

Analysis the Mobius Function Graph $(\mathcal{M}^0)^c$ in Graph Theory

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Abstract: This study introduces a novel graph called the Mobius Function Graph $(\mathcal{M}^0)^c$. It presents three approaches to computing the prime-counting function based on this graph and proves various properties associated with it. Furthermore, the domination number, independence number, chromatic number, and clique number of the graph are analyzed. Finally, the connection between the domination number and the independence number is examined in detail.

Keywords: Mobius Function Graph $(\mathcal{M}^0)^c$, domination number, independence number, and chromatic number.

1. INTRODUCTION

Recently graph theory has become very popular, emerging as a universal language applicable across various disciplines, including medicine, engineering, physics, chemistry, and more. A graph $G(V,E)$ consists of two fundamental sets: The groups of lines (E) with vertices (V). Graph theory spans multiple fields such as topological graphs, [1-4].

Key PRINCIPLES IN GRAPH THEORY include the dominating set, independent set, and vertex coloring. These concepts lead to critical parameters, namely, the domination number, independence number, and chromatic number. A subset of the vertex set is considered dominating if every vertex outside this subset is adjacent to at least one vertex within it. The domination number [5], denoted as $\gamma(G)$, is the minimum size of all dominating sets. An independent set is a subset of the vertex set where no two vertices are adjacent. The independence number [6], denoted as $\beta(G)$, is the maximum size of all independent sets. A vertex coloring of G assigns colors to its vertices such that no two adjacent vertices share the same color [7]. The chromatic number, $\chi(G)$, represents the minimum number of colors required for such an assignment. Additionally, the clique number of a graph is the size of the largest complete sub graph, where every vertex is connected to all others within the sub graph [8].

In this paper, analyze the correlation of graph theory and number theory by offering a novel graph. The vertices of this graph are based on Actual numbers and their edges are determined by mathematical function is a recognized as (Mobius function). We demonstrate three methods for calculating the prime-counting function, $\pi(n)$, which counts the prime numbers \mathbf{N} smaller than or equivalent to the order of a given graph. Additionally, explore additional aspects of this graph, which include independence number. [9, 10].

2. Complement the Mobius Function Graph $(\mathcal{M}^0)^c$ in Graph Theory

The purpose of this research is introduced new graphs which are called the complete of Mobius Function Graph $(\mathcal{M}^0)^c$. The relationship between these graphs are clarified. Moreover, the clique number, independence number, the domination number associated with each of these graphs has been founded. Additionally, compare the results of these graphs (\mathcal{M}^i) , $i = 0, 1, -1$ with its complement. Also, the chromatic number, the $\text{Cent}((\mathcal{M}^i)^c)$, and $\text{Per}((\mathcal{M}^i)^c)$ graphs are founded with some properties.

2.1 Main results.

Proposition 2.1.1. Let G be a graph of order n , then the graphs $\mathcal{M}^0, \mathcal{M}^1$, and \mathcal{M}^{-1} are decomposition of complete graph of order n .
Proof.

Let e_{ij} be an edge that joint the two vertices v_i and v_j , then

1) If $e_{ij} \in E(\mathcal{M}^0) \cap E(\mathcal{M}^1)$ or $e_{ij} \in E(\mathcal{M}^0) \cap E(\mathcal{M}^{-1})$, then $e_{ij} \in E(\mathcal{M}^0)$, this means ij has a square prime factor. Moreover, the

Edge $e_{ij} \in E(\mathcal{M}^1)$ or $e_{ij} \in E(\mathcal{M}^{-1})$, in each cases ij is a free square prime factor and this is a contradiction. Thus,

$$E(\mathcal{M}^0) \cap E(\mathcal{M}^1) = E(\mathcal{M}^0) \cap E(\mathcal{M}^{-1}) = \emptyset.$$

2) If $e_{ij} \in E(\mathcal{M}^1) \cap E(\mathcal{M}^{-1})$, then $e_{ij} \in \mathcal{M}^1$, this means $\mathcal{M}(v_i v_j) = \prod_{j=1}^k P_j, k$ is even and $e_{ij} \in \mathcal{M}^{-1}$, this means $\mathcal{M}(v_i v_j) = \prod_{j=1}^k P_j, k$ is odd and this is a contradiction. Thus, $E(\mathcal{M}^1) \cap E(\mathcal{M}^{-1}) = \emptyset$.

3) Let e_{ij} be any edge in the graph G , then ij take three options according to the definition of MFG . The first option that ij has a square prime factor so

This edge belongs to the $MFG \mathcal{M}^0$. The second option that ij has a free square prime factor so there are two cases as follows.

Case 1. If ij can be written as the form $\prod_{j=1}^k P_j, k$ is even, then the edge e_{ij} belong to the $MFG \mathcal{M}^1$.

Case 2. If ij can be written as the form $\prod_{j=1}^k P_j, k$ is odd, then the edge e_{ij} belong to the $MFG \mathcal{M}^{-1}$. Thus, $E(\mathcal{M}^0) \cup E(\mathcal{M}^1) \cup E(\mathcal{M}^{-1}) = E(K_n)$ (for example, see Fig 1.)

Therefore, depending of all cases above, the result is obtained. \square

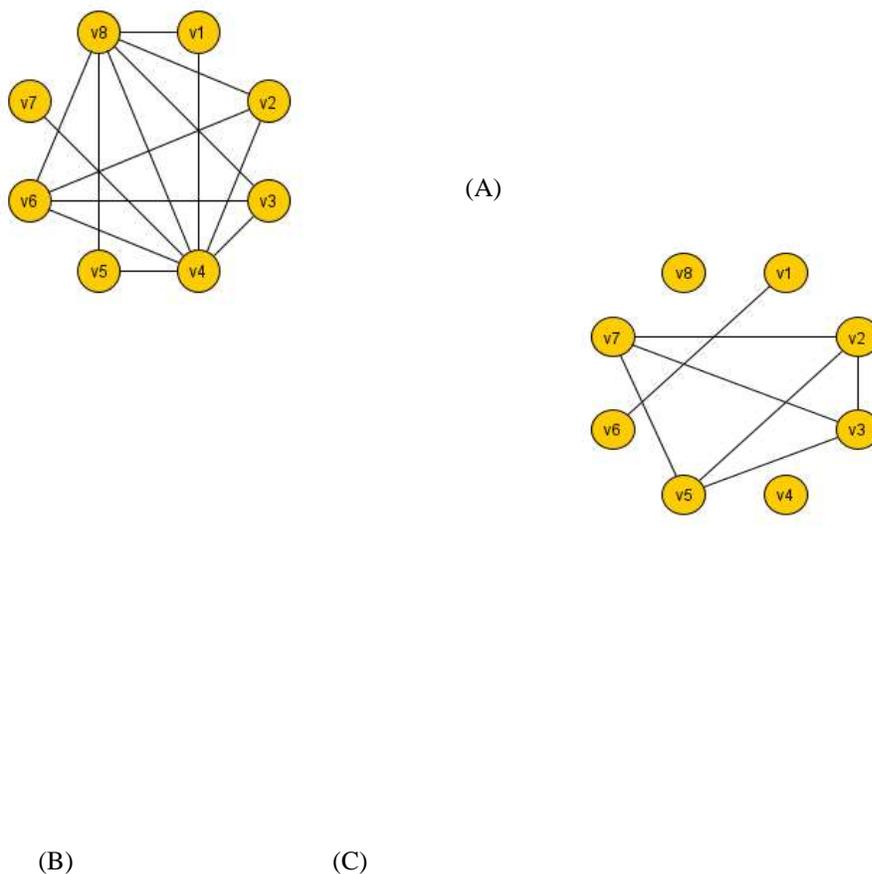


Fig 1. $MFG \mathcal{M}^0$ (A), \mathcal{M}^1 (B), and \mathcal{M}^{-1} (C)

Theorem 2.1.2. Assume that G is the $MFG (\mathcal{M}^0)^c$, then $\omega(G) = \pi(n) + 1$.

Proof. Take any two vertices V_i and V_j such that i and j are primes, so $M(ij) = 1$, thus these vertices are adjacent. Therefore, an induced sub graph is made up of vertices annotated with prime integers as well. Which isomorphic to the whole graph, so $\omega(G) \geq \pi(n)$. Moreover, the vertex of labeled one is adjacent to all vertices mentioned above, so $\omega(G) \geq \pi(n)+1$. Let V_r be any other vertex that adjacent to all vertices mentioned above, so there are at least two prime numbers say p and q such that $r=pq$. Thus, this vertex not adjacent to two vertices V_p and V_q and this is a contradiction (as an example, see Fig 2.). Therefore, $\omega(G)=\pi(n)+1$. \square

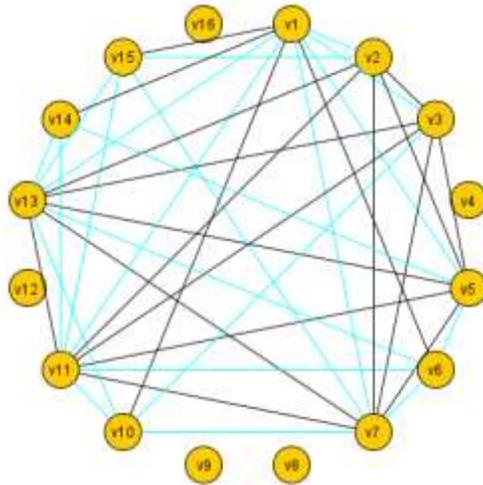


Fig 2. The $MFG(\mathcal{M}^0)^c$ of order 16, Blue represent \mathcal{M}^{-1} , Black represent \mathcal{M}^1

Proposition 2.1.3. If G is a $MFG(\mathcal{M}^0)^c$ is non-trivial, then $\chi(G) = \pi(n) + 1$.

Proof. Depending on Theorem 3.2.3, $\omega((\mathcal{M}^0)^c) = \pi(n) + 1$, then $\pi(n) + 1$ Colors which are Color the vertices of the greatest subgraph that is isomorphic to complete graph of order $\pi(n) + 1$. By the same colors that used previously, we can color the remain vertices. Thus, $\chi(G) = \pi(n) + 1$. \square

Proposition 2.1.4. Assume that G is a $MFG(\mathcal{M}^0)^c$ of order n , then

- 1) Each vertex has labeled not free square prime is an isolated vertex.
- 2) Assume that G is a $MFG(\mathcal{M}^0)^c$ of order $n \geq 4$, then $MFG(\mathcal{M}^0)^c$ is disconnected, otherwise, the $MFG(\mathcal{M}^0)^c$ is connected.

Proof. Let $f(v_i) = i$, so

1) It is obvious that every vertex has a square prime factor is an isolated depending on the definition of $MFG(\mathcal{M}^0)^c$. Thus, the result is obtained.

2) If $n = 4$, then $f(v_4) = 4 = 2^2$, thus the vertex v_4 is not adjacent to all other vertices in the $(\mathcal{M}^0)^c$, so this graph is disconnected. Now, if

$n \leq 3$, it is obvious that the $(\mathcal{M}^0)^c$ is connected. Therefore, the result is obtained. \square

Proposition 2.1.5. Assume that G is a $MFG(\mathcal{M}^0)^c$ of order n , then

$$\beta((\mathcal{M}^0)^c) = \begin{cases} 1, & \text{if } n = 1 \\ |N_e^n| + |S - N_e^n|, & \text{otherwise} \end{cases}$$

Proof. Two cases depending on the order of the $MFG(\mathcal{M}^0)^c$ are discussed as follows:

Case1. If $n=1$ then $\beta((\mathcal{M}^0)^c) = 1$.

Case2. Consider $u, v \in N_e^n$, so $f(uv) = 2^2 P_z P_w$. Therefore, the two vertices u and v are not adjacent, then the set N_e^n is an independent, so $\beta(G) \geq |N_e^n|$. Each vertex in the set S belong to every independent set. Note that $S \cap N_e^n \neq \emptyset$, so

$$\beta(G) \geq |N_e^n| + |S - N_e^n|.$$

There is at least one P_i , $i = 1, \dots, r$ such that P_i splits one of the labeled of a vertex in the set N_e^n , since $2 < P_j$ (as an example, see Fig 3).

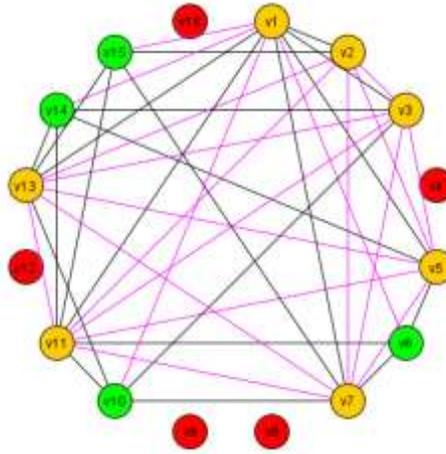


Fig 3. $MFG(\mathcal{M}^0)^c$, red represent independence number and isolated, Green represent even number

Proposition 2.1.6. Assume that G is a non-trivial $MFG(\mathcal{M}^0)^c, \mathcal{M}^1$, and \mathcal{M}^{-1} , then

- 1- $\beta((\mathcal{M}^0)^c) \neq \beta(\mathcal{M}^1) + \beta(\mathcal{M}^{-1})$
- 2- $\beta((\mathcal{M}^0)^c) \neq \beta(\mathcal{M}^1)$ if $n = 2, 3, 4, 5$, otherwise $\beta((\mathcal{M}^0)^c) = \beta(\mathcal{M}^1)$
- 3- $\beta((\mathcal{M}^0)^c) \leq \beta(\mathcal{M}^1)$
- 4- $\beta((\mathcal{M}^0)^c) = \beta(\mathcal{M}^{-1})$ if $n = 1, 2$, otherwise $\beta((\mathcal{M}^0)^c) \neq \beta(\mathcal{M}^{-1})$.
- 5- $\beta((\mathcal{M}^0)^c) \leq \beta(\mathcal{M}^{-1})$

Proof.

(1-4) are straightforward from Proposition 2.1.5. There are two cases as follows

Case 1. If $n = 1, 2, \dots, 6$, it is obvious by using the Proposition 2.1.5.

Case 2. If $n \geq 7$, then $\beta((\mathcal{M}^0)^c) = |N_e^n| + |S - N_e^n| = |S \cup N_e^n| = |S| + |N_e^n - S|$ and $\beta(\mathcal{M}^{-1}) = |S \cup S_1| = |S| + |S_1|$, since $S \cap S_1 = \emptyset$. Since $|N_e^n - S| \leq |S_1|$, then $\beta((\mathcal{M}^0)^c) \leq \beta(\mathcal{M}^{-1})$. \square

Proposition 2.1.7. Assume that G is a $MFG(\mathcal{M}^0)^c$ of order n , then

$$\gamma(\mathcal{M}^0)^c = |S| + 1.$$

Proof.

Each an isolated vertex belongs to every dominating set, so $\gamma(\mathcal{M}^0)^c \geq |S|$. The vertex v_1 is adjacent to all vertices which are not isolated. Thus,

$$\gamma(\mathcal{M}^0)^c = |S| + 1. \square$$

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