|V(\Gamma(Z_{36}))| = 17 + 11 - 5 = 23$ as in the following figure

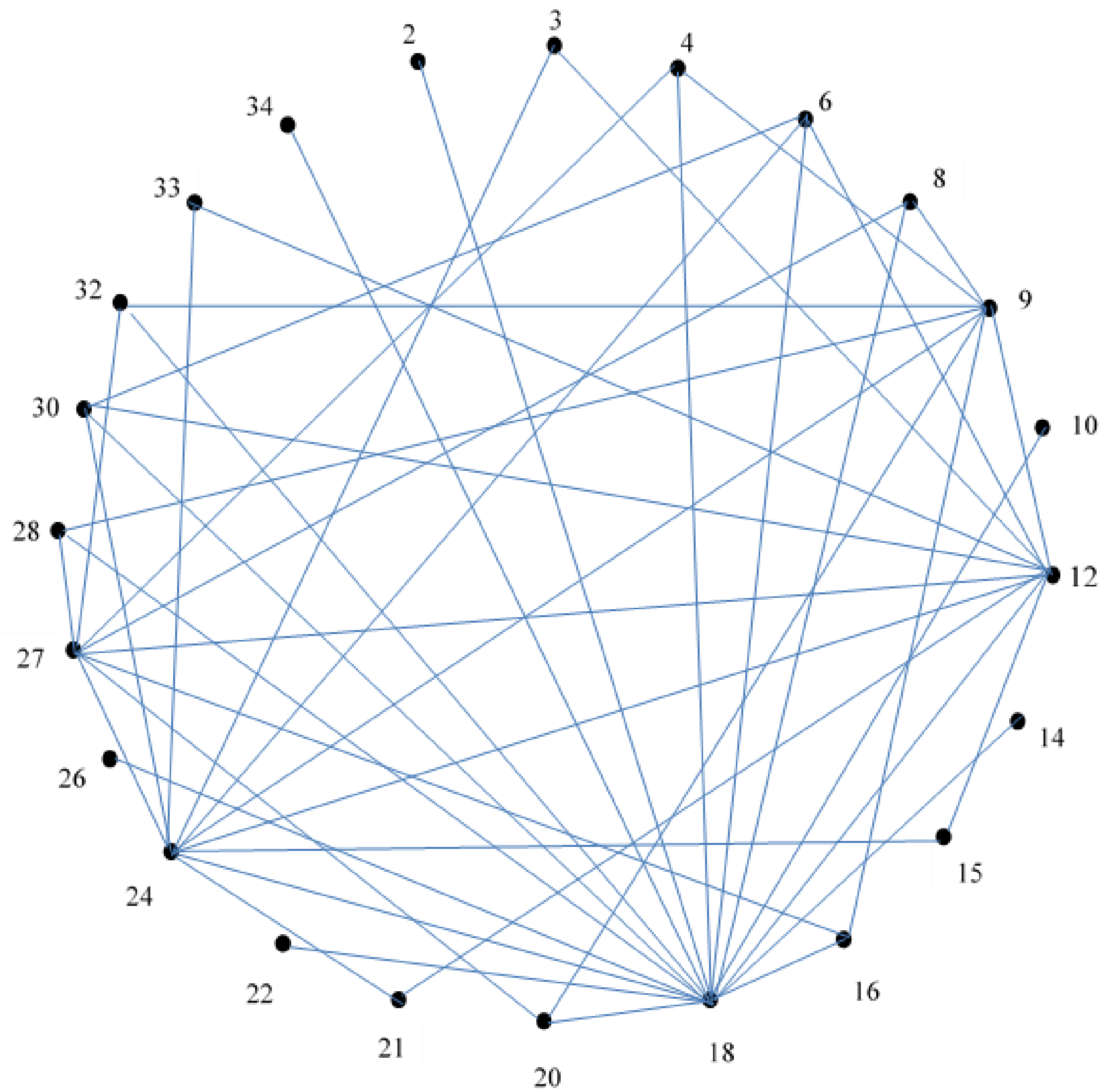

FIGURE 1. The graph $\Gamma(Z_{p^2q^2})$.

Theorem 4. The clique of the graph $\Gamma(Z_{p^2q^2})$ is complete graph of order $pq - 1$.

Proof. Let $D = \{pq, 2pq, \ldots, (pq-1)pq\}$, then for all two vertices in the set D say u and v, $uv = ipq\ jpq = ij\ p^2q^2$, so u and v are adjacent. Thus, the induced subgraph $\langle D \rangle$ is complete (as an example, in Figure 1 , $D = \{6, 12, 18, 24,30\}$). Now, to prove that $\langle D \rangle$ has the maximum cardinality of all complete as subgraph from graph $\Gamma(Z_{p^2q^2})$ such that $\langle D \rangle$ is complete. Let $H \neq D$ is a complete subgraph of the graph $\Gamma(Z_{p^2q^2})$, so for all two disjoint vertices u_1 and u_2 belong to the set H, u_1u_2 must divides p^2q^2, thus there are only two cases as follows.

Case 1. If pq^2 divides all vertices of the set H, then $\langle H\rangle$ is the induced subgraph isomorphic to complete graph. The vertices of this graph is $\{pq^2, 2pq^2, \dots, (p-1)pq^2\}$ and the cardinal of this set is $p-1$.

Case 2. If p^2q divides all vertices of the set H, then by the same technique in Case1, the order of $\langle H\rangle$ is $q-1$.

For all two cases above $|H| < |D|$, Therefore, $\langle D\rangle$ is clique in the graph $\Gamma(Z_{p^2q^2})$.

Theorem 5. $\Delta\left(\Gamma(Z_{p^2q^2})\right) = pq^2 - 2$.

Proof. From Theorem2, the set $S_1 = \{p, 2p, \dots., p(pq^2-1)\}$ contains the vertex $pq^2$and all other vertices are adjacent to this vertex, since each vertex in the set S_1 can be written as $\{ip,\ i = 1,2,\dots, pq^2-1\}$. Thus, $pq^2\ ip = ip^2q^2$, and so this vertex divides p^2q^2. Therefore, the vertex pq^2 is adjacent to vertices of form ip except the vertex pq^2, since the graph is simple. Then $deg(pq^2) \geq pq^2 - 2$, since $|S_1| = pq^2 - 1$. The remain vertices not in the set S_1 are the form $jq\ and\ j \neq p$, so $pq^2(jq) = jpq^3$ and this vertex not divides p^2q^2. Therefore, $deg(pq^2) = pq^2 - 2$.

Now, assume that there is a vertex say v such that $deg(v) > deg(pq^2)$, then the vertex v is written as one of the following.

1) If $v = iq^2, i \neq p$, the vertices which are adjacent to the vertex v are the form jp^2 and it is obvious that all these vertices belong to the set S_1 and not equal to whole set, then the cardinal of these vertices is less than $|S_1|$. Therefore, $deg(v) < deg(pq^2)$ and this is a contradiction.

2) If $v = p^2q$, then all vertices adjacent to v must be of order jq. Thus, in the same manner above the vertices of the set $S_2 = \{q, 2q, \dots., q(p^2q-1)\}$ are the only adjacent to v, so $deg(v) = p^2q - 2 < pq^2 - 2 = deg(pq^2)$, Again this is a contradiction.

3) If $v = ip^2, i \neq q$, the vertices which are adjacent to the vertex v are the form jq^2 and it is obvious that all these vertices belong to the set S_2 and not equal to whole set, then the cardinal of these vertices is less than $|S_2|$. Therefore, $deg(v) < deg(pq^2)$ and this is a contradiction.

From all cases above, the result is proved.

Theorem 6. $\gamma\left(\Gamma(Z_{p^2q^2})\right) = 2$.

Proof. Let $D = \{pq^2, p^2q\}$, it is clear that $pq^2, p^2q \in V\left(\Gamma(Z_{p^2q^2})\right)$. From Theorem 5, the vertex pq^2 dominates all vertices in the set S_1 that defined in Theorem 2. Also, form Theorem 5, the remain vertices do not dominate by the vertex pq^2 have the form $jq\ and\ j \neq p$. One can easily conclude that the vertex p^2q dominates all vertices of the form $jq\ and\ j \neq p$. Thus, D is a dominating set. From Theorem 5 and Theorem2, $\Delta\left(\Gamma(Z_{pq^2})\right) = pq^2 - 2$ and $\left|V\left(\Gamma(Z_{p^2q^2})\right)\right| = pq^2 + qp^2 - 2 - \left\lfloor\frac{p^2q^2-q}{pq}\right\rfloor$, so $pq^2 - 2 < pq^2 + qp^2 - \left\lfloor\frac{p^2q^2-q}{pq}\right\rfloor - 1$. Therefore, there is no vertex in the graph $\Gamma(Z_{p^2q^2})$ is adjacent to all other vertices. Thus, the set D has the minimum cardinality of a dominating set and the prove is done.

Proposition 7. The graph $\Gamma(Z_{p^2q^2})$; (whenever p and q are primes number and $p < q$) has pendants vertices $\{pr; r = 1\ or\ r\ is\ prime\ number\ such\ that\ r \neq q\}$ if $p = 2$, otherwise this graph has no pendant vertex.

Proof. There are two cases depend on the value of p as the following.
Case 1. If $p = 2$, then let $S = \{pr; r = 1 \text{ or r is a prime number such that } r \neq q\}$. Again there are two subcases:
Subcase 1. If $r = 1$, then the labeled of the first vertex that adjacent to the vertex of labeled p is pq^2. The second vertex that adjacent to the vertex of labeled p is $2pq^2$, but $2pq^2 = p^2q^2$, and this number is not exist in the graph $\Gamma(Z_{p^2q^2})$. Thus, the vertex of labeled p is a pendant vertex (as an example, see Figure 1).
Subcase 2. If r is a prime number such that $r \neq q$, by the same technique in Subcase 1, one can conclude that the vertices labeled pr are pendants (as an example, see Figure 1).

International Journal of Engineering and Information Systems (IJEAIS)
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Case 2. If $p \neq 2$, then the labeled of the first vertex that adjacent to the vertex of labeled p is pq^2. The second vertex that adjacent to the vertex of labeled p is $2pq^2 < p^2q^2$, then the vertex of labeled p is adjacent to the second vertex. Thus, the vertex of labeled p is not pendant. In the same manner, one can prove that there is no other pendants vertices.
From two cases above, the required is obtained.

Theorem 8. $\Delta\left(\Gamma(Z_{pqr})\right) = qr - 1$

Proof. From Theorem 1, the vertices of the graph $\Gamma(Z_{pqr})$ is $\{p, 2p, \dots, p(qr-1), q, 2q, \dots, q(pr-1), r, 2r, \dots, r(qp-1)\}$. There are many duplicate numbers in this collection so repetition should be excluded to get the vertex set of the graph $\Gamma(Z_{pqr})$. If we divided the previous collection into three sets, each section is based on the prime number multiplied by as follows: $V_p = \{p, 2p, \dots, p(qr-1)\}$, $V_q = \{q, 2q, \dots, q(pr-1)\}$, and $V_r = \{r, 2r, \dots, r(qp-1)\}$. Thus, $V_p \cap V_q = \{pq, 2pq, \dots, pq(r-1)\}$, $V_p \cap V_r = \{pr, 2pr, \dots, pr(q-1)\}$, and $V_r \cap V_q = \{rq, 2rq, \dots, rq(p-1)\}$. Thus, $V\left(\Gamma(Z_{pqr})\right) = \{p, 2p, \dots, p(qr-1), q, 2q, \dots, q(pr-1), r, 2r, \dots, r(qp-1)\} - \{pq, 2pq, \dots, pq(r-1), pr, 2pr, \dots, pr(q-1), rq, 2rq, \dots, rq(p-1)\}$.
The vertex of labeled qr is adjacent to all vertices in the set V_p and $qr \notin V_p$, since each vertex in this set can be written as ip, thus, $qr\,(ip) = ipqr$. In same manner in the Theorem 5 the vertex of labeled qr is not adjacent to other vertices in $V\left(\Gamma(Z_{pqr})\right)$ and it is maximum degree. Therefore, $\Delta\left(\Gamma(Z_{pqr})\right) = qr - 1$.

Theorem 9. $\gamma\left(\Gamma(Z_{pqr})\right) = 3$..

Proof. Let $D = \{pq, pr, rq\}$, it is clear that $pq, pr, rq \in V\left(\Gamma(Z_{p^2q^2})\right)$. The vertex labeled pq dominates all vertices in the set $V_r = \{r, 2r, \dots, r(qp-1)\}$ (from Theorem 8), since each vertex in the set V_r can be written as ir, so the multiple $pq(ir)$ divides pqr. The vertex of labeled pq cannot adjacent to other vertices in $V\left(\Gamma(Z_{pqr})\right)$ that mentioned in Theorem 8, since the multiple the vertex of labeled pq with the other vertices in the $V\left(\Gamma(Z_{pqr})\right)$ not divides pqr. In same manner the vertex of labeled pr is dominates all vertices in the set $V_q = \{q, 2q, \dots, q(pr-1)\}$ that mentioned in Theorem 8. Also, the vertex of labeled rq is dominates all vertices in the set $V_p = \{p, 2p, \dots, p(qr-1)\}$ that mentioned in Theorem 8. It is clear that $V\left(\Gamma(Z_{pqr})\right) = V_p \cup V_q \cup V_r$. Therefore the set D is a dominating set of the graph, so $\gamma\left(\Gamma(Z_{pqr})\right) \leq 3$. Now, if delete a vertex from the set D, then there are three cases depend on the deletion as follows.
I) If delete the vertex of labeled pq, then the set $D - \{pq\}$ cannot dominates some vertices of the set V_r as an example the vertex of labeled r, since $pr(r) = pr^2$ is not divides pqr and $rq(r) = qr^2$ and again qr^2 is not divides pqr.
II) If delete the vertex of labeled pr, then the set $D - \{pr\}$ cannot dominates some vertices of the set V_q as an example the vertex of labeled q, since $pq(q) = pq^2$ is not divides pqr and $rq(q) = rq^2$ and again rq^2 is not divides pqr.
III) If delete the vertex of labeled rq, then the set $D - \{rq\}$ cannot dominates some vertices of the set V_p as an example the vertex of labeled p, since $pq(p) = qp^2$ is not divides pqr and $rp(p) = rp^2$ and again rp^2 is not divides pqr.
From three cases above the minimality of the set D is proved. Finally, it is clear that the set D is minimum by followed the same technique in Theorem 6. Thus, $\gamma\left(\Gamma(Z_{pqr})\right) = 3$.

Proposition 10. The set of pendants vertices in the graph $\Gamma(Z_{pqr})$ is non-empty and equal to $S = \{p^k\,;\ 1 \leq p^k < pqr\} \cup \{p^m s;\ p^m s < pqr\}$ whenever s is prime number greater that r if and only if $p = 2$.

Proof. Let $p = 2$, then each vertex adjacent to the vertex of labeled p must as form multiple of qr. Thus, the only vertex of labeled qr is adjacent to the vertex of labeled p, since $2qr \equiv 0 (mod\ pqr)$, so, the vertex of labeled $p = 2$ is the pendant vertex. By using the same technique mentioned above, one can be proved that the vertices $\{p^k\,;\ 1 \leq p^k < pqr\} \cup \{p^m s;\ p^m s < pqr\}$ whenever s is prime number greater that r are pendants.
Conversely, let $S = \{p^k\,;\ 1 \leq p^k < pqr\} \cup \{p^m s;\ p^m s < pqr\}$ whenever s is prime number greater that r is the non-empty set of pendants vertices. As in above the vertex labeled p is adjacent to only vertices which are labeled multiple of the number qr. Thus, the vertex of labeled p is adjacent to the vertex of labeled qr. Now, if $2qr < pqr$, then the vertex of labeled p is adjacent to it and this is a contradiction with our assumption. Therefore, $p = 2$ and the prove is done.
□

Example 11. In the Figure 2., the graph $\Gamma(Z_{30}) \equiv \Gamma(Z_{2\,3\,5})$ the vertices of the set $V_2 = \{2,4,6,8,10,12,14,16,18,20,22,24,26,28\}$, $V_3 = \{3,6,9,12,15,18,21,24,27\}$, and $V_5 = \{5,10,15,20,25\}$. The pairwise intersection of these sets are $V_2 \cap V_3 = \{6,12,18,24\}$, $V_2 \cap V_5 = \{10,20\}$, and $V_3 \cap V_5 = \{15\}$. Thus, $V(\Gamma(Z_{2\,3\,5})) = \{2,3,4,5,6,8,9,10,12,14,15,16,18,20,21,22,24,25,26,27,28\}$, so $|V(\Gamma(Z_{2\,3\,5}))| = 21$. The maximum degree of the graph $\Gamma(Z_{30}) = deg(v) = 14; v$ is labeled $qr = 15$. A minimum dominating set is D which it contains the vertices of labeled $\{6,10,15\}$ colored by green. The pendants vertices in the graph $\Gamma(Z_{30})$ is the vertices of labeled $\{2,4,8,14,16,22,26\}$ colored by red.

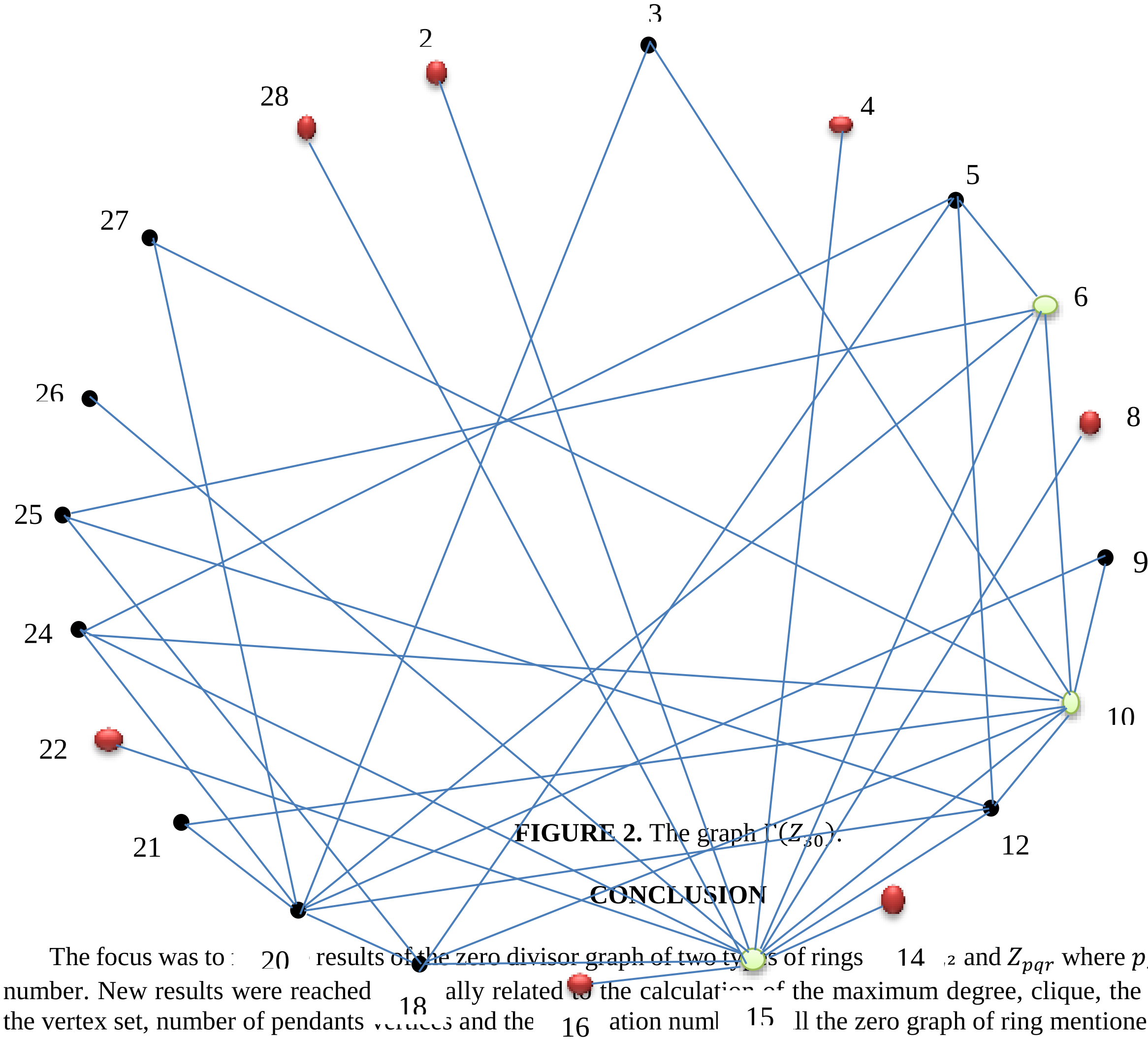

FIGURE 2. The graph $\Gamma(Z_{30})$.

CONCLUSION

The focus was to [illegible] results of the zero divisor graph of two types of rings [illegible] and Z_{pqr} where p, q, and r are primes number. New results were reached [illegible]ally related to the calculation of the maximum degree, clique, the cardinal number of the vertex set, number of pendants vertices and the [illegible]ation num[illegible]ll the zero graph of ring mentioned above.

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