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First Accuracy in Generalized Approximation Space Using Admixture Vertex Edges Systems

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Abstract: *First lower and higher approximations of* $\mathfrak{P}$ *using incidence vertex edge systems (or, alternatively, first lower and upper approximations of* $\mathfrak{P}$ *using non-incidece vertex edge systems are the main idea in this study. This study is in charge of creating and examining the first lower and upper approximations of* $\mathfrak{P}$ *using admixture vertex edge systems. Furthermore, We made use of Some of its properties are examined, along with the first accuracy of approximating a subgraph* $\mathfrak{P} \subseteq \xi$ *utilizing (incidence, non-incidence, and admixture) vertex edge systems.*

Keywords: graph , topology, generalized approximation space, accuracy, lower and upper approximations.

1. INTRODUCTION:

The part of graph theory known as combinatorics has close ties to other branches of mathematics, including matrix theory, group theory, and topology. The second is that when graphs are used to empirically depict a variety of concepts, they will be extremely helpful in practice. Numerous theoretical and practical applications can be found in the field of topological graph theory [1, 2, 3, 4, 5, 8, and 9]. We predict that topological graph structure will play an important role in bridging the gap between topology and applications. For all graph theory words and nomenclature, we refer to Harary [6], and for all topology terms and notation, we refer to Moller [7]. The following are some basic concepts in graph theory [10].
A graph is pair $\boldsymbol{\xi} = (\mathbb{V}(\boldsymbol{\xi}), \mathbb{E}(\boldsymbol{\xi}))$ where $\mathbb{V}(\boldsymbol{\xi})$ is a non-empty set whose elements are called points or vertices (called vertex set) and $\mathbb{E}(\boldsymbol{\xi})$ is the set of unordered pairs of elements of $\mathbb{V}(\boldsymbol{\xi})$ (called edge set). An edge of a graph that joins a vertex to itself is called a loop.
A subgraph of a graph $\boldsymbol{\xi}$ is a graph each of whose vertices belong to $\mathbb{V}(\boldsymbol{\xi})$ and each of whose edges belong to $\mathbb{E}(\boldsymbol{\xi})$. An empty graph if the vertices set and edge set is empty. A degree of a vertex $\mathfrak{i}$ in a graph $\boldsymbol{\xi}$ is the number of edges of $\boldsymbol{\xi}$ incident with $\mathfrak{i}$. Let $\xi = (\mathbb{V}(\boldsymbol{\xi}), \mathbb{E}(\boldsymbol{\xi}))$ be a graph then the incidence vertex edges set of $\mathfrak{i}$ is denoted by $\mathrm{INVE}(\mathfrak{i})$ and defined by: $\mathrm{INVE}(\mathfrak{i}) = \{\mathfrak{e} \in \mathbb{E}(\boldsymbol{\xi}) : \mathfrak{e} = (\mathfrak{i}, \mathfrak{d}) \text{ for some } \mathfrak{d} \in \mathbb{V}(\boldsymbol{\xi})\}$. The non-incidence vertex edges set of $\mathfrak{i}$ is denoted by $\mathrm{NINVE}(\mathfrak{i})$ and defined by: $\mathrm{NINVE}(\mathfrak{i}) = \{\mathfrak{e} \in \mathbb{E}(\boldsymbol{\xi}) : \mathfrak{e} = (\mathfrak{d}, \mathfrak{r}) \text{ and } \mathfrak{d}, \mathfrak{r} \neq \mathfrak{i} \text{ for som } \mathfrak{d}, \mathfrak{r} \in \mathbb{V}(\boldsymbol{\xi})\}$. The incidence vertex edges system (resp. non-incidence vertex edges system) of a vertex $\mathfrak{i} \in \mathbb{V}(\boldsymbol{\xi})$ is denoted by $\mathrm{INVES}(\mathfrak{i})$ (resp. $\mathrm{NINVES}(\mathfrak{i})$) and defined by : $\mathrm{INVES}(\mathfrak{i}) = \{\mathrm{INVE}(\mathfrak{i})\}$ (resp. $\mathrm{NINVES}(\mathfrak{i}) = \{\mathrm{NINVE}(\mathfrak{i})\}$) .The admixture vertex edges system of a vertex $\mathfrak{i} \in \mathbb{V}(\boldsymbol{\xi})$ is denoted by $\mathrm{AVES}(\mathfrak{i})$ and defined by: $\mathrm{AVES}(\mathfrak{i}) = \{\mathrm{INVES}(\mathfrak{i}), \mathrm{NINVES}(\mathfrak{i})\}$, such that such that $\mathrm{AVE}(\mathfrak{i}) \in \mathrm{AVES}(\mathfrak{i})$. The A-space is the pair $(\boldsymbol{\xi}, \Upsilon_{\mathrm{a}})$ such that $\boldsymbol{\xi}$ is a graph and $\Upsilon_{\mathrm{a}} : \mathbb{V}(\boldsymbol{\xi}) \longrightarrow \mathrm{P}(\mathrm{P}(\mathbb{E}(\boldsymbol{\xi})))$ is a mapping which assigns for each $\mathfrak{i}$ in $\mathbb{V}(\boldsymbol{\xi})$ its admixture vertex edges system in $\mathrm{P}(\mathrm{P}(\mathbb{E}(\boldsymbol{\xi})))$.

2. First New Approximation Operators Using Admixture Vertex Edges Systems

Introducing a set theoretic foundation for granular computing with admixture vertex edges systems is the main goal of this section. There are several varieties of arbitrary edges systems in use. Using Admixture vertex edges systems, we proposed new definitions for the lower and upper approximation operators in the generalized approximation space. We obtain the attributes of the proposed operators. Additionally, we defined and examined the features of accuracy for the introduced approximations.

Definition 2.1: Let $\boldsymbol{\Gamma} = (\mathbb{V}(\boldsymbol{\xi}), \mathbb{E}(\boldsymbol{\xi}))$ be a generalized approximation space and $\mathfrak{P} \subseteq \boldsymbol{\xi}$. Then :
(1) The first lower and upper approximations of $\mathfrak{P}$ using incidence vertex edges systems are denoted by $\mathrm{L}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P}))$ and $\mathrm{U}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P}))$ and defined by:

$$\mathrm{L}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) = \left\{ \begin{matrix} \mathfrak{e} \in \mathbb{E}(\mathfrak{P}) \text{ there exsist } \mathfrak{i} \text{ incidence on } \mathfrak{e} \\ \text{such that } \mathrm{INVE}(\mathfrak{i}) \subseteq \mathbb{E}(\mathfrak{P}) \end{matrix} \right\}.$$

$$\mathrm{U}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{P}) \cup \left\{ \begin{matrix} \mathfrak{e} \in \mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P}); \mathrm{INVE}(\mathfrak{i}) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset \\ \text{where } \mathfrak{i} \text{ incidence on } \mathfrak{e} \end{matrix} \right\}.$$

(2) The first lower and upper approximations of $\mathfrak{P}$ using non incidence vertex edges systems are denoted by $\mathrm{L}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$ and $\mathrm{U}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$ and defined by:

$$L_n^1(\mathbb{E}(\mathfrak{P})) = \left\{\begin{matrix} \mathbf{e} \in \mathbb{E}(\mathfrak{P}) \text{ there exsist } \mathbf{i} \text{ incidence on } \mathbf{e} \\ \text{such that } \mathrm{NINVE}(\mathbf{i}) \subseteq \mathbb{E}(\mathfrak{P}) \end{matrix}\right\}.$$

$$U_n^1(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{P}) \cup \left\{\begin{matrix} \mathbf{e} \in \mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P}); \mathrm{NINVE}(\mathbf{i}) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset \\ \text{where } \mathbf{i} \text{ incidence on } \mathbf{e} \end{matrix}\right\}.$$

(3) The first lower and upper approximations of $\mathfrak{P}$ using admixture vertex edges systems are denoted by $L_a^1(\mathbb{E}(\mathfrak{P}))$ and $U_a^1(\mathbb{E}(\mathfrak{P}))$ and defined by:

$$L_a^1(\mathbb{E}(\mathfrak{P})) = \left\{\begin{matrix} \mathbf{e} \in \mathbb{E}(\mathfrak{P}) \text{ there exsist } \mathbf{i} \text{ incidence on } \mathbf{e} \\ \text{such that } \mathrm{AVE}(\mathbf{i}) \subseteq \mathbb{E}(\mathfrak{P}) \end{matrix}\right\}.$$

$$U_a^1(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{P}) \cup \left\{\begin{matrix} \mathbf{e} \in \mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P}) \text{where } \mathbf{i} \text{ incidence on } \mathbf{e} \\ \text{for all } \mathrm{AVE}(\mathbf{i}) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset \end{matrix}\right\}.$$

Definition 2.2: Let $\boldsymbol{\Gamma} = (\mathbb{V}(\boldsymbol{\xi}), \mathbb{E}(\boldsymbol{\xi}))$ be an generalized approximation space and $\mathfrak{P} \subseteq \boldsymbol{\xi}$. Then :

(1) The first boundary, positive and negative regions of $\mathfrak{P}$ using incidence vertex edges systems are denoted by: $Bd_i^1(\mathbb{E}(\mathfrak{P}))$, $POS_i^1(\mathbb{E}(\mathfrak{P}))$ and $NEG_i^1(\mathbb{E}(\mathfrak{P}))$ and defined by:

$$Bd_i^1(\mathbb{E}(\mathfrak{P})) = U_i^1(\mathbb{E}(\mathfrak{P})) - L_i^1(\mathbb{E}(\mathfrak{P})).$$
$$POS_i^1(\mathbb{E}(\mathfrak{P})) = L_i^1(\mathbb{E}(\mathfrak{P})).$$
$$NEG_i^1(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\boldsymbol{\xi}) - U_i^1(\mathbb{E}(\mathfrak{P})).$$

(2) The first boundary, positive and negative regions of $\mathfrak{P}$ using non incidence vertex edges systems are denoted by $Bd_n^1(\mathbb{E}(\mathfrak{P}))$, $POS_n^1(\mathbb{E}(\mathfrak{P}))$ and $NEG_n^1(\mathbb{E}(\mathfrak{P}))$ and defined by:

$$Bd_n^1(\mathbb{E}(\mathfrak{P})) = U_n^1(\mathbb{E}(\mathfrak{P})) - L_n^1(\mathbb{E}(\mathfrak{P})).$$
$$POS_n^1(\mathbb{E}(\mathfrak{P})) = L_n^1(\mathbb{E}(\mathfrak{P})).$$
$$NEG_n^1(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\boldsymbol{\xi}) - U_n^1(\mathbb{E}(\mathfrak{P})).$$

(3) The first boundary, positive and negative regions of $\mathfrak{P}$ using admixture vertex edges systems are denoted by: $Bd_a^1(\mathbb{E}(\mathfrak{P}))$, $POS_a^1(\mathbb{E}(\mathfrak{P}))$ and $NEG_a^1(\mathbb{E}(\mathfrak{P}))$ and defined by:

$$Bd_a^1(\mathbb{E}(\mathfrak{P})) = U_a^1(\mathbb{E}(\mathfrak{P})) - L_a^1(\mathbb{E}(\mathfrak{P})).$$
$$POS_a^1(\mathbb{E}(\mathfrak{P})) = L_a^1(\mathbb{E}(\mathfrak{P})).$$
$$NEG_a^1(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\boldsymbol{\xi}) - U_a^1(\mathbb{E}(\mathfrak{P})).$$

Definition 2.3: Let $\boldsymbol{\Gamma} = (\mathbb{V}(\boldsymbol{\xi}), \mathbb{E}(\boldsymbol{\xi}))$ be a generalized approximation space. The first accuracy of the approximation of $\mathfrak{P} \subseteq \boldsymbol{\xi}$ using (incidence, non-incidence and admixture) vertex edges systems are denoted by $(\mathfrak{f}_i^1(\mathbb{E}(\mathfrak{P})), \mathfrak{f}_n^1(\mathbb{E}(\mathfrak{P}))$ and $\mathfrak{f}_a^1(\mathbb{E}(\mathfrak{P})))$ and defined respectively by:

$$\mathfrak{f}_i^1(\mathbb{E}(\mathfrak{P})) = 1 - \frac{|Bd_i^1(\mathbb{E}(\mathfrak{P}))|}{|\mathbb{E}(\boldsymbol{\xi})|},$$

$$\mathfrak{f}_n^1(\mathbb{E}(\mathfrak{P})) = 1 - \frac{|Bd_n^1(\mathbb{E}(\mathfrak{P}))|}{|\mathbb{E}(\boldsymbol{\xi})|},$$

$$\mathfrak{f}_a^1(\mathbb{E}(\mathfrak{P})) = 1 - \frac{|Bd_a^1(\mathbb{E}(\mathfrak{P}))|}{|\mathbb{E}(\boldsymbol{\xi})|}.$$

It is obvious that $0 \leq \mathfrak{f}_i^1(\mathbb{E}(\mathfrak{P})) \leq 1$, $0 \leq \mathfrak{f}_n^1(\mathbb{E}(\mathfrak{P})) \leq 1$ and $0 \leq \mathfrak{f}_a^1(\mathbb{E}(\mathfrak{P})) \leq 1$. Moreover, if $\mathfrak{f}_i^1(\mathbb{E}(\mathfrak{P})) = 1$ or $\mathfrak{f}_n^1(\mathbb{E}(\mathfrak{P})) = 1$ or $\mathfrak{f}_a^1(\mathbb{E}(\mathfrak{P})) = 1$ then $\mathfrak{P}$ is called $\mathfrak{P}$-definable ($\mathfrak{P}$-exact) graph otherwise, it is called $\mathfrak{P}$-rough."

Proposition 2.4: Let $\boldsymbol{\Gamma} = (\mathbb{V}(\boldsymbol{\xi}), \mathbb{E}(\boldsymbol{\xi}))$ be a generalized approximation space $L_a^1(\mathbb{E}(\mathfrak{P})) \subseteq \mathbb{E}(\mathfrak{P}) \subseteq U_a^1(\mathbb{E}(\mathfrak{P}))$.
Proof: Obvious direct from definition (2.1).

Proposition 2.5: Let $\boldsymbol{\Gamma} = (\mathbb{V}(\boldsymbol{\xi}), \mathbb{E}(\boldsymbol{\xi}))$ be a generalized approximation space and $\mathfrak{P} \subseteq \boldsymbol{\xi}$. Then:

(i) $\mathfrak{P}$ is exact graph if and only if $L_a(\mathbb{E}(\mathfrak{P})) = U_a(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{P})$.

(ii) $\mathfrak{P}$ is rough graph if and only if $L_a(\mathbb{E}(\mathfrak{P})) \neq U_a(\mathbb{E}(\mathfrak{P}))$.

Example 2.6: Let $\boldsymbol{\Gamma} = (\mathbb{V}(\boldsymbol{\xi}), \mathbb{E}(\boldsymbol{\xi}))$ be a generalized approximation space, $\boldsymbol{\Gamma}$ induced of figure(2.1) such that $\mathbb{V}(\boldsymbol{\xi}) = \{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3, \mathbf{i}_4\}$, $\mathbb{E}(\boldsymbol{\xi}) = \{\mathbf{e}_1 = (\mathbf{i}_1, \mathbf{i}_1), \mathbf{e}_2 = (\mathbf{i}_1, \mathbf{i}_4), \mathbf{e}_3 = (\mathbf{i}_2, \mathbf{i}_3), \mathbf{e}_4 = (\mathbf{i}_2, \mathbf{i}_4), \mathbf{e}_5 = (\mathbf{i}_3, \mathbf{i}_4)\}$.

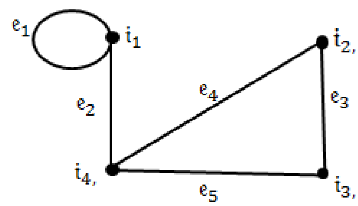

Figure 2.1: graph $\mathfrak{G}$ given in example (2.6).

Then $\mathrm{INVE}(\mathbf{i}_1) = \{\mathbf{e}_1, \mathbf{e}_2\}$, $\mathrm{INVE}(\mathbf{i}_2) = \{\mathbf{e}_3, \mathbf{e}_4\}$, $\mathrm{INVE}(\mathbf{i}_3) = \{\mathbf{e}_3, \mathbf{e}_5\}$, $\mathrm{INVE}(\mathbf{i}_4) = \{\mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5\}$. And $\mathrm{NINVE}(\mathbf{i}_1) = \{\mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$, $\mathrm{NINVE}(\mathbf{i}_2) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_5\}$, $\mathrm{NINVE}(\mathbf{i}_3) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4\}$, $\mathrm{NINVE}(\mathbf{i}_4) = \{\mathbf{e}_1, \mathbf{e}_3\}$. And $\mathrm{AVE}(\mathbf{i}_1) = \{\{\mathbf{e}_1, \mathbf{e}_2\}, \{\mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}\}$, $\mathrm{AVE}(\mathbf{i}_2) = \{\{\mathbf{e}_3, \mathbf{e}_4\}, \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_5\}\}$, $\mathrm{AVE}(\mathbf{i}_3) = \{\{\mathbf{e}_3, \mathbf{e}_5\}, \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4\}\}$, $\mathrm{AVE}(\mathbf{i}_4) = \{\{\mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5\}, \{\mathbf{e}_1, \mathbf{e}_3\}\}$.

Accordingly, can be obtain the following tables

Table 2.1: $L_i^1(\mathbb{E}(\mathfrak{P}))$, $L_n^1(\mathbb{E}(\mathfrak{P}))$ and $L_a^1(\mathbb{E}(\mathfrak{P}))$ for all $\mathfrak{P} \subseteq \xi$

$\mathbb{E}(\mathfrak{P})$	$L_i^1(\mathbb{E}(\mathfrak{P}))$	$L_n^1(\mathbb{E}(\mathfrak{P}))$	$L_a^1(\mathbb{E}(\mathfrak{P}))$
$\{\mathbf{e}_1\}$	ϕ	ϕ	ϕ
$\{\mathbf{e}_2\}$	ϕ	ϕ	ϕ
$\{\mathbf{e}_3\}$	ϕ	ϕ	ϕ
$\{\mathbf{e}_4\}$	ϕ	ϕ	ϕ
$\{\mathbf{e}_5\}$	ϕ	ϕ	ϕ
$\{\mathbf{e}_1, \mathbf{e}_2\}$	$\{\mathbf{e}_1, \mathbf{e}_2\}$	ϕ	$\{\mathbf{e}_1, \mathbf{e}_2\}$
$\{\mathbf{e}_1, \mathbf{e}_3\}$	ϕ	ϕ	ϕ
$\{\mathbf{e}_1, \mathbf{e}_4\}$	ϕ	ϕ	ϕ
$\{\mathbf{e}_1, \mathbf{e}_5\}$	ϕ	ϕ	ϕ
$\{\mathbf{e}_2, \mathbf{e}_3\}$	ϕ	ϕ	ϕ
$\{\mathbf{e}_2, \mathbf{e}_4\}$	ϕ	ϕ	ϕ
$\{\mathbf{e}_2, \mathbf{e}_5\}$	ϕ	ϕ	ϕ
$\{\mathbf{e}_3, \mathbf{e}_4\}$	$\{\mathbf{e}_3, \mathbf{e}_4\}$	ϕ	$\{\mathbf{e}_3, \mathbf{e}_4\}$
$\{\mathbf{e}_3, \mathbf{e}_5\}$	$\{\mathbf{e}_3, \mathbf{e}_5\}$	ϕ	$\{\mathbf{e}_3, \mathbf{e}_5\}$
$\{\mathbf{e}_4, \mathbf{e}_5\}$	ϕ	ϕ	ϕ
$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$	$\{\mathbf{e}_1, \mathbf{e}_2\}$	$\{\mathbf{e}_2\}$	$\{\mathbf{e}_1, \mathbf{e}_2\}$
$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4\}$	$\{\mathbf{e}_1, \mathbf{e}_2\}$	ϕ	$\{\mathbf{e}_1, \mathbf{e}_2\}$
$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_5\}$	$\{\mathbf{e}_1, \mathbf{e}_2\}$	ϕ	$\{\mathbf{e}_1, \mathbf{e}_2\}$
$\{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$	$\{\mathbf{e}_3, \mathbf{e}_4\}$	ϕ	$\{\mathbf{e}_3, \mathbf{e}_4\}$

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$\{e_2, e_3, e_5\}$	$\{e_3, e_5\}$	ϕ	$\{e_3, e_5\}$
$\{e_3, e_4, e_1\}$	$\{e_3, e_4\}$	$\{e_4\}$	$\{e_3, e_4\}$
$\{e_3, e_4, e_5\}$	$\{e_3, e_4, e_5\}$	ϕ	$\{e_3, e_4, e_5\}$
$\{e_4, e_5, e_1\}$	ϕ	ϕ	ϕ
$\{e_4, e_5, e_2\}$	$\{e_4, e_5, e_2\}$	ϕ	$\{e_4, e_5, e_2\}$
$\{e_1, e_3, e_5\}$	$\{e_3, e_5\}$	$\{e_5\}$	$\{e_3, e_5\}$
$\{e_1, e_2, e_3, e_4\}$	$\{e_1, e_2, e_3, e_4\}$	$\{e_2, e_3, e_4\}$	$\{e_1, e_2, e_3, e_4\}$
$\{e_1, e_2, e_3, e_5\}$	$\{e_1, e_2, e_3, e_5\}$	$\{e_2, e_3, e_5\}$	$\{e_1, e_2, e_3, e_5\}$
$\{e_2, e_3, e_4, e_5\}$	$\{e_2, e_3, e_4, e_5\}$	$\{e_2\}$	$\{e_2, e_3, e_4, e_5\}$
$\{e_1, e_3, e_4, e_5\}$	$\{e_3, e_4, e_5\}$	$\{e_1, e_4, e_5\}$	$\{e_1, e_3, e_4, e_5\}$
$\{e_1, e_2, e_4, e_5\}$	$\{e_1, e_2, e_4, e_5\}$	$\{e_4, e_5\}$	$\{e_1, e_2, e_4, e_5\}$
$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
ϕ	ϕ	ϕ	ϕ

Table 2.2: $U_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P}))$, $U_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$ and $U_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))$ for all $\mathfrak{P} \subseteq \xi$.

$\mathbb{E}(\mathfrak{P})$	$\mathrm{U}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P}))$	$\mathrm{U}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$	$\mathrm{U}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))$
$\{e_1\}$	$\{e_1\}$	$\{e_1, e_3, e_4, e_5\}$	$\{e_1\}$
$\{e_2\}$	$\{e_1, e_2\}$	$\{e_2, e_3\}$	$\{e_2\}$
$\{e_3\}$	$\{e_3\}$	$\{e_1, e_2, e_3\}$	$\{e_3\}$
$\{e_4\}$	$\{e_4\}$	$\{e_1, e_4\}$	$\{e_4\}$
$\{e_5\}$	$\{e_5\}$	$\{e_1, e_5\}$	$\{e_5\}$
$\{e_1, e_2\}$	$\{e_1, e_2\}$	$\mathbb{E}(\xi)$	$\{e_1, e_2\}$
$\{e_1, e_3\}$	$\{e_1, e_3\}$	$\mathbb{E}(\xi)$	$\{e_1, e_3\}$
$\{e_1, e_4\}$	$\{e_1, e_4, e_2\}$	$\mathbb{E}(\xi)$	$\{e_1, e_4, e_2\}$
$\{e_1, e_5\}$	$\{e_1, e_5, e_2\}$	$\mathbb{E}(\xi)$	$\{e_1, e_5, e_2\}$
$\{e_2, e_3\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_2, e_4\}$	$\{e_1, e_2, e_4\}$	$\{e_1, e_2, e_3, e_4\}$	$\{e_1, e_2, e_4\}$
$\{e_2, e_5\}$	$\{e_1, e_2, e_5\}$	$\{e_1, e_2, e_3, e_5\}$	$\{e_1, e_2, e_5\}$
$\{e_3, e_4\}$	$\{e_3, e_4, e_5\}$	$\mathbb{E}(\xi)$	$\{e_3, e_4, e_5\}$
$\{e_3, e_5\}$	$\{e_3, e_4, e_5\}$	$\mathbb{E}(\xi)$	$\{e_3, e_4, e_5\}$
$\{e_4, e_5\}$	$\{e_3, e_4, e_5\}$	$\{e_1, e_3, e_4, e_5\}$	$\{e_3, e_4, e_5\}$
$\{e_1, e_2, e_3\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_1, e_2, e_4\}$	$\{e_1, e_2, e_4\}$	$\mathbb{E}(\xi)$	$\{e_1, e_2, e_4\}$
$\{e_1, e_2, e_5\}$	$\{e_1, e_2, e_5\}$	$\mathbb{E}(\xi)$	$\{e_1, e_2, e_5\}$
$\{e_2, e_3, e_4\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_2, e_3, e_5\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_3, e_4, e_1\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_3, e_4, e_5\}$	$\{e_3, e_4, e_5\}$	$\mathbb{E}(\xi)$	$\{e_3, e_4, e_5\}$
$\{e_4, e_5, e_1\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_4, e_5, e_2\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_1, e_3, e_5\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_1, e_2, e_3, e_4\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_1, e_2, e_3, e_5\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_2, e_3, e_4, e_5\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_1, e_3, e_4, e_5\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_1, e_2, e_4, e_5\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
ϕ	ϕ	ϕ	ϕ

Table 2.3: $Bd_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P}))$, $Bd_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$ and $Bd_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))$ for all $\mathfrak{P} \subseteq \xi$.

$\mathbb{E}(\mathfrak{P})$	$\mathrm{Bd}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P}))$	$\mathrm{Bd}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$	$\mathrm{Bd}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))$
$\{e_1\}$	$\{e_1\}$	$\{e_1, e_3, e_4, e_5\}$	$\{e_1\}$
$\{e_2\}$	$\{e_1, e_2\}$	$\{e_2, e_3\}$	$\{e_2\}$
$\{e_3\}$	$\{e_3\}$	$\{e_1, e_2, e_3\}$	$\{e_3\}$
$\{e_4\}$	$\{e_4\}$	$\{e_1, e_4\}$	$\{e_4\}$
$\{e_5\}$	$\{e_5\}$	$\{e_1, e_5\}$	$\{e_5\}$
$\{e_1, e_2\}$	$\emptyset$	$\mathbb{E}(\xi)$	$\emptyset$
$\{e_1, e_3\}$	$\{e_1, e_3\}$	$\mathbb{E}(\xi)$	$\{e_1, e_3\}$
$\{e_1, e_4\}$	$\{e_1, e_4, e_2\}$	$\mathbb{E}(\xi)$	$\{e_1, e_4, e_2\}$
$\{e_1, e_5\}$	$\{e_1, e_5, e_2\}$	$\mathbb{E}(\xi)$	$\{e_1, e_5, e_2\}$
$\{e_2, e_3\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_2, e_4\}$	$\{e_1, e_2, e_4\}$	$\{e_1, e_2, e_3, e_4\}$	$\{e_1, e_2, e_4\}$
$\{e_2, e_5\}$	$\{e_1, e_2, e_5\}$	$\{e_1, e_2, e_3, e_5\}$	$\{e_1, e_2, e_5\}$
$\{e_3, e_4\}$	$\{e_5\}$	$\mathbb{E}(\xi)$	$\{e_5\}$
$\{e_3, e_5\}$	$\{e_4\}$	$\mathbb{E}(\xi)$	$\{e_4\}$
$\{e_4, e_5\}$	$\{e_3, e_4, e_5\}$	$\{e_1, e_3, e_4, e_5\}$	$\{e_3, e_4, e_5\}$
$\{e_1, e_2, e_3\}$	$\{e_3, e_4, e_5\}$	$\{e_1, e_3, e_4, e_5\}$	$\{e_3, e_4, e_5\}$
$\{e_1, e_2, e_4\}$	$\{e_4\}$	$\mathbb{E}(\xi)$	$\{e_4\}$
$\{e_1, e_2, e_5\}$	$\{e_5\}$	$\mathbb{E}(\xi)$	$\{e_5\}$
$\{e_2, e_3, e_4\}$	$\{e_1, e_2, e_5\}$	$\mathbb{E}(\xi)$	$\{e_1, e_2, e_5\}$
$\{e_2, e_3, e_5\}$	$\{e_1, e_2, e_4\}$	$\mathbb{E}(\xi)$	$\{e_1, e_2, e_4\}$
$\{e_3, e_4, e_1\}$	$\{e_1, e_2, e_5\}$	$\{e_1, e_2, e_3, e_5\}$	$\{e_1, e_2, e_5\}$
$\{e_3, e_4, e_5\}$	$\emptyset$	$\mathbb{E}(\xi)$	$\emptyset$
$\{e_4, e_5, e_1\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_4, e_5, e_2\}$	$\{e_1, e_3\}$	$\mathbb{E}(\xi)$	$\{e_1, e_3\}$
$\{e_1, e_3, e_5\}$	$\{e_1, e_2, e_4\}$	$\{e_1, e_2, e_3, e_4\}$	$\{e_1, e_2, e_4\}$
$\{e_1, e_2, e_3, e_4\}$	$\{e_5\}$	$\{e_1, e_5\}$	$\{e_5\}$
$\{e_1, e_2, e_3, e_5\}$	$\{e_4\}$	$\{e_1, e_4\}$	$\{e_4\}$
$\{e_2, e_3, e_4, e_5\}$	$\{e_1\}$	$\{e_1, e_3, e_4, e_5\}$	$\{e_1\}$
$\{e_1, e_3, e_4, e_5\}$	$\{e_1, e_2\}$	$\{e_2, e_3\}$	$\{e_2\}$
$\{e_1, e_2, e_4, e_5\}$	$\{e_3\}$	$\{e_1, e_2, e_3\}$	$\{e_3\}$
$\mathbb{E}(\xi)$	ϕ	ϕ	ϕ
ϕ	ϕ	ϕ	ϕ

Table 2.2: $NEG_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P}))$, $NEG_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$ and $NEG_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))$ for all $\mathfrak{P} \subseteq \xi$.

$\mathbb{E}(\mathfrak{P})$	$\mathrm{NEG}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P}))$	$\mathrm{NEG}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$	$\mathrm{NEG}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))$
$\{e_1\}$	$\{e_2, e_3, e_4, e_5\}$	$\{e_2\}$	$\{e_2, e_3, e_4, e_5\}$
$\{e_2\}$	$\{e_3, e_4, e_5\}$	$\{e_1, e_4, e_5\}$	$\{e_1, e_3, e_4, e_5\}$
$\{e_3\}$	$\{e_1, e_2, e_4, e_5\}$	$\{e_4, e_5\}$	$\{e_1, e_2, e_4, e_5\}$
$\{e_4\}$	$\{e_1, e_2, e_3, e_5\}$	$\{e_2, e_3, e_5\}$	$\{e_1, e_2, e_3, e_5\}$

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$\{\mathbf{e}_5\}$	$\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}$	$\{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}$	$\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}$
$\{\mathbf{e}_1,\mathbf{e}_2\}$	$\{\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_5\}$	$\emptyset$	$\{\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_5\}$
$\{\mathbf{e}_1,\mathbf{e}_3\}$	$\{\mathbf{e}_2,\mathbf{e}_4,\mathbf{e}_5\}$	$\emptyset$	$\{\mathbf{e}_2,\mathbf{e}_4,\mathbf{e}_5\}$
$\{\mathbf{e}_1,\mathbf{e}_4\}$	$\{\mathbf{e}_3,\mathbf{e}_5\}$	$\emptyset$	$\{\mathbf{e}_3,\mathbf{e}_5\}$
$\{\mathbf{e}_1,\mathbf{e}_5\}$	$\{\mathbf{e}_3,\mathbf{e}_4\}$	$\emptyset$	$\{\mathbf{e}_3,\mathbf{e}_4\}$
$\{\mathbf{e}_2,\mathbf{e}_3\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_2,\mathbf{e}_4\}$	$\{\mathbf{e}_3,\mathbf{e}_5\}$	$\{\mathbf{e}_5\}$	$\{\mathbf{e}_3,\mathbf{e}_5\}$
$\{\mathbf{e}_2,\mathbf{e}_5\}$	$\{\mathbf{e}_3,\mathbf{e}_4\}$	$\{\mathbf{e}_4\}$	$\{\mathbf{e}_3,\mathbf{e}_4\}$
$\{\mathbf{e}_3,\mathbf{e}_4\}$	$\{\mathbf{e}_1,\mathbf{e}_2\}$	$\emptyset$	$\{\mathbf{e}_1,\mathbf{e}_2\}$
$\{\mathbf{e}_3,\mathbf{e}_5\}$	$\{\mathbf{e}_1,\mathbf{e}_2\}$	$\emptyset$	$\{\mathbf{e}_1,\mathbf{e}_2\}$
$\{\mathbf{e}_4,\mathbf{e}_5\}$	$\{\mathbf{e}_1,\mathbf{e}_2\}$	$\{\mathbf{e}_2\}$	$\{\mathbf{e}_1,\mathbf{e}_2\}$
$\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_4\}$	$\{\mathbf{e}_3,\mathbf{e}_5\}$	$\emptyset$	$\{\mathbf{e}_3,\mathbf{e}_5\}$
$\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_5\}$	$\{\mathbf{e}_3,\mathbf{e}_4\}$	$\emptyset$	$\{\mathbf{e}_3,\mathbf{e}_4\}$
$\{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_5\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_1\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_5\}$	$\{\mathbf{e}_1,\mathbf{e}_2\}$	$\emptyset$	$\{\mathbf{e}_1,\mathbf{e}_2\}$
$\{\mathbf{e}_4,\mathbf{e}_5,\mathbf{e}_1\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_4,\mathbf{e}_5,\mathbf{e}_2\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_5\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_5\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_5\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_5\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_4,\mathbf{e}_5\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathbb{E}(\boldsymbol{\xi})$	$\emptyset$	$\emptyset$	$\emptyset$
ϕ	$\mathbb{E}(\boldsymbol{\xi})$	$\mathbb{E}(\boldsymbol{\xi})$	$\mathbb{E}(\boldsymbol{\xi})$

Table 2.4: $\mathfrak{f}_{\mathrm{i}}^1(\mathbb{E}(\mathfrak{P}))$, $\mathfrak{f}_{\mathrm{n}}^1(\mathbb{E}(\mathfrak{P}))$ and $\mathfrak{f}_{\mathrm{a}}^1(\mathbb{E}(\mathfrak{P}))$ for all $\mathfrak{P} \subseteq \boldsymbol{\xi}$.

$\mathbb{E}(\mathfrak{P})$	$\mathfrak{f}_{\mathrm{i}}^1(\mathbb{E}(\mathfrak{P}))$	$\mathfrak{f}_{\mathrm{n}}^1(\mathbb{E}(\mathfrak{P}))$	$\mathfrak{f}_{\mathrm{a}}^1(\mathbb{E}(\mathfrak{P}))$
$\{\mathbf{e}_1\}$	4/5	1/5	4/5
$\{\mathbf{e}_2\}$	3/5	3/5	4/5
$\{\mathbf{e}_3\}$	4/5	2/5	4/5
$\{\mathbf{e}_4\}$	4/5	3/5	4/5
$\{\mathbf{e}_5\}$	4/5	3/5	4/5
$\{\mathbf{e}_1,\mathbf{e}_2\}$	1	0	1
$\{\mathbf{e}_1,\mathbf{e}_3\}$	3/5	0	3/5
$\{\mathbf{e}_1,\mathbf{e}_4\}$	2/5	0	2/5
$\{\mathbf{e}_1,\mathbf{e}_5\}$	2/5	0	2/5
$\{\mathbf{e}_2,\mathbf{e}_3\}$	0	0	0
$\{\mathbf{e}_2,\mathbf{e}_4\}$	2/5	1/5	2/5
$\{\mathbf{e}_2,\mathbf{e}_5\}$	2/5	1/5	2/5
$\{\mathbf{e}_3,\mathbf{e}_4\}$	4/5	0	4/5
$\{\mathbf{e}_3,\mathbf{e}_5\}$	4/5	0	4/5
$\{\mathbf{e}_4,\mathbf{e}_5\}$	2/5	1/5	2/5
$\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$	2/5	1/5	2/5
$\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_4\}$	4/5	0	4/5
$\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_5\}$	4/5	0	4/5
$\{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}$	2/5	0	2/5
$\{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_5\}$	2/5	0	2/5
$\{\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_1\}$	2/5	1/5	2/5
$\{\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_5\}$	1	0	1
$\{\mathbf{e}_4,\mathbf{e}_5,\mathbf{e}_1\}$	0	0	0
$\{\mathbf{e}_4,\mathbf{e}_5,\mathbf{e}_2\}$	3/5	0	3/5
$\{\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_5\}$	2/5	1/5	2/5
$\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}$	4/5	3/5	4/5
$\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_5\}$	4/5	3/5	4/5
$\{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_5\}$	4/5	1/5	4/5
$\{\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_5\}$	3/5	3/5	4/5
$\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_4,\mathbf{e}_5\}$	4/5	2/5	4/5
$\mathbb{E}(\boldsymbol{\xi})$	1	1	1
ϕ	1	1	1

Theorem 2.7: Let $\boldsymbol{\Gamma} = (\mathbb{V}(\boldsymbol{\xi}), \mathbb{E}(\boldsymbol{\xi}))$ be a generalized approximation space and $\mathfrak{P} \subseteq \boldsymbol{\xi}$. Then:

(1) $\mathrm{L}_{\mathrm{a}}^1(\mathbb{E}(\mathfrak{P})) = \mathrm{L}_{\mathrm{i}}^1(\mathbb{E}(\mathfrak{P})) \cup \mathrm{L}_{\mathrm{n}}^1\big(\mathbb{E}(\mathfrak{P})\big)$.

(2) $\mathrm{U}_{\mathrm{a}}^1(\mathbb{E}(\mathfrak{P})) = \mathrm{U}_{\mathrm{i}}^1(\mathbb{E}(\mathfrak{P})) \cap \mathrm{U}_{\mathrm{n}}^1(\mathbb{E}(\mathfrak{P}))$.

(3) $\mathrm{Bd}_{\mathrm{a}}^1(\mathbb{E}(\mathfrak{P})) = \mathrm{Bd}_{\mathrm{i}}^1(\mathbb{E}(\mathfrak{P})) \cap \mathrm{Bd}_{\mathrm{n}}^1(\mathbb{E}(\mathfrak{P}))$.

(4) $\mathrm{NEG}_{\mathrm{a}}^1(\mathbb{E}(\mathfrak{P})) = \mathrm{NEG}_{\mathrm{i}}^1(\mathbb{E}(\mathfrak{P})) \cup \mathrm{NEG}_{\mathrm{n}}^1(\mathbb{E}(\mathfrak{P}))$.

(5) $\mathfrak{f}_{\mathrm{a}}^1(\mathbb{E}(\mathfrak{P})) \geq \max\{\mathfrak{f}_{\mathrm{i}}^1(\mathbb{E}(\mathfrak{P})), \mathfrak{f}_{\mathrm{n}}^1(\mathbb{E}(\mathfrak{P}))\}$.

Proof :

(1) Let $\mathbf{e} \in \mathrm{L}_{\mathrm{a}}^1\big(\mathbb{E}(\mathfrak{P})\big) \Longrightarrow \exists\ \mathrm{AVE}(\mathfrak{i}) \subseteq \mathbb{E}(\mathfrak{P})$ such that $\mathfrak{i}$ incidence on $\mathbf{e} \Longrightarrow$
$\mathrm{INVE}(\mathfrak{i}) \subseteq \mathbb{E}(\mathfrak{P}) \ \vee\ \mathrm{NINVE}(\mathfrak{i}) \subseteq \mathbb{E}(\mathfrak{P}) \Longrightarrow$
$\mathbf{e} \in \mathrm{L}_{\mathrm{i}}^1\big(\mathbb{E}(\mathfrak{P})\big) \ \vee\ \mathbf{e} \in \mathrm{L}_{\mathrm{n}}^1\big(\mathbb{E}(\mathfrak{P})\big) \Longrightarrow$
$\mathbf{e} \in [\mathrm{L}_{\mathrm{i}}^1(\mathbb{E}(\mathfrak{P})) \cup \mathrm{L}_{\mathrm{n}}^1(\mathbb{E}(\mathfrak{P}))]$, thus
$\mathrm{L}_{\mathrm{a}}^1(\mathbb{E}(\mathfrak{P})) \subseteq [\mathrm{L}_{\mathrm{i}}^1(\mathbb{E}(\mathfrak{P})) \cup \mathrm{L}_{\mathrm{n}}^1(\mathbb{E}(\mathfrak{P}))] - - - (1)$.
Let $\mathbf{e} \in [\mathrm{L}_{\mathrm{i}}^1(\mathbb{E}(\mathfrak{P})) \cup \mathrm{L}_{\mathrm{n}}^1(\mathbb{E}(\mathfrak{P}))] \Longrightarrow \mathbf{e} \in \mathrm{L}_{\mathrm{i}}^1(\mathbb{E}(\mathfrak{P})) \ \vee\ \mathbf{e} \in \mathrm{L}_{\mathrm{n}}^1(\mathbb{E}(\mathfrak{P})) \Longrightarrow$ there exist $\mathfrak{i}$ incidence on $\mathbf{e}$ such that $\mathrm{INVE}(\mathfrak{i}) \subseteq \mathbb{E}(\mathfrak{P}) \vee \mathrm{NINVE}(\mathfrak{i}) \subseteq \mathbb{E}(\mathfrak{P}) \Longrightarrow$
there exist $\mathfrak{i}$ incidence on $\mathbf{e}$ such that $\mathrm{AVE}(\mathfrak{i}) \subseteq \mathbb{E}(\mathfrak{P})$
$\Longrightarrow\ \mathbf{e} \in \mathrm{L}_{\mathrm{a}}^1\big(\mathbb{E}(\mathfrak{P})\big)$, thus
$[\mathrm{L}_{\mathrm{i}}^1(\mathbb{E}(\mathfrak{P})) \cup \mathrm{L}_{\mathrm{n}}^1(\mathbb{E}(\mathfrak{P}))] \subseteq \mathrm{L}_{\mathrm{a}}^1\big(\mathbb{E}(\mathfrak{P})\big) - - - (2)$.
From (1) and (2) we get $\mathrm{L}_{\mathrm{a}}^1(\mathbb{E}(\mathfrak{P})) = \mathrm{L}_{\mathrm{i}}^1(\mathbb{E}(\mathfrak{P})) \cup \mathrm{L}_{\mathrm{n}}^1\big(\mathbb{E}(\mathfrak{P})\big)$.

(2) let $\mathbf{e} \in \mathrm{U}_{\mathrm{a}}^1(\mathbb{E}(\mathfrak{P}))$, then there are two cases :
Case1: $\mathbf{e} \in \mathbb{E}(\mathfrak{P}) \ \Longrightarrow\ \mathbf{e} \in \mathrm{U}_{\mathrm{i}}^1\big(\mathbb{E}(\mathfrak{P})\big) \ \wedge\ \mathbf{e} \in\ \mathrm{U}_{\mathrm{n}}^1\big(\mathbb{E}(\mathfrak{P})\big) \Longrightarrow \mathbf{e} \in \mathrm{U}_{\mathrm{i}}^1\big(\mathbb{E}(\mathfrak{P})\big) \cap \mathrm{U}_{\mathrm{n}}^1\big(\mathbb{E}(\mathfrak{P})\big)$. Case2: $\mathbf{e} \in \mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P})$.
Since $\mathbf{e} \in \mathrm{U}_{\mathrm{a}}^1\big(\mathbb{E}(\mathfrak{P})\big) \Longrightarrow \forall\ \mathrm{AVE}(\mathfrak{i})$; where $\mathfrak{i}$ incidence on $\mathbf{e}$ $\Longrightarrow \mathrm{AVE}(\mathfrak{i}) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset \Longrightarrow (\mathrm{INVE}(\mathfrak{i}) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset) \wedge (\mathrm{NINVE}(\mathfrak{i}) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset) \Longrightarrow \mathbf{e} \in \mathrm{U}_{\mathrm{i}}^1\big(\mathbb{E}(\mathfrak{P})\big) \ \wedge \mathbf{e} \in$

$\mathrm{U}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathfrak{e} \in \mathrm{U}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cap \mathrm{U}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$. So in case1 and case2 $\mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \subseteq \mathrm{L}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cup \mathrm{L}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P})) - - - (1)$.

Conversely, $\mathfrak{e} \in (\mathrm{L}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cap \mathrm{U}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P})))$, then there are two cases: Case1: $\mathfrak{e} \in \mathbb{E}(\mathfrak{P}) \Rightarrow \mathfrak{e} \in \mathrm{U}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))$.

Case2: $\in \mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P})$. Since $\mathfrak{e} \in \left(\mathrm{L}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cap \mathrm{U}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))\right) \Rightarrow (\mathrm{INVE}(\mathfrak{i}) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset) \wedge (\mathrm{NINVE}(\mathfrak{i}) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset)$ where $\mathfrak{i}$ incidence on $\mathfrak{e} \Rightarrow$ for all $\mathrm{AVE}(\mathfrak{i})$, $\mathrm{AVE}(\mathfrak{i}) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset \Rightarrow \mathfrak{e} \in \mathrm{U}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))$. So in case1 and case2 $\mathrm{L}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cup \mathrm{L}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P})) \subseteq \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) - - - (2)$.

Consequently, $\mathrm{U}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) = \mathrm{U}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cap \mathrm{U}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$.

(3) Let $\mathfrak{e} \in \mathrm{Bd}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathfrak{e} \in \mathrm{U}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \wedge \mathfrak{e} \notin \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))$ since $\mathfrak{e} \in \mathrm{U}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))$ by (2) above we get $\mathfrak{e} \in \left(\mathrm{U}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cap \mathrm{U}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))\right) \Rightarrow \mathfrak{e} \in \mathrm{U}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P}))$ and $\mathfrak{e} \in \mathrm{U}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$. Since $\mathfrak{e} \notin \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))$ by (1) above we get $\mathfrak{e} \notin \left(\mathrm{L}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cup \mathrm{L}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))\right) \Rightarrow \mathfrak{e} \notin \mathrm{L}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P}))$ and $\mathfrak{e} \notin \mathrm{L}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$ so $\mathfrak{e} \in \mathrm{U}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P}))$ and $\mathfrak{e} \notin \mathrm{L}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \wedge \mathfrak{e} \in \mathrm{U}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$ and $\mathfrak{e} \notin \mathrm{L}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$ hence $\mathfrak{e} \in \mathrm{Bd}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \wedge \mathfrak{e} \in \mathrm{Bd}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathfrak{e} \in \left(\mathrm{Bd}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cap \mathrm{Bd}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))\right)$. thus $\mathrm{Bd}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \subseteq \mathrm{Bd}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cap \mathrm{Bd}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P})) - - - (1)$.

Conversely, $\mathfrak{e} \in \left(\mathrm{Bd}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cap \mathrm{Bd}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))\right) \Rightarrow \mathfrak{e} \in \mathrm{Bd}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \wedge \mathfrak{e} \in \mathrm{Bd}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$, since $\mathfrak{e} \in \mathrm{Bd}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathfrak{e} \in \mathrm{U}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P}))$ and $\mathfrak{e} \notin \mathrm{L}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P}))$ and since $\mathfrak{e} \in \mathrm{Bd}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathfrak{e} \in \mathrm{U}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$ and $\mathfrak{e} \notin \mathrm{L}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$ and hence $\mathfrak{e} \in \left(\mathrm{U}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cap \mathrm{U}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))\right)$ by (2) above we get $\mathfrak{e} \in \mathrm{U}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))$ and $\mathfrak{e} \notin \left(\mathrm{L}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cup \mathrm{L}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))\right)$ by (1) above we get $\mathfrak{e} \notin \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))$ then $\mathfrak{e} \in \mathrm{Bd}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))$. thus $\mathrm{Bd}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cap \mathrm{Bd}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P})) \subseteq \mathrm{Bd}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) - - - (2)$.

Consequently, $\mathrm{Bd}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) = \mathrm{Bd}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cap \mathrm{Bd}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$.

(4) Let $\mathfrak{e} \in \mathrm{NEG}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathfrak{e} \notin \mathrm{U}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))$ by (2) above we $\mathfrak{e} \notin \left[\mathrm{U}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cap \mathrm{U}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))\right] \Rightarrow \mathfrak{e} \notin \mathrm{U}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \vee \mathfrak{e} \notin \mathrm{U}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathfrak{e} \in \mathrm{NEG}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \vee \mathfrak{e} \in \mathrm{NEG}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathfrak{e} \in \mathrm{NEG}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cup \mathrm{NEG}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$. Thus $\mathrm{NEG}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \subseteq \mathrm{NEG}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cup \mathrm{NEG}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P})) - - - (1)$.

Let $\mathfrak{e} \in [\mathrm{NEG}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cup \mathrm{NEG}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))]$
$\Rightarrow \mathfrak{e} \in \mathrm{NEG}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \vee \mathfrak{e} \in \mathrm{NEG}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P})) \Rightarrow$
$\mathfrak{e} \notin \mathrm{U}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \vee \mathfrak{e} \notin \mathrm{U}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$
$\Rightarrow \mathfrak{e} \notin [\mathrm{U}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cap \mathrm{U}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))]$ by (2) above we get $\mathfrak{e} \notin \mathrm{U}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathfrak{e} \in \mathrm{NEG}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))$, thus $\mathrm{NEG}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cup \mathrm{NEG}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P})) \subseteq \mathrm{NEG}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) - - - (2)$.

From (1) and (2) we get $\mathrm{NEG}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) = \mathrm{NEG}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})) \cup \mathrm{NEG}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$.

(5) By (3) above we get $\mathrm{Bd}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \subseteq \mathrm{Bd}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P}))$ and hence $\left|\mathrm{Bd}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))\right| \leq \left|\mathrm{Bd}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P}))\right| \Rightarrow \frac{\left|\mathrm{Bd}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))\right|}{|\mathbb{E}(\boldsymbol{\xi})|} \leq \frac{\left|\mathrm{Bd}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P}))\right|}{|\mathbb{E}(\boldsymbol{\xi})|} \Rightarrow 1 - \frac{\left|\mathrm{Bd}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))\right|}{|\mathbb{E}(\boldsymbol{\xi})|} \geq 1 - \frac{\left|\mathrm{Bd}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P}))\right|}{|\mathbb{E}(\boldsymbol{\xi})|} \Rightarrow$ $\mathfrak{f}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \geq \mathfrak{f}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P}))$. In the same way we get $\mathfrak{f}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \geq \mathfrak{f}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))$, thus $\mathfrak{f}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \geq \max\left\{\mathfrak{f}_{\mathrm{i}}^{1}(\mathbb{E}(\mathfrak{P})), \mathfrak{f}_{\mathrm{n}}^{1}(\mathbb{E}(\mathfrak{P}))\right\}$.

Proposition2.8: (Lower 1 properties)

Let $\boldsymbol{\Gamma} = (\mathbb{V}(\boldsymbol{\xi}), \mathrm{E}(\xi))$ be a generalized approximation space and $\mathfrak{P}, \mathrm{K} \subseteq \boldsymbol{\xi}$. Then:

$(\mathbf{L_1^1})$ $\mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \subseteq \mathbb{E}(\mathfrak{P})$.

$(\mathbf{L_2^1})$ $\mathrm{L}_{\mathrm{a}}^{1}(\mathrm{E}(\xi)) = \mathrm{E}(\xi)$.

$(\mathbf{L_3^1})$ $\mathrm{L}_{\mathrm{a}}^{1}(\emptyset) = \emptyset$.

$(\mathbf{L_4^1})$ If $\mathbb{E}(\mathfrak{P}) \subseteq \mathbb{E}(\mathrm{K})$,then $\mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \subseteq \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathrm{K}))$.

$(\mathbf{L_5^1})$ $\mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(\mathrm{K})) \subseteq \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \cap \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathrm{K}))$.

$(\mathbf{L_6^1})$ $\mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \cup \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathrm{K})) \subseteq \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(\mathrm{K}))$.

$(\mathbf{L_7^1})$ $\mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\boldsymbol{\xi}) - \left[\mathrm{U}_{\mathrm{a}}^{1}(\mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P}))\right]$.

Proof:

The proof $(\mathbf{L_1^1})$,$(\mathbf{L_2^1})$ and $(\mathbf{L_3^1})$ from definition (2.1(3)).

$(\mathbf{L_4^1})$ Let $\mathbb{E}(\mathfrak{P}) \subseteq \mathbb{E}(\mathrm{K})$ and $\mathfrak{e} \in \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))$, then $\exists\ \mathfrak{i}$ incidence on $\mathfrak{e}$ such that $\mathrm{AVE}(\mathfrak{i}) \subseteq \mathbb{E}(\mathfrak{P})$ so $\mathfrak{e} \in \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \subseteq \mathbb{E}(\mathfrak{P}) \subseteq \mathbb{E}(\mathrm{K})$. Thus we have $\mathfrak{e} \in \mathbb{E}(\mathrm{K})$ and there exist $\mathfrak{i}$ incidence on $\mathfrak{e}$ such that $\mathrm{AVE}(\mathfrak{i}) \subseteq \mathbb{E}(\mathfrak{P}) \subseteq \mathbb{E}(\mathrm{K})$. Hence, $\mathfrak{e} \in \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathrm{K}))$ and thus $\mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \subseteq \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathrm{K}))$.

$(\mathbf{L_5^1})$ Since $(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(\mathrm{K})) \subseteq \mathbb{E}(\mathfrak{P})$ by $(\mathrm{L_4^1})$ above we get $\mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(\mathrm{K})) \subseteq \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) - - - (1)$. And since $(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(\mathrm{K})) \subseteq \mathbb{E}(\mathrm{K})$ by $(\mathrm{L_4^1})$ above we get $\mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(\mathrm{K})) \subseteq \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathrm{K})) - - - (2)$. From (1) and (2) we get $\mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(\mathrm{K})) \subseteq \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \cap \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathrm{K}))$.

$(\mathbf{L_6^1})$ Since $\mathbb{E}(\mathfrak{P}) \subseteq (\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(\mathrm{K}))$ by $(\mathrm{L_4^1})$ above we get $\mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \subseteq \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(\mathrm{K})) - - - (1)$. And since $\mathbb{E}(\mathrm{K}) \subseteq (\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(\mathrm{K}))$ by $(\mathrm{L_4^1})$ above we get $\mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathrm{K})) \subseteq \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(\mathrm{K})) - - - (2)$. From (1) and (2) we get $\mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \cup \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathrm{K})) \subseteq \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(\mathrm{K}))$.

$(\mathbf{L_7^1})$ Let $\mathfrak{e} \in \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \Rightarrow$ there exist $\mathfrak{i}$ incidence on $\mathfrak{e}$ such that $\mathrm{AVE}(\mathfrak{i}) \subseteq \mathbb{E}(\mathfrak{P})$ thus $\mathrm{AVE}(\mathfrak{i}) \cap [\mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P})] = \emptyset$ and $\mathfrak{e} \in \mathbb{E}(\boldsymbol{\xi}) - [\mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P})]$ by definition (4.1.1(3)) we get $\mathfrak{e} \notin \mathrm{U}_{\mathrm{a}}^{1}(\mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P}))$, thus

$\mathfrak{e} \in \mathbb{E}(\boldsymbol{\xi}) - \mathrm{U}_{\mathrm{a}}^{1}(\mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P}))$. Hence

$\Rightarrow \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \subseteq \mathbb{E}(\boldsymbol{\xi}) - \mathrm{U}_{\mathrm{a}}^{1}(\mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P})) - - - (1)$.

$\mathfrak{e} \in \mathbb{E}(\boldsymbol{\xi}) - \mathrm{U}_{\mathrm{a}}^{1}(\mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P})) \Rightarrow \mathfrak{e} \in \mathbb{E}(\boldsymbol{\xi})$ and $\mathfrak{e} \notin \mathrm{U}_{\mathrm{a}}^{1}(\mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P})) \Rightarrow \mathfrak{e} \in \mathbb{E}(\boldsymbol{\xi}) - [\mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P})]$ and there exist $\mathfrak{i}$ incidence on $\mathfrak{e}$ such that $\mathrm{AVE}(\mathfrak{i}) \cap [\mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P})] = \emptyset$ then $\mathrm{AVE}(\mathfrak{i}) \subseteq \mathbb{E}(\mathfrak{P}) \Rightarrow \mathfrak{e} \in \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) \Rightarrow$ $\mathbb{E}(\boldsymbol{\xi}) - \mathrm{U}_{\mathrm{a}}^{1}(\mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P})) \subseteq \mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) - - - (2)$. From (1) and (2) we get $\mathrm{L}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\boldsymbol{\xi}) - \left[\mathrm{U}_{\mathrm{a}}^{1}(\mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P}))\right]$.

Proposition 2.9: (Upper 1 properties)

Let $\boldsymbol{\Gamma} = (\mathbb{V}(\boldsymbol{\xi}), \mathrm{E}(\xi))$ be a generalized approximation space and $\mathfrak{P}, \mathrm{K} \subseteq \boldsymbol{\xi}$. Then:

$(\mathbf{U_1^1})$ $\mathbb{E}(\mathfrak{P}) \subseteq \mathrm{U}_{\mathrm{a}}^{1}(\mathbb{E}(\mathfrak{P}))$.

$(\mathbf{U_2^1})$ $\mathrm{U}_{\mathrm{a}}^{1}(\mathrm{E}(\xi)) = \mathrm{E}(\xi)$.

$(\mathbf{U_3^1})$ $U_a^1(\emptyset) = \emptyset$.

$(\mathbf{U_4^1})$ If $\mathbb{E}(\mathfrak{P}) \subseteq \mathbb{E}(K)$, then $U_a^1(\mathbb{E}(\mathfrak{P})) \subseteq U_a^1(\mathbb{E}(K))$.

$(\mathbf{U_5^1})$ $U_a^1(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) \subseteq U_a^1(\mathbb{E}(\mathfrak{P})) \cap U_a^1(\mathbb{E}(K))$.

$(\mathbf{U_6^1})$ $U_a^1(\mathbb{E}(\mathfrak{P})) \cup U_a^1(\mathbb{E}(K)) \subseteq U_a^1(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K))$.

$(\mathbf{U_7^1})$ $U_a^1(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\boldsymbol{\xi}) - [L_a^1(\mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P}))]$.

Proof :

The proof $(\mathbf{U_1^1})$, $(\mathbf{U_2^1})$ and $(\mathbf{U_3^1})$ from definition (2.1(3)).

$(\mathbf{U_4^1})$ Let $\mathbb{E}(\mathfrak{P}) \subseteq \mathbb{E}(K)$ and $\mathfrak{e} \in U_a^1(\mathbb{E}(\mathfrak{P}))$, we have:

Case1: If $\mathfrak{e} \in \mathbb{E}(\mathfrak{P}) \Rightarrow \mathfrak{e} \in \mathbb{E}(\mathfrak{P}) \subseteq \mathbb{E}(K) \Rightarrow \mathfrak{e} \in \mathbb{E}(K) \Rightarrow \mathfrak{e} \in U_a^1(\mathbb{E}(K))$.

Case2: If $\mathfrak{e} \in \mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P})$. Since $\mathfrak{e} \in U_a^1(\mathbb{E}(\mathfrak{P})) \Rightarrow \forall\, AVE(\mathfrak{i})$ where $\mathfrak{i}$ incidence on $\mathfrak{e}$ then $AVE(\mathfrak{i}) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset$ and since $\mathbb{E}(\mathfrak{P}) \subseteq \mathbb{E}(K)$
$\Rightarrow \forall\, AVE(\mathfrak{i})$: $AVE(\mathfrak{i}) \cap \mathbb{E}(K) \neq \emptyset - - - (*)$ and hence

(i) If $\mathfrak{e} \in \mathbb{E}(K) - \mathbb{E}(\mathfrak{P}) \Rightarrow \mathfrak{e} \in \mathbb{E}(K) \Rightarrow \mathfrak{e} \in U_a^1(\mathbb{E}(K))$.

(ii) If $\mathfrak{e} \in \mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(K)$ and by $(*)$ we get $\mathfrak{e} \in U_a^1(\mathbb{E}(K))$. Thus
$U_a^1(\mathbb{E}(\mathfrak{P})) \subseteq U_a^1(\mathbb{E}(K))$.

$(\mathbf{U_5^1})$ Since $(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) \subseteq \mathbb{E}(\mathfrak{P})$ by (U_4^1) above we get $U_a^1(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) \subseteq U_a^1(\mathbb{E}(\mathfrak{P})) - - - (1)$.And since $(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) \subseteq \mathbb{E}(K)$ by (U_4^1) above we get $U_a^1(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) \subseteq U_a^1(\mathbb{E}(K)) - - - (2)$. From (1) and (2) we get $U_a^1(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) \subseteq U_a^1(\mathbb{E}(\mathfrak{P})) \cap U_a^1(\mathbb{E}(K))$.

$(\mathbf{U_6^1})$ Since $\mathbb{E}(\mathfrak{P}) \subseteq (\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K))$ by (U_4^1) above we get $U_a^1(\mathbb{E}(\mathfrak{P})) \subseteq U_a^1(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) - - - (1)$. And since $\mathbb{E}(K) \subseteq (\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K))$ by (U_4^1) above we get $U_a^1(\mathbb{E}(K)) \subseteq U_a^1(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) - - - (2)$. From (1) and (2) we get $U_a^1(\mathbb{E}(\mathfrak{P})) \cup U_a^1(\mathbb{E}(K)) \subseteq U_a^1(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K))$.

$(\mathbf{U_7^1})$ By Lower 1 properties (L_7^1)

$L_a^1(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\boldsymbol{\xi}) - [U_a^1(\mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P}))] \Rightarrow$

$$\mathbb{E}(\boldsymbol{\xi}) - L_a^1(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\boldsymbol{\xi}) - (\mathbb{E}(\boldsymbol{\xi}) - [U_a^1(\mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P}))])$$

$\Rightarrow \mathbb{E}(\boldsymbol{\xi}) - L_a^1(\mathbb{E}(\mathfrak{P})) = U_a^1(\mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P}))$. Now we replace $\mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P})$ for $\mathbb{E}(\mathfrak{P})$ we get $U_a^1(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\boldsymbol{\xi}) - L_a^1(\mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P}))$.

Proposition2.10: Let $\boldsymbol{\Gamma} = (\mathbb{V}(\boldsymbol{\xi}), E(\xi))$ be a generalized approximation space and $\mathfrak{P}, K \subseteq \boldsymbol{\xi}$. Then:

(1) $NEG_a^1(\emptyset) = \mathbb{E}(\boldsymbol{\xi})$.

(2) $NEG_a^1(\mathbb{E}(\mathfrak{P})) \cap NEG_a^1(\mathbb{E}(K)) \subseteq NEG_a^1(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K))$.

(3) $NEG_a^1(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) \subseteq NEG_a^1(\mathbb{E}(\mathfrak{P})) \cup NEG_a^1(\mathbb{E}(K))$.

(4) $Bd_a^1(\emptyset) = \emptyset$.

Proof:

(1) By definition $(2.2.(3))$ we get $NEG_a^1(\emptyset) = \mathbb{E}(\boldsymbol{\xi}) - U_a^1(\emptyset)$ and from Upper[1] properties (U_3^1) we get $NEG_a^1(\emptyset) = \mathbb{E}(\boldsymbol{\xi})$.

(2) Let $\mathfrak{e} \in [NEG_a^1(\mathbb{E}(\mathfrak{P})) \cap NEG_a^1(\mathbb{E}(K))]$ $\Rightarrow \mathfrak{e} \in NEG_a^1(\mathbb{E}(\mathfrak{P})) \wedge \mathfrak{e} \in NEG_a^1(\mathbb{E}(K)) \Rightarrow \mathfrak{e} \notin U_a^1(\mathbb{E}(\mathfrak{P})) \wedge \mathfrak{e} \notin U_a^1(\mathbb{E}(K)) \Rightarrow \mathfrak{e} \notin [U_a^1(\mathbb{E}(\mathfrak{P})) \cap U_a^1(\mathbb{E}(K))] \Rightarrow \mathfrak{e} \notin \mathbb{E}(\mathfrak{P})$, there exist $\mathfrak{i}$ incidence on $\mathfrak{e}$ such that $AVE(\mathfrak{i}) \cap \mathbb{E}(\mathfrak{P}) = \emptyset$ and $\mathfrak{e} \notin \mathbb{E}(K)$, there exist $\mathfrak{i}$ incidence on $\mathfrak{e}$ such that $AVE(\mathfrak{i}) \cap \mathbb{E}(K) = \emptyset$,
since $(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) \subseteq \mathbb{E}(\mathfrak{P}) \Rightarrow AVE(\mathfrak{i}) \cap (\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) = \emptyset$ also $(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) \subseteq \mathbb{E}(K) \Rightarrow AVE(\mathfrak{i}) \cap (\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) = \emptyset$ since $\mathfrak{e} \notin (\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K))$ then $\mathfrak{e} \notin U_a^1(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K))$ and hence $\mathfrak{e} \in NEG_a^1(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K))$, thus$NEG_a^1(\mathbb{E}(\mathfrak{P})) \cap NEG_a^1(\mathbb{E}(K)) \subseteq NEG_a^1(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K))$.

(3) Let $\mathfrak{e} \in NEG_a^1(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) \Rightarrow \mathfrak{e} \notin U_a^1(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) \Rightarrow \mathfrak{e} \notin (\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K))$ and there exist $\mathfrak{i}$ incidence on $\mathfrak{e}$ such that $AVE(\mathfrak{i}) \cap (\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) = \emptyset$, so $\mathfrak{e} \notin \mathbb{E}(\mathfrak{P}) \wedge \mathfrak{e} \notin \mathbb{E}(K)$ and since $\mathbb{E}(\mathfrak{P}) \subseteq (\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) \Rightarrow AVE(\mathfrak{i}) \cap \mathbb{E}(\mathfrak{P}) = \emptyset \Rightarrow \mathfrak{e} \notin U_a^1(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathfrak{e} \in NEG_a^1(\mathbb{E}(\mathfrak{P}))$.
Or since $\mathbb{E}(K) \subseteq (\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) \Rightarrow AVE(\mathfrak{i}) \cap \mathbb{E}(K) = \emptyset \Rightarrow \mathfrak{e} \notin U_a^1(\mathbb{E}(K)) \Rightarrow \mathfrak{e} \in NEG_a^1(\mathbb{E}(K))$ Thus $\mathfrak{e} \in NEG_a^1(\mathbb{E}(\mathfrak{P})) \cup NEG_a^1(\mathbb{E}(K))$.
So, $NEG_a^1(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) \subseteq NEG_a^1(\mathbb{E}(\mathfrak{P})) \cup NEG_a^1(\mathbb{E}(K))$.

(4) Since $Bd_a^1(\emptyset) = U_a^1(\emptyset) - L_a^1(\emptyset)$ from Lower[1] properties (L_3^1) and Upper[1] properties (U_3^1) we get $Bd_a^1(\emptyset) = \emptyset$.

Proposition2.11: Let $\boldsymbol{\Gamma} = (\mathbb{V}(\boldsymbol{\xi}), E(\xi))$ be a generalized approximation space, if $\boldsymbol{\Gamma}$ induced of is antisymmetric graph $\boldsymbol{\xi}$ and $\mathfrak{P}, K \subseteq \boldsymbol{\xi}$. Then:

(1) $L_a^1(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{P})$.

(2) $L_a^1(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) = L_a^1(\mathbb{E}(\mathfrak{P})) \cap L_a^1(\mathbb{E}(K))$.

(3) $L_a^1(\mathbb{E}(\mathfrak{P})) \cup L_a^1(\mathbb{E}(K)) = L_a^1(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K))$.

(4) $U_a^1(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{P})$.

(5) $U_a^1(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) = U_a^1(\mathbb{E}(\mathfrak{P})) \cap U_a^1(\mathbb{E}(K))$.

(6) $U_a^1(\mathbb{E}(\mathfrak{P})) \cup U_a^1(\mathbb{E}(K)) = U_a^1(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K))$.

(7) $L_a^1(\mathbb{E}(\mathfrak{P})) = U_a^1(\mathbb{E}(\mathfrak{P}))$.

(8) $Bd_a^1(\mathbb{E}(\mathfrak{P})) = \emptyset$.

(9) $NEG_a^1(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P})$.

Proof :

Let $\boldsymbol{\xi}$ be an antisymmetric graph and $\mathfrak{P}, K \subseteq \boldsymbol{\xi}$:

(1) By Lower[1] properties (L_1^1) we get $L_a^1(\mathbb{E}(\mathfrak{P})) \subseteq \mathbb{E}(\mathfrak{P}) - - - (1)$. Let $\mathfrak{e} \in \mathbb{E}(\mathfrak{P})$ and $\mathfrak{e} \notin L_a^1(\mathbb{E}(\mathfrak{P})) \Rightarrow \forall\, AVE(\mathfrak{i}) \nsubseteq \mathbb{E}(\mathfrak{P})$ where $\mathfrak{i}$ incidence on $\mathfrak{e}$ and this contradiction because $\boldsymbol{\xi}$ is antisymmetric $[\forall\, \mathfrak{e} \in \mathbb{E}(\mathfrak{P}) \Rightarrow INVE(\mathfrak{i}) \subseteq \mathbb{E}(\mathfrak{P})$ where $\mathfrak{i}$ incidence on $\mathfrak{e}$ $]$ and hence $\mathfrak{e} \in L_a^1(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathbb{E}(\mathfrak{P}) \subseteq L_a^1(\mathbb{E}(\mathfrak{P})) - - - (2)$. From (1) and (2) we get $L_a^1(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{P})$.

(2) By Lower1 properties (L_5^1) we get $L_a^1(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) \subseteq L_a^1(\mathbb{E}(\mathfrak{P})) \cap L_a^1(\mathbb{E}(K)) - - - (1)$. Let $\mathfrak{e} \in [L_a^1(\mathbb{E}(\mathfrak{P})) \cap L_a^1(\mathbb{E}(K))] \Rightarrow \mathfrak{e} \in L_a^1(\mathbb{E}(\mathfrak{P})) \wedge \mathfrak{e} \in L_a^1(\mathbb{E}(K))$ by (1) above we get $\mathfrak{e} \in \mathbb{E}(\mathfrak{P}) \wedge \mathfrak{e} \in \mathbb{E}(K) \Rightarrow \mathfrak{e} \in (\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K))$ by (1) above we get $\mathfrak{e} \in L_a^1(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) \Rightarrow L_a^1(\mathbb{E}(\mathfrak{P})) \cap$

$L_a^1(\mathbb{E}(K)) \subseteq L_a^1(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) - - - (2)$. From (1) and (2) we get $L_a^1(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) = L_a^1(\mathbb{E}(\mathfrak{P})) \cap L_a^1(\mathbb{E}(K))$.

(3) By Lower1 properties (L_6^1) we get $L_a^1(\mathbb{E}(\mathfrak{P})) \cup L_a^1(\mathbb{E}(K)) \subseteq L_a^1(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) - - - (1)$. Let $\mathfrak{e} \in L_a^1(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K))$ by (1) above we get $\mathfrak{e} \in (\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) \Longrightarrow \mathfrak{e} \in \mathbb{E}(\mathfrak{P}) \vee \mathfrak{e} \in \mathbb{E}(K)$ by (1) above we get $\mathfrak{e} \in L_a^1(\mathbb{E}(\mathfrak{P})) \vee \mathfrak{e} \in L_a^1(\mathbb{E}(K)) \Longrightarrow \mathfrak{e} \in [L_a^1(\mathbb{E}(\mathfrak{P})) \cup L_a^1(\mathbb{E}(K))] \Longrightarrow L_a^1(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) \subseteq L_a^1(\mathbb{E}(\mathfrak{P})) \cup L_a^1(\mathbb{E}(K)) - - - (2)$. From (1) and (2) we get $L_a^1(\mathbb{E}(\mathfrak{P})) \cup L_a^1(\mathbb{E}(K)) = L_a^1(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K))$.

(4) By Upper1 properties (U_1^1) we get $\mathbb{E}(\mathfrak{P}) \subseteq U_a^1(\mathbb{E}(\mathfrak{P})) - - - (1)$. Let $\mathfrak{e} \in U_a^1(\mathbb{E}(\mathfrak{P})) \Longrightarrow \mathfrak{e} \in \mathbb{E}(\mathfrak{P}) \vee \mathfrak{e} \in \mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P})$. If $\mathfrak{e} \in \mathbb{E}(\mathfrak{P})$ thus $U_a^1(\mathbb{E}(\mathfrak{P})) \subseteq \mathbb{E}(\mathfrak{P})$.

If $\mathfrak{e} \in \mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P})$; $\forall$ $AVE(\mathfrak{i}) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset$ where $\mathfrak{i}$ incidence on $\mathfrak{e} \Longrightarrow \exists\, \mathfrak{e}_1 \in INVE(\mathfrak{i})$ and $\mathfrak{e}_1 \in \mathbb{E}(\mathfrak{P})$, since $\boldsymbol{\xi}$ is antisymmetric graph so $INVE(\mathfrak{i}) = \{\mathfrak{e}\}$ and $\mathfrak{e} \notin \mathbb{E}(\mathfrak{P})$ and this mean $AVE(\mathfrak{i}) \cap \mathbb{E}(\mathfrak{P}) = \emptyset$ when $AVE(\mathfrak{i}) = INVE(\mathfrak{i})$ because $\mathfrak{e} \notin \mathbb{E}(\mathfrak{P})$ so when $\boldsymbol{\xi}$ is antisymmetric graph must $\mathfrak{e} \in \mathbb{E}(\mathfrak{P})$ thus $U_a^1(\mathbb{E}(\mathfrak{P})) \subseteq \mathbb{E}(\mathfrak{P}) - - - (2)$ From (1) and (2) we get $\mathbb{E}(\mathfrak{P}) = U_a^1(\mathbb{E}(\mathfrak{P}))$.

(5) By Upper[1] properties (U_5^1) we get $U_a^1(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) \subseteq U_a^1(\mathbb{E}(\mathfrak{P})) \cap U_a^1(\mathbb{E}(K)) - - - (1)$. Let $\mathfrak{e} \in [U_a^1(\mathbb{E}(\mathfrak{P})) \cap U_a^1(\mathbb{E}(K))] \Longrightarrow \mathfrak{e} \in U_a^1(\mathbb{E}(\mathfrak{P})) \wedge \mathfrak{e} \in U_a^1(\mathbb{E}(K))$ by (4) above we get $\mathfrak{e} \in \mathbb{E}(\mathfrak{P}) \wedge \mathfrak{e} \in \mathbb{E}(K) \Longrightarrow \mathfrak{e} \in (\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K))$ by (4) above we get $\mathfrak{e} \in U_a^1(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) \Longrightarrow U_a^1(\mathbb{E}(\mathfrak{P})) \cap U_a^1(\mathbb{E}(K)) \subseteq U_a^1(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) - - - (2)$. From (1) and (2) we get $U_a^1(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) = U_a^1(\mathbb{E}(\mathfrak{P})) \cap U_a^1(\mathbb{E}(K))$.

(6) By Upper[1] properties (U_6^1) we get $U_a^1(\mathbb{E}(\mathfrak{P})) \cup U_a^1(\mathbb{E}(K)) \subseteq U_a^1(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) - - - (1)$. Let $\mathfrak{e} \in U_a^1(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K))$ by (4) above we get $\mathfrak{e} \in (\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) \Longrightarrow \mathfrak{e} \in \mathbb{E}(\mathfrak{P}) \vee \mathfrak{e} \in \mathbb{E}(K)$ by (4) above we get $\mathfrak{e} \in U_a^1(\mathbb{E}(\mathfrak{P})) \vee \mathfrak{e} \in U_a^1(\mathbb{E}(K)) \Longrightarrow \mathfrak{e} \in [U_a^1(\mathbb{E}(\mathfrak{P})) \cup U_a^1(\mathbb{E}(K))] \Longrightarrow U_a^1(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) \subseteq U_a^1(\mathbb{E}(\mathfrak{P})) \cup U_a^1(\mathbb{E}(K)) - - - (2)$. From (1) and (2) we get $U_a^1(\mathbb{E}(\mathfrak{P})) \cup U_a^1(\mathbb{E}(K)) = U_a^1(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K))$.

(7) By (1) above we get $L_a^1(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{P}) - - - (1)$. And by (4) above we get $U_a^1(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{P}) - - - (2)$. From (1) and (2) we get $L_a^1(\mathbb{E}(\mathfrak{P})) = U_a^1(\mathbb{E}(\mathfrak{P}))$.

(8) Since $Bd_a^1(\mathbb{E}(\mathfrak{P})) = U_a^1(\mathbb{E}(\mathfrak{P})) - L_a^1(\mathbb{E}(\mathfrak{P}))$, by (7) above we get $Bd_a^1(\mathbb{E}(\mathfrak{P})) = \emptyset$.

(9) Since $NEG_a^1(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\boldsymbol{\xi}) - U_a^1(\mathbb{E}(\mathfrak{P}))$ and by (4) above we get $NEG_a^1(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\boldsymbol{\xi}) - \mathbb{E}(\mathfrak{P})$.

Proposition 2.12: Let $\boldsymbol{\Gamma} = (\mathbb{V}(\boldsymbol{\xi}), E(\xi))$ be a generalized approximation space, if $\boldsymbol{\Gamma}$ induced of antisymmetric graph $\boldsymbol{\xi}$ then for all $\mathbb{E}(\mathfrak{P}) \subseteq E(\xi)$ is ($\mathfrak{P}$-exact).

Proof: Let $\boldsymbol{\xi}$ be an antisymmetric graph and $\mathfrak{P} \subseteq \boldsymbol{\xi}$. By proposition(2.11(8)) we get $Bd_a^1(\mathbb{E}(\mathfrak{P})) = \emptyset \Longrightarrow |Bd_a^1(\mathbb{E}(\mathfrak{P}))| = 0$ and by definition (2.3) we get $\mathfrak{f}_a^1(\mathbb{E}(\mathfrak{P})) = 1 - \frac{|Bd_a^1(\mathbb{E}(\mathfrak{P}))|}{|\mathbb{E}(\boldsymbol{\xi})|} = 1 - \frac{0}{|\mathbb{E}(\boldsymbol{\xi})|} = 1$, thus $\mathfrak{P}$-exact.

3. References

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