

Second Accuracy Approximation Using AVE Systems

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Abstract: In contrast to second lower and upper approximations of $\mathfrak{P}$ using non incident edges systems, the primary idea of this work is to generate and investigate second lower and upper approximations of $\mathfrak{P}$ admixture vertex edges systems as well as second lower and upper approximations operators of $\mathfrak{P}$ using incidence vertex edges systems. Furthermore, we made use of The second accuracy of the approximation of a subgraph. $\mathfrak{P} \subseteq \xi$ is investigated, along with some of its features, using (incidence, non-incidence, and admixture) vertex edges systems.

Keywords: graph , topology, accuracy of lower and upper approximations.

1. INTRODUCTION:

Two factors make graph theory a significant and intriguing area of mathematics that is primarily utilized in discrete mathematics. From a mathematical perspective, the graph is visually appealing. Even though they are simple relation graphs, they can be used to depict a variety of mathematical graphs, such as harmonic objects and topographic space. The second reason is that graphs will be very useful in practice when many concepts are empirically represented by them. Numerous theoretical and practical applications can be found in the branch of mathematics known as topological graph theory [1, 2, 3, 4, 5, 8, 09]. In order to bridge the gap between topology and applications, we believe that topological graph structure will be crucial. We consult Harary [6] for all graph theory jargon and nomenclature, and Moller [7] for all topology language and notation. A few fundamental ideas of graph theory [10] are introduced. A graph is pair $\xi = (\mathbb{V}(\xi), \mathbb{E}(\xi))$ where $\mathbb{V}(\xi)$ is a non-empty set whose elements are called points or vertices (called vertex set) and $\mathbb{E}(\xi)$ is the set of unordered pairs of elements of $\mathbb{V}(\xi)$ (called edge set). An edge of a graph that joins a vertex to itself is called a loop.

A subgraph of a graph ξ is a graph each of whose vertices belong to $\mathbb{V}(\xi)$ and each of whose edges belong to $\mathbb{E}(\xi)$. An empty graph if the vertices set and edge set is empty. A degree of a vertex i in a graph ξ is the number of edges of ξ incident with i . Let $\xi = (\mathbb{V}(\xi), \mathbb{E}(\xi))$ be a graph then the incidence vertex edges set of i is denoted by $\text{INVE}(i)$ and defined by: $\text{INVE}(i) = \{e \in \mathbb{E}(\xi) : e = (i, \vartheta) \text{ for some } \vartheta \in \mathbb{V}(\xi)\}$. The non-incidence vertex edges set of i is denoted by $\text{NINVE}(i)$ and defined by: $\text{NINVE}(i) = \{e \in \mathbb{E}(\xi) : e = (\vartheta, \gamma) \text{ and } \vartheta, \gamma \neq i \text{ for some } \vartheta, \gamma \in \mathbb{V}(\xi)\}$. The incidence vertex edges system (resp. non-incidence vertex edges system) of a vertex $i \in \mathbb{V}(\xi)$ is denoted by $\text{INVES}(i)$ (resp. $\text{NINVES}(i)$) and defined by:

$$\text{INVES}(i) = \{\text{INVE}(i)\} \text{ (resp. } \text{NINVES}(i) = \{\text{NINVE}(i)\}) .$$

The admixture vertex edges system of a vertex $i \in \mathbb{V}(\xi)$ is denoted by $\text{AVES}(i)$ and defined by: $\text{AVES}(i) = \{\text{INVES}(i), \text{NINVES}(i)\}$, such that $\text{AVE}(i) \in \text{AVES}(i)$. The A-space is the pair (ξ, γ_a) such that ξ is a graph and $\gamma_a: \mathbb{V}(\xi) \rightarrow \mathcal{P}(\mathcal{P}(\mathbb{E}(\xi)))$ is a mapping which assigns for each i in $\mathbb{V}(\xi)$ its admixture vertex edges system in $\mathcal{P}(\mathcal{P}(\mathbb{E}(\xi)))$.

2. Second New Approximation Operators Using Admixture Vertex Edges Systems:

Presenting a set-theoretic basis for granular computing with admixture vertex edges systems is the goal of this section. Taking the generalized approximation space into account $\Gamma = (\mathbb{V}(\xi), \mathbb{E}(\xi))$, we created a new definition of the lower and upper approximation operators using admixture vertex edges systems. The approximations are made using (incidence, non-incidence, and admixture) vertex edges systems.

Definition 2.1. Let $\Gamma = (\mathbb{V}(\xi), \mathbb{E}(\xi))$ be a generalized approximation space and $\mathfrak{P} \subseteq \xi$. Then:

(1) The second lower and upper approximations of $\mathfrak{P}$ using incidence vertex edges system are denoted by $L_i^2(\mathbb{E}(\mathfrak{P}))$ and $U_i^2(\mathbb{E}(\mathfrak{P}))$ and defined by:

$$L_i^2(\mathbb{E}(\mathfrak{P})) = \left\{ e \in \mathbb{E}(\xi) \text{ there exist } i \text{ incidence on } e \text{ such that } \text{INVE}(i) \subseteq \mathbb{E}(\mathfrak{P}) \right\}.$$

$$U_i^2(\mathbb{E}(\mathfrak{P})) = \left\{ e \in \mathbb{E}(\xi) \text{ such that } \text{INVE}(i) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset \text{ where } i \text{ incidence on } e \right\}.$$

(2) The second lower and upper approximations of $\mathfrak{P}$ using non-incidence vertex edges system are denoted by $L_n^2(\mathbb{E}(\mathfrak{P}))$ and $U_n^2(\mathbb{E}(\mathfrak{P}))$ and defined by:

$$L_n^2(\mathbb{E}(\mathfrak{P})) = \left\{ e \in \mathbb{E}(\xi) \text{ there exist } i \text{ incidence on } e \text{ such that } \text{NINVE}(i) \subseteq \mathbb{E}(\mathfrak{P}) \right\}.$$

$$U_n^2(\mathbb{E}(\mathfrak{P})) = \left\{ e \in \mathbb{E}(\xi) \text{ such that } \text{NINVE}(i) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset \text{ where } i \text{ incidence on } e \right\}.$$

(3) The second lower and upper approximations of $\mathfrak{P}$ using admixture vertex edges system are denoted by $L_a^2(\mathbb{E}(\mathfrak{P}))$ and $U_a^2(\mathbb{E}(\mathfrak{P}))$ and defined by:

$$L_a^2(\mathbb{E}(\mathfrak{P})) = \left\{ \mathbf{e} \in \mathbb{E}(\mathfrak{P}) \text{ there exist } \mathbf{i} \text{ incidence on } \mathbf{e} \right. \\ \left. \text{such that } \text{AVE}(\mathbf{i}) \subseteq \mathbb{E}(\mathfrak{P}) \right\}.$$

$$U_a^2(\mathbb{E}(\mathfrak{P})) = \left\{ \mathbf{e} \in \mathbb{E}(\mathfrak{P}) \text{ where } \mathbf{i} \text{ incidence on } \mathbf{e} \right. \\ \left. \text{for all } \text{AVE}(\mathbf{i}) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset \right\}.$$

Definition 2.2. Let $\mathcal{F} = (\mathbb{V}(\mathfrak{X}), \mathbb{E}(\mathfrak{X}))$ be a generalization approximation space and $\mathfrak{P} \subseteq \mathfrak{X}$. Then:

(1) The second boundary, positive and negative regions of $\mathfrak{P}$ using incidence vertex edges system are denoted by $\text{Bd}_i^2(\mathbb{E}(\mathfrak{P}))$, $\text{POS}_i^2(\mathbb{E}(\mathfrak{P}))$ and $\text{NEG}_i^2(\mathbb{E}(\mathfrak{P}))$ and defined by:

$$\text{Bd}_i^2(\mathbb{E}(\mathfrak{P})) = U_i^2(\mathbb{E}(\mathfrak{P})) - L_i^2(\mathbb{E}(\mathfrak{P})).$$

$$\text{POS}_i^2(\mathbb{E}(\mathfrak{P})) = L_i^2(\mathbb{E}(\mathfrak{P})).$$

$$\text{NEG}_i^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{X}) - U_i^2(\mathbb{E}(\mathfrak{P})).$$

(2) The second boundary, positive and negative regions of $\mathfrak{P}$ using non-incidence vertex edges system are denoted by $\text{Bd}_n^2(\mathbb{E}(\mathfrak{P}))$, $\text{POS}_n^2(\mathbb{E}(\mathfrak{P}))$ and $\text{NEG}_n^2(\mathbb{E}(\mathfrak{P}))$ and defined by:

$$\text{Bd}_n^2(\mathbb{E}(\mathfrak{P})) = U_n^2(\mathbb{E}(\mathfrak{P})) - L_n^2(\mathbb{E}(\mathfrak{P})).$$

$$\text{POS}_n^2(\mathbb{E}(\mathfrak{P})) = L_n^2(\mathbb{E}(\mathfrak{P})).$$

$$\text{NEG}_n^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{X}) - U_n^2(\mathbb{E}(\mathfrak{P})).$$

(3) The second boundary, positive and negative regions of $\mathfrak{P}$ using admixture vertex edges system are denoted by $\text{Bd}_a^2(\mathbb{E}(\mathfrak{P}))$, $\text{POS}_a^2(\mathbb{E}(\mathfrak{P}))$ and $\text{NEG}_a^2(\mathbb{E}(\mathfrak{P}))$ and defined by:

$$\text{Bd}_a^2(\mathbb{E}(\mathfrak{P})) = U_a^2(\mathbb{E}(\mathfrak{P})) - L_a^2(\mathbb{E}(\mathfrak{P})).$$

$$\text{POS}_a^2(\mathbb{E}(\mathfrak{P})) = L_a^2(\mathbb{E}(\mathfrak{P})).$$

$$\text{NEG}_a^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{X}) - U_a^2(\mathbb{E}(\mathfrak{P})).$$

Definition 2.3. Let $\mathcal{F} = (\mathbb{V}(\mathfrak{X}), \mathbb{E}(\mathfrak{X}))$ be a generalized approximation space. The second accuracy of the approximation of $\mathfrak{P} \subseteq \mathfrak{X}$ using (incidence, non-incidence, and admixture) vertex edges system are denoted by $f_i^2(\mathbb{E}(\mathfrak{P}))$, $f_n^2(\mathbb{E}(\mathfrak{P}))$ and $f_a^2(\mathbb{E}(\mathfrak{P}))$ and defined respectively by:

$$f_i^2(\mathbb{E}(\mathfrak{P})) = 1 - \frac{|\text{Bd}_i^2(\mathbb{E}(\mathfrak{P}))|}{|\mathbb{E}(\mathfrak{X})|},$$

$$f_n^2(\mathbb{E}(\mathfrak{P})) = 1 - \frac{|\text{Bd}_n^2(\mathbb{E}(\mathfrak{P}))|}{|\mathbb{E}(\mathfrak{X})|},$$

$$f_a^2(\mathbb{E}(\mathfrak{P})) = 1 - \frac{|\text{Bd}_a^2(\mathbb{E}(\mathfrak{P}))|}{|\mathbb{E}(\mathfrak{X})|}.$$

It is obvious that $0 \leq f_i^2(\mathbb{E}(\mathfrak{P})) \leq 1$, $0 \leq f_n^2(\mathbb{E}(\mathfrak{P})) \leq 1$ and $0 \leq f_a^2(\mathbb{E}(\mathfrak{P})) \leq 1$. Moreover, if $f_i^2(\mathbb{E}(\mathfrak{P})) = 1$ or $f_n^2(\mathbb{E}(\mathfrak{P})) = 1$ or $f_a^2(\mathbb{E}(\mathfrak{P})) = 1$ then $\mathfrak{P}$ is called $\mathfrak{P}$ -definable ($\mathfrak{P}$ -exact) otherwise, it is called $\mathfrak{P}$ -rough.

Proposition 2.4. Let $\mathcal{F} = (\mathbb{V}(\mathfrak{X}), \mathbb{E}(\mathfrak{X}))$ be a generalized approximation space and $\mathfrak{P} \subseteq \mathfrak{X}$. Then $\mathfrak{P}$ is exact if and only if $U_a^2(\mathbb{E}(\mathfrak{P})) \subseteq L_a^2(\mathbb{E}(\mathfrak{P}))$.

Proof: Clear.

Example 2.5. Let $\mathcal{F} = (\mathbb{V}(\mathfrak{X}), \mathbb{E}(\mathfrak{X}))$ be a generalized approximation space, $\mathcal{F}$ induced of figure(2.1) such that $\mathbb{V}(\mathfrak{X}) = \{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3, \mathbf{i}_4\}$, $\mathbb{E}(\mathfrak{X}) = \{\mathbf{e}_1 = (\mathbf{i}_1, \mathbf{i}_1), \mathbf{e}_2 = (\mathbf{i}_1, \mathbf{i}_4), \mathbf{e}_3 = (\mathbf{i}_2, \mathbf{i}_3), \mathbf{e}_4 = (\mathbf{i}_2, \mathbf{i}_4), \mathbf{e}_5 = (\mathbf{i}_3, \mathbf{i}_4)\}$.

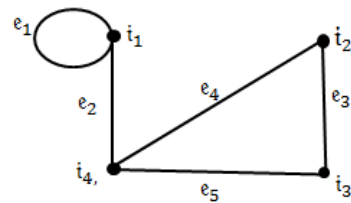


Figure 2.1: graph $\mathfrak{X}$ given in example 2.5

Then $\text{INVE}(\mathbf{i}_1) = \{\mathbf{e}_1, \mathbf{e}_2\}$, $\text{INVE}(\mathbf{i}_2) = \{\mathbf{e}_3, \mathbf{e}_4\}$, $\text{INVE}(\mathbf{i}_3) = \{\mathbf{e}_3, \mathbf{e}_5\}$, $\text{INVE}(\mathbf{i}_4) = \{\mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5\}$. And $\text{NINVE}(\mathbf{i}_1) = \{\mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$, $\text{NINVE}(\mathbf{i}_2) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_5\}$, $\text{NINVE}(\mathbf{i}_3) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4\}$, $\text{NINVE}(\mathbf{i}_4) = \{\mathbf{e}_1, \mathbf{e}_3\}$. And $\text{AVE}(\mathbf{i}_1) = \{\{\mathbf{e}_1, \mathbf{e}_2\}, \{\mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}\}$, $\text{AVE}(\mathbf{i}_2) = \{\{\mathbf{e}_3, \mathbf{e}_4\}, \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_5\}\}$, $\text{AVE}(\mathbf{i}_3) = \{\{\mathbf{e}_3, \mathbf{e}_5\}, \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4\}\}$, $\text{AVE}(\mathbf{i}_4) = \{\{\mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5\}, \{\mathbf{e}_1, \mathbf{e}_3\}\}$.

Accordingly, can be obtain the following tables

Table 2.1: $L_i^2(\mathbb{E}(\mathfrak{P}))$, $L_n^2(\mathbb{E}(\mathfrak{P}))$ and $L_a^2(\mathbb{E}(\mathfrak{P}))$ for all $\mathfrak{P} \subseteq \mathfrak{X}$

$\mathbb{E}(\mathfrak{P})$	$L_i^2(\mathbb{E}(\mathfrak{P}))$	$L_n^2(\mathbb{E}(\mathfrak{P}))$	$L_a^2(\mathbb{E}(\mathfrak{P}))$
$\{\mathbf{e}_1\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_2\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_3\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_4\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_5\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_1, \mathbf{e}_2\}$	$\{\mathbf{e}_1, \mathbf{e}_2\}$	$\emptyset$	$\{\mathbf{e}_1, \mathbf{e}_2\}$
$\{\mathbf{e}_1, \mathbf{e}_3\}$	$\emptyset$	$\{\mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5\}$	$\{\mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5\}$
$\{\mathbf{e}_1, \mathbf{e}_4\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_1, \mathbf{e}_5\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_2, \mathbf{e}_3\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_2, \mathbf{e}_4\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_2, \mathbf{e}_5\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_3, \mathbf{e}_4\}$	$\{\mathbf{e}_3, \mathbf{e}_4\}$	$\emptyset$	$\{\mathbf{e}_3, \mathbf{e}_4\}$
$\{\mathbf{e}_3, \mathbf{e}_5\}$	$\{\mathbf{e}_3, \mathbf{e}_5\}$	$\emptyset$	$\{\mathbf{e}_3, \mathbf{e}_5\}$
$\{\mathbf{e}_4, \mathbf{e}_5\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$	$\{\mathbf{e}_1, \mathbf{e}_2\}$	$\{\mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5\}$	$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5\}$
$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4\}$	$\{\mathbf{e}_1, \mathbf{e}_2\}$	$\{\mathbf{e}_3, \mathbf{e}_5\}$	$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_5\}$
$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_5\}$	$\{\mathbf{e}_1, \mathbf{e}_2\}$	$\{\mathbf{e}_3, \mathbf{e}_4\}$	$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$
$\{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$	$\{\mathbf{e}_3, \mathbf{e}_4\}$	$\emptyset$	$\{\mathbf{e}_3, \mathbf{e}_4\}$

$\{e_2, e_3, e_5\}$	$\{e_3, e_5\}$	ϕ	$\{e_3, e_5\}$
$\{e_3, e_4, e_1\}$	$\{e_3, e_4\}$	$\{e_2, e_4, e_5\}$	$\{e_2, e_3, e_4, e_5\}$
$\{e_3, e_4, e_5\}$	$\{e_3, e_4, e_5\}$	$\{e_1, e_2\}$	$\mathbb{E}(\xi)$
$\{e_4, e_5, e_1\}$	ϕ	ϕ	ϕ
$\{e_4, e_5, e_2\}$	$\{e_4, e_5, e_2\}$	ϕ	$\{e_4, e_5, e_2\}$
$\{e_1, e_3, e_5\}$	$\{e_3, e_5\}$	$\{e_2, e_4, e_5\}$	$\{e_2, e_3, e_4, e_5\}$
$\{e_1, e_2, e_3, e_4\}$	$\{e_1, e_2, e_3, e_4\}$	$\{e_2, e_3, e_4, e_5\}$	$\mathbb{E}(\xi)$
$\{e_1, e_2, e_3, e_5\}$	$\{e_1, e_2, e_3, e_5\}$	$\{e_2, e_3, e_4, e_5\}$	$\mathbb{E}(\xi)$
$\{e_2, e_3, e_4, e_5\}$	$\{e_2, e_3, e_4, e_5\}$	$\{e_1, e_2\}$	$\mathbb{E}(\xi)$
$\{e_1, e_3, e_4, e_5\}$	$\{e_3, e_4, e_5\}$	$\{e_1, e_2, e_4, e_5\}$	$\mathbb{E}(\xi)$
$\{e_1, e_2, e_4, e_5\}$	$\{e_1, e_2, e_4, e_5\}$	$\{e_3, e_4, e_5\}$	$\mathbb{E}(\xi)$
$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
ϕ	ϕ	ϕ	ϕ

Table 2.2: $U_1^2(\mathbb{E}(\mathfrak{P}))$, $U_n^2(\mathbb{E}(\mathfrak{P}))$ and $U_a^2(\mathbb{E}(\mathfrak{P}))$ for all $\mathfrak{P} \subseteq \xi$.

$\mathbb{E}(\mathfrak{P})$	$U_1^2(\mathbb{E}(\mathfrak{P}))$	$U_n^2(\mathbb{E}(\mathfrak{P}))$	$U_a^2(\mathbb{E}(\mathfrak{P}))$
$\{e_1\}$	$\{e_1\}$	$\{e_3, e_4, e_5\}$	ϕ
$\{e_2\}$	$\{e_1, e_2\}$	$\{e_3\}$	ϕ
$\{e_3\}$	$\{e_3\}$	$\{e_1, e_2\}$	ϕ
$\{e_4\}$	$\{e_4\}$	$\{e_1\}$	ϕ
$\{e_5\}$	$\{e_5\}$	$\{e_1\}$	ϕ
$\{e_1, e_2\}$	$\{e_1, e_2\}$	$\{e_3, e_4, e_5\}$	ϕ
$\{e_1, e_3\}$	$\{e_1, e_3\}$	$\mathbb{E}(\xi)$	$\{e_1, e_3\}$
$\{e_1, e_4\}$	$\{e_1, e_2, e_4\}$	$\mathbb{E}(\xi)$	$\{e_1, e_2, e_4\}$
$\{e_1, e_5\}$	$\{e_1, e_2, e_5\}$	$\mathbb{E}(\xi)$	$\{e_1, e_2, e_5\}$
$\{e_2, e_3\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_2, e_4\}$	$\{e_1, e_2, e_4\}$	$\{e_1, e_3\}$	$\{e_1\}$
$\{e_2, e_5\}$	$\{e_1, e_2, e_5\}$	$\{e_1, e_3\}$	$\{e_1\}$
$\{e_3, e_4\}$	$\{e_3, e_4, e_5\}$	$\{e_1, e_2, e_5\}$	$\{e_5\}$
$\{e_3, e_5\}$	$\{e_3, e_4, e_5\}$	$\{e_1, e_2, e_4\}$	$\{e_4\}$
$\{e_4, e_5\}$	$\{e_3, e_4, e_5\}$	$\{e_1, e_3\}$	$\{e_3\}$
$\{e_1, e_2, e_3\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_1, e_2, e_4\}$	$\{e_1, e_2, e_4\}$	$\mathbb{E}(\xi)$	$\{e_1, e_2, e_4\}$
$\{e_1, e_2, e_5\}$	$\{e_1, e_2, e_5\}$	$\mathbb{E}(\xi)$	$\{e_1, e_2, e_5\}$
$\{e_2, e_3, e_4\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_2, e_3, e_5\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_1, e_3, e_4\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_3, e_4, e_5\}$	$\{e_3, e_4, e_5\}$	$\mathbb{E}(\xi)$	$\{e_3, e_4, e_5\}$
$\{e_1, e_4, e_5\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_2, e_4, e_5\}$	$\mathbb{E}(\xi)$	$\{e_1, e_3\}$	$\{e_1, e_3\}$
$\{e_1, e_3, e_5\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$

$\{e_1, e_2, e_3, e_4\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_1, e_2, e_3, e_5\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_2, e_3, e_4, e_5\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_1, e_3, e_4, e_5\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_1, e_2, e_4, e_5\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
ϕ	ϕ	ϕ	ϕ

Table 2.3: $Bd_1^2(\mathbb{E}(\mathfrak{P}))$, $Bd_n^2(\mathbb{E}(\mathfrak{P}))$ and $Bd_a^2(\mathbb{E}(\mathfrak{P}))$ for all $\mathfrak{P} \subseteq \xi$.

$\mathbb{E}(\mathfrak{P})$	$Bd_1^2(\mathbb{E}(\mathfrak{P}))$	$Bd_n^2(\mathbb{E}(\mathfrak{P}))$	$Bd_a^2(\mathbb{E}(\mathfrak{P}))$
$\{e_1\}$	$\{e_1\}$	$\{e_3, e_4, e_5\}$	ϕ
$\{e_2\}$	$\{e_1, e_2\}$	$\{e_3\}$	ϕ
$\{e_3\}$	$\{e_3\}$	$\{e_1, e_2\}$	ϕ
$\{e_4\}$	$\{e_4\}$	$\{e_1\}$	ϕ
$\{e_5\}$	$\{e_5\}$	$\{e_1\}$	ϕ
$\{e_1, e_2\}$	ϕ	$\{e_3, e_4, e_5\}$	ϕ
$\{e_1, e_3\}$	$\{e_1, e_3\}$	$\{e_1, e_3\}$	$\{e_1, e_3\}$
$\{e_1, e_4\}$	$\{e_1, e_2, e_4\}$	$\mathbb{E}(\xi)$	$\{e_1, e_2, e_4\}$
$\{e_1, e_5\}$	$\{e_1, e_2, e_5\}$	$\mathbb{E}(\xi)$	$\{e_1, e_2, e_5\}$
$\{e_2, e_3\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_2, e_4\}$	$\{e_1, e_2, e_4\}$	$\{e_1, e_3\}$	$\{e_1\}$
$\{e_2, e_5\}$	$\{e_1, e_2, e_5\}$	$\{e_1, e_3\}$	$\{e_1\}$
$\{e_3, e_4\}$	$\{e_5\}$	$\{e_1, e_2, e_5\}$	$\{e_5\}$
$\{e_3, e_5\}$	$\{e_4\}$	$\{e_1, e_2, e_4\}$	$\{e_4\}$
$\{e_4, e_5\}$	$\{e_3, e_4, e_5\}$	$\{e_1, e_3\}$	$\{e_3\}$
$\{e_1, e_2, e_3\}$	$\{e_3, e_4, e_5\}$	$\{e_1, e_3\}$	$\{e_3\}$
$\{e_1, e_2, e_4\}$	$\{e_4\}$	$\{e_1, e_2, e_4\}$	$\{e_4\}$
$\{e_1, e_2, e_5\}$	$\{e_5\}$	$\{e_1, e_2, e_5\}$	$\{e_5\}$
$\{e_2, e_3, e_4\}$	$\{e_1, e_2, e_5\}$	$\mathbb{E}(\xi)$	$\{e_1, e_2, e_5\}$
$\{e_2, e_3, e_5\}$	$\{e_1, e_2, e_4\}$	$\mathbb{E}(\xi)$	$\{e_1, e_2, e_4\}$
$\{e_3, e_4, e_1\}$	$\{e_1, e_2, e_5\}$	$\{e_1, e_3\}$	$\{e_1\}$
$\{e_3, e_4, e_5\}$	$\emptyset$	$\{e_3, e_4, e_5\}$	$\emptyset$
$\{e_4, e_5, e_1\}$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$
$\{e_4, e_5, e_2\}$	$\{e_1, e_3\}$	$\{e_1, e_3\}$	$\{e_1, e_3\}$
$\{e_1, e_3, e_5\}$	$\{e_1, e_2, e_4\}$	$\{e_1, e_3\}$	$\{e_1\}$
$\{e_1, e_2, e_3, e_4\}$	$\{e_5\}$	$\{e_1\}$	$\emptyset$
$\{e_1, e_2, e_3, e_5\}$	$\{e_4\}$	$\{e_1\}$	$\emptyset$
$\{e_2, e_3, e_4, e_5\}$	$\{e_1\}$	$\{e_3, e_4, e_5\}$	$\emptyset$
$\{e_1, e_3, e_4, e_5\}$	$\{e_1, e_2\}$	$\{e_3\}$	$\emptyset$
$\{e_1, e_2, e_4, e_5\}$	$\{e_3\}$	$\{e_1, e_2\}$	$\emptyset$
$\mathbb{E}(\xi)$	ϕ	ϕ	ϕ
ϕ	ϕ	ϕ	ϕ

Table 2.2: $NEG_1^2(\mathbb{E}(\mathfrak{P}))$, $NEG_n^2(\mathbb{E}(\mathfrak{P}))$ and $NEG_a^2(\mathbb{E}(\mathfrak{P}))$ for all $\mathfrak{P} \subseteq \xi$.

$\mathbb{E}(\mathfrak{P})$	$\text{NEG}_i^2(\mathbb{E}(\mathfrak{P}))$	$\text{NEG}_n^2(\mathbb{E}(\mathfrak{P}))$	$\text{NEG}_a^2(\mathbb{E}(\mathfrak{P}))$
$\{e_1\}$	$\{e_2, e_3, e_4, e_5\}$	$\{e_1, e_2\}$	$\mathbb{E}(\xi)$
$\{e_2\}$	$\{e_3, e_4, e_5\}$	$\{e_1, e_2, e_4, e_5\}$	$\mathbb{E}(\xi)$
$\{e_3\}$	$\{e_1, e_2, e_4, e_5\}$	$\{e_3, e_4, e_5\}$	$\mathbb{E}(\xi)$
$\{e_4\}$	$\{e_1, e_2, e_3, e_5\}$	$\{e_2, e_3, e_4, e_5\}$	$\mathbb{E}(\xi)$
$\{e_5\}$	$\{e_1, e_2, e_3, e_4\}$	$\{e_2, e_3, e_4, e_5\}$	$\mathbb{E}(\xi)$
$\{e_1, e_2\}$	$\{e_3, e_4, e_5\}$	$\{e_1, e_2\}$	$\mathbb{E}(\xi)$
$\{e_1, e_3\}$	$\{e_2, e_4, e_5\}$	$\emptyset$	$\{e_2, e_4, e_5\}$
$\{e_1, e_4\}$	$\{e_3, e_5\}$	$\emptyset$	$\{e_3, e_5\}$
$\{e_1, e_5\}$	$\{e_3, e_4\}$	$\emptyset$	$\{e_3, e_4\}$
$\{e_2, e_3\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{e_2, e_4\}$	$\{e_3, e_5\}$	$\{e_5\}$	$\{e_3, e_5\}$
$\{e_2, e_5\}$	$\{e_3, e_4\}$	$\{e_2, e_4, e_5\}$	$\{e_2, e_3, e_4, e_5\}$
$\{e_3, e_4\}$	$\{e_1, e_2\}$	$\{e_3, e_4\}$	$\{e_1, e_2, e_3, e_4\}$
$\{e_3, e_5\}$	$\{e_1, e_2\}$	$\{e_3, e_5\}$	$\{e_1, e_2, e_3, e_5\}$
$\{e_4, e_5\}$	$\{e_1, e_2\}$	$\{e_2, e_4, e_5\}$	$\{e_1, e_2, e_4, e_5\}$
$\{e_1, e_2, e_3\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{e_1, e_2, e_4\}$	$\{e_3, e_5\}$	$\emptyset$	$\{e_3, e_5\}$
$\{e_1, e_2, e_5\}$	$\{e_3, e_4\}$	$\emptyset$	$\{e_3, e_4\}$
$\{e_2, e_3, e_4\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{e_2, e_3, e_5\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{e_3, e_4, e_1\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{e_3, e_4, e_5\}$	$\{e_1, e_2\}$	$\emptyset$	$\{e_1, e_2\}$
$\{e_4, e_5, e_1\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{e_4, e_5, e_2\}$	$\emptyset$	$\{e_4, e_5, e_2\}$	$\{e_4, e_5, e_2\}$
$\{e_1, e_3, e_5\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{e_1, e_2, e_3, e_4\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{e_1, e_2, e_3, e_5\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{e_2, e_3, e_4, e_5\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{e_1, e_3, e_4, e_5\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{e_1, e_2, e_4, e_5\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathbb{E}(\xi)$	$\emptyset$	$\emptyset$	$\emptyset$
ϕ	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{E}(\xi)$

Table 2.4: $f_i^2(\mathbb{E}(\mathfrak{P}))$, $f_n^2(\mathbb{E}(\mathfrak{P}))$ and $f_a^2(\mathbb{E}(\mathfrak{P}))$ for all $\mathfrak{P} \subseteq \xi$.

$\mathbb{E}(\mathfrak{P})$	$f_i^2(\mathbb{E}(\mathfrak{P}))$	$f_n^2(\mathbb{E}(\mathfrak{P}))$	$f_a^2(\mathbb{E}(\mathfrak{P}))$
$\{e_1\}$	4/5	2/5	1
$\{e_2\}$	3/5	4/5	1
$\{e_3\}$	4/5	3/5	1
$\{e_4\}$	4/5	4/5	1
$\{e_5\}$	4/5	4/5	1
$\{e_1, e_2\}$	1	2/5	1
$\{e_1, e_3\}$	3/5	3/5	3/5
$\{e_1, e_4\}$	2/5	0	2/5
$\{e_1, e_5\}$	2/5	0	2/5
$\{e_2, e_3\}$	0	0	0

$\{e_2, e_4\}$	2/5	3/5	4/5
$\{e_2, e_5\}$	2/5	3/5	4/5
$\{e_3, e_4\}$	4/5	2/5	4/5
$\{e_3, e_5\}$	4/5	2/5	4/5
$\{e_4, e_5\}$	2/5	3/5	4/5
$\{e_1, e_2, e_3\}$	2/5	3/5	4/5
$\{e_1, e_2, e_4\}$	4/5	2/5	4/5
$\{e_1, e_2, e_5\}$	4/5	2/5	4/5
$\{e_2, e_3, e_4\}$	2/5	0	2/5
$\{e_2, e_3, e_5\}$	2/5	0	2/5
$\{e_3, e_4, e_1\}$	2/5	3/5	4/5
$\{e_3, e_4, e_5\}$	1	2/5	1
$\{e_4, e_5, e_1\}$	0	0	0
$\{e_4, e_5, e_2\}$	3/5	3/5	3/5
$\{e_1, e_3, e_5\}$	2/5	3/5	4/5
$\{e_1, e_2, e_3, e_4\}$	4/5	4/5	1
$\{e_1, e_2, e_3, e_5\}$	4/5	4/5	1
$\{e_2, e_3, e_4, e_5\}$	4/5	2/5	1
$\{e_1, e_3, e_4, e_5\}$	3/5	4/5	1
$\{e_1, e_2, e_4, e_5\}$	4/5	3/5	1
$\mathbb{E}(\xi)$	1	1	1
ϕ	1	1	1

Theorem 2.6. Let $\Gamma = (\mathbb{V}(\xi), \mathbb{E}(\xi))$ be a generalized approximation space and $\mathfrak{P} \subseteq \xi$. Then:

- (1) $L_a^2(\mathbb{E}(\mathfrak{P})) = L_i^2(\mathbb{E}(\mathfrak{P})) \cup L_n^2(\mathbb{E}(\mathfrak{P}))$.
- (2) $U_a^2(\mathbb{E}(\mathfrak{P})) = U_i^2(\mathbb{E}(\mathfrak{P})) \cap U_n^2(\mathbb{E}(\mathfrak{P}))$.
- (3) $\text{Bd}_a^2(\mathbb{E}(\mathfrak{P})) = \text{Bd}_i^2(\mathbb{E}(\mathfrak{P})) \cap \text{Bd}_n^2(\mathbb{E}(\mathfrak{P}))$.
- (4) $\text{NEG}_a^2(\mathbb{E}(\mathfrak{P})) = \text{NEG}_i^2(\mathbb{E}(\mathfrak{P})) \cup \text{NEG}_n^2(\mathbb{E}(\mathfrak{P}))$.
- (5) $f_a^2(\mathbb{E}(\mathfrak{P})) \geq \max\{f_i^2(\mathbb{E}(\mathfrak{P})), f_n^2(\mathbb{E}(\mathfrak{P}))\}$.

Proof:

- (1) Let $e \in L_a^2(\mathbb{E}(\mathfrak{P})) \Leftrightarrow \exists i$ incidence on e such that $\text{AVE}(i) \subseteq \mathbb{E}(\mathfrak{P}) \Leftrightarrow \text{INVE}(i) \subseteq \mathbb{E}(\mathfrak{P}) \vee \text{NINVE}(i) \subseteq \mathbb{E}(\mathfrak{P}) \Leftrightarrow e \in L_i^2(\mathbb{E}(\mathfrak{P})) \vee e \in L_n^2(\mathbb{E}(\mathfrak{P})) \Leftrightarrow e \in (L_i^2(\mathbb{E}(\mathfrak{P})) \cup L_n^2(\mathbb{E}(\mathfrak{P})))$, hence $L_a^2(\mathbb{E}(\mathfrak{P})) = L_i^2(\mathbb{E}(\mathfrak{P})) \cup L_n^2(\mathbb{E}(\mathfrak{P}))$.
- (2) Let $e \in U_a^2(\mathbb{E}(\mathfrak{P})) \Leftrightarrow$ where i incidence on e for all $\text{AVE}(i)$, $\text{AVE}(i) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset \Leftrightarrow (\text{INVE}(i) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset) \wedge (\text{NINVE}(i) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset) \Leftrightarrow e \in U_i^2(\mathbb{E}(\mathfrak{P})) \wedge e \in U_n^2(\mathbb{E}(\mathfrak{P})) \Leftrightarrow e \in (U_i^2(\mathbb{E}(\mathfrak{P})) \cap U_n^2(\mathbb{E}(\mathfrak{P})))$ hence, $U_a^2(\mathbb{E}(\mathfrak{P})) = U_i^2(\mathbb{E}(\mathfrak{P})) \cap U_n^2(\mathbb{E}(\mathfrak{P}))$.
- (3) Let $e \in \text{Bd}_a^2(\mathbb{E}(\mathfrak{P})) \Rightarrow e \in U_a^2(\mathbb{E}(\mathfrak{P})) \wedge e \notin L_a^2(\mathbb{E}(\mathfrak{P}))$ since $e \in U_a^2(\mathbb{E}(\mathfrak{P}))$ by (2) above we get $e \in (U_i^2(\mathbb{E}(\mathfrak{P})) \cap U_n^2(\mathbb{E}(\mathfrak{P}))) \Rightarrow e \in U_i^2(\mathbb{E}(\mathfrak{P}))$ and $e \in U_n^2(\mathbb{E}(\mathfrak{P}))$. Since $e \notin L_a^2(\mathbb{E}(\mathfrak{P}))$ by (1) above we get $e \notin$

$(L_1^2(\mathbb{E}(\mathfrak{P})) \cup L_n^2(\mathbb{E}(\mathfrak{P}))) \Rightarrow \mathbf{e} \notin L_1^2(\mathbb{E}(\mathfrak{P}))$ and $\mathbf{e} \notin L_n^2(\mathbb{E}(\mathfrak{P}))$ hence $\mathbf{e} \in \text{Bd}_1^2(\mathbb{E}(\mathfrak{P}))$ and $\mathbf{e} \in \text{Bd}_n^2(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathbf{e} \in (\text{Bd}_1^2(\mathbb{E}(\mathfrak{P})) \cap \text{Bd}_n^2(\mathbb{E}(\mathfrak{P})))$. So $\text{Bd}_a^2(\mathbb{E}(\mathfrak{P})) \subseteq (\text{Bd}_1^2(\mathbb{E}(\mathfrak{P})) \cap \text{Bd}_n^2(\mathbb{E}(\mathfrak{P})))$ — — — (1)

Conversely, $\mathbf{e} \in (\text{Bd}_1^2(\mathbb{E}(\mathfrak{P})) \cap \text{Bd}_n^2(\mathbb{E}(\mathfrak{P}))) \Rightarrow \mathbf{e} \in \text{Bd}_1^2(\mathbb{E}(\mathfrak{P}))$ and $\mathbf{e} \in \text{Bd}_n^2(\mathbb{E}(\mathfrak{P}))$, since $\mathbf{e} \in \text{Bd}_1^2(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathbf{e} \in U_1^2(\mathbb{E}(\mathfrak{P}))$ and $\mathbf{e} \notin L_1^2(\mathbb{E}(\mathfrak{P}))$ and since $\mathbf{e} \in \text{Bd}_n^2(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathbf{e} \in U_n^2(\mathbb{E}(\mathfrak{P}))$ and $\mathbf{e} \notin L_n^2(\mathbb{E}(\mathfrak{P}))$, hence $\mathbf{e} \in (U_1^2(\mathbb{E}(\mathfrak{P})) \cap U_n^2(\mathbb{E}(\mathfrak{P})))$ by (2) above we get $\mathbf{e} \in U_a^2(\mathbb{E}(\mathfrak{P}))$ and $\mathbf{e} \notin (L_1^2(\mathbb{E}(\mathfrak{P})) \cup L_n^2(\mathbb{E}(\mathfrak{P})))$ by (1) above we get $\mathbf{e} \notin L_a^2(\mathbb{E}(\mathfrak{P}))$ then $\mathbf{e} \in \text{Bd}_a^2(\mathbb{E}(\mathfrak{P}))$.

So $(\text{Bd}_1^2(\mathbb{E}(\mathfrak{P})) \cap \text{Bd}_n^2(\mathbb{E}(\mathfrak{P}))) \subseteq \text{Bd}_a^2(\mathbb{E}(\mathfrak{P}))$ — — — (2).

From (1) and (2) $\text{Bd}_a^2(\mathbb{E}(\mathfrak{P})) =$

$$(\text{Bd}_1^2(\mathbb{E}(\mathfrak{P})) \cap \text{Bd}_n^2(\mathbb{E}(\mathfrak{P}))).$$

(4) Let $\mathbf{e} \in \text{NEG}_a^2(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathbf{e} \notin U_a^2(\mathbb{E}(\mathfrak{P}))$ by (2) above we get $\Rightarrow \mathbf{e} \notin [U_1^2(\mathbb{E}(\mathfrak{P})) \cap U_n^2(\mathbb{E}(\mathfrak{P}))] \Rightarrow \mathbf{e} \notin U_1^2(\mathbb{E}(\mathfrak{P})) \vee \mathbf{e} \notin U_n^2(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathbf{e} \in \text{NEG}_1^2(\mathbb{E}(\mathfrak{P})) \vee \mathbf{e} \in \text{NEG}_n^2(\mathbb{E}(\mathfrak{P}))$

thus $\text{NEG}_a^2(\mathbb{E}(\mathfrak{P})) \subseteq \text{NEG}_1^2(\mathbb{E}(\mathfrak{P})) \cup \text{NEG}_n^2(\mathbb{E}(\mathfrak{P}))$ — — —

(1). Let $\mathbf{e} \in [\text{NEG}_1^2(\mathbb{E}(\mathfrak{P})) \cup \text{NEG}_n^2(\mathbb{E}(\mathfrak{P}))] \Rightarrow \mathbf{e} \in \text{NEG}_1^2(\mathbb{E}(\mathfrak{P})) \vee \mathbf{e} \in \text{NEG}_n^2(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathbf{e} \notin U_1^2(\mathbb{E}(\mathfrak{P})) \vee \mathbf{e} \notin U_n^2(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathbf{e} \notin [U_1^2(\mathbb{E}(\mathfrak{P})) \cap U_n^2(\mathbb{E}(\mathfrak{P}))]$ by (2) above we get $\Rightarrow \mathbf{e} \notin U_a^2(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathbf{e} \in \text{NEG}_a^2(\mathbb{E}(\mathfrak{P}))$ thus $\text{NEG}_1^2(\mathbb{E}(\mathfrak{P})) \cup \text{NEG}_n^2(\mathbb{E}(\mathfrak{P})) \subseteq \text{NEG}_a^2(\mathbb{E}(\mathfrak{P}))$ — — — (2). From (1) and (2) we get $\text{NEG}_a^2(\mathbb{E}(\mathfrak{P})) = \text{NEG}_1^2(\mathbb{E}(\mathfrak{P})) \cup \text{NEG}_n^2(\mathbb{E}(\mathfrak{P}))$.

(5) By (3) above we get $\text{Bd}_a^2(\mathbb{E}(\mathfrak{P})) =$

$$\text{Bd}_1^2(\mathbb{E}(\mathfrak{P})) \cap \text{Bd}_n^2(\mathbb{E}(\mathfrak{P}))$$

$$\Rightarrow \text{Bd}_a^2(\mathbb{E}(\mathfrak{P})) \subseteq \text{Bd}_1^2(\mathbb{E}(\mathfrak{P})) \text{ and hence } |\text{Bd}_a^2(\mathbb{E}(\mathfrak{P}))| \leq$$

$$|\text{Bd}_1^2(\mathbb{E}(\mathfrak{P}))| \Rightarrow \frac{|\text{Bd}_a^2(\mathbb{E}(\mathfrak{P}))|}{|\mathbb{E}(\mathfrak{P})|} \leq \frac{|\text{Bd}_1^2(\mathbb{E}(\mathfrak{P}))|}{|\mathbb{E}(\mathfrak{P})|} \Rightarrow 1 -$$

$$\frac{|\text{Bd}_a^2(\mathbb{E}(\mathfrak{P}))|}{|\mathbb{E}(\mathfrak{P})|} \geq 1 - \frac{|\text{Bd}_1^2(\mathbb{E}(\mathfrak{P}))|}{|\mathbb{E}(\mathfrak{P})|} \Rightarrow f_a^2(\mathbb{E}(\mathfrak{P})) \geq f_1^2(\mathbb{E}(\mathfrak{P})).$$
 In

the same way we get $f_a^2(\mathbb{E}(\mathfrak{P})) \geq f_n^2(\mathbb{E}(\mathfrak{P}))$ thus $f_a^2(\mathbb{E}(\mathfrak{P})) \geq \max\{f_1^2(\mathbb{E}(\mathfrak{P})), f_n^2(\mathbb{E}(\mathfrak{P}))\}$.

Proposition 2.7. (Lower ² properties)

Let $\mathcal{F} = (\mathbb{V}(\mathfrak{X}), \mathbb{E}(\mathfrak{X}))$ be a generalized approximation space and $\mathfrak{P}, k \subseteq \mathfrak{X}$. Then:

$$(L_1^2) L_a^2(\mathbb{E}(\mathfrak{X})) = \mathbb{E}(\mathfrak{X}).$$

$$(L_2^2) \text{ If } \mathbb{E}(\mathfrak{P}) \subseteq \mathbb{E}(k), \text{ then } L_a^2(\mathbb{E}(\mathfrak{P})) \subseteq L_a^2(\mathbb{E}(k)).$$

$$(L_3^2) L_a^2(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(k)) \subseteq L_a^2(\mathbb{E}(\mathfrak{P})) \cap L_a^2(\mathbb{E}(k)).$$

$$(L_4^2) L_a^2(\mathbb{E}(\mathfrak{P})) \cup L_a^2(\mathbb{E}(k)) \subseteq L_a^2(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(k)).$$

$$(L_5^2) L_a^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{X}) - [U_a^2(\mathbb{E}(\mathfrak{X}) - \mathbb{E}(\mathfrak{P}))].$$

Proof:

$$(L_1^2) \text{ Clear from definition (2.1.(3)).}$$

(L₂²) Let $\mathbb{E}(\mathfrak{P}) \subseteq \mathbb{E}(k)$ and $\mathbf{e} \in L_a^2(\mathbb{E}(\mathfrak{P}))$, then there exist $\mathbf{i}$ incidence on $\mathbf{e}$ such that $\text{AVE}(\mathbf{i}) \subseteq \mathbb{E}(\mathfrak{P})$, since

$\mathbb{E}(\mathfrak{P}) \subseteq \mathbb{E}(k)$. Thus we have $\text{AVE}(\mathbf{i}) \subseteq \mathbb{E}(k)$ hence $\mathbf{e} \in L_a^2(\mathbb{E}(k))$ and so $L_a^2(\mathbb{E}(\mathfrak{P})) \subseteq L_a^2(\mathbb{E}(k))$.

(L₃²) Since $(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(k)) \subseteq \mathbb{E}(\mathfrak{P})$ by (L₂²) above we get $L_a^2(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(k)) \subseteq L_a^2(\mathbb{E}(\mathfrak{P}))$ — — — (1). And since $(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(k)) \subseteq \mathbb{E}(k)$ by (L₂²) above we get $L_a^2(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(k)) \subseteq L_a^2(\mathbb{E}(k))$ — — — (2). From (1) and (2) we have $L_a^2(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(k)) \subseteq L_a^2(\mathbb{E}(\mathfrak{P})) \cap L_a^2(\mathbb{E}(k))$.

(L₄²) Since $\mathbb{E}(\mathfrak{P}) \subseteq (\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(k))$ by (L₂²) above we get $L_a^2(\mathbb{E}(\mathfrak{P})) \subseteq L_a^2(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(k))$ — — — (1). And since $\mathbb{E}(k) \subseteq (\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(k))$ by (L₂²) above we get $L_a^2(\mathbb{E}(k)) \subseteq L_a^2(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(k))$ — — — (2). From (1) and (2) we have $L_a^2(\mathbb{E}(\mathfrak{P})) \cup L_a^2(\mathbb{E}(k)) \subseteq L_a^2(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(k))$.

(L₅²) Let $\mathbf{e} \in L_a^2(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathbf{e} \in \mathbb{E}(\mathfrak{X})$ and $\exists \mathbf{i}$ incidence on $\mathbf{e}$ such that $\text{AVE}(\mathbf{i}) \subseteq \mathbb{E}(\mathfrak{P}) \Rightarrow \text{AVE}(\mathbf{i}) \cap [\mathbb{E}(\mathfrak{X}) - \mathbb{E}(\mathfrak{P})] = \emptyset \Rightarrow \mathbf{e} \notin U_a^2(\mathbb{E}(\mathfrak{X}) - \mathbb{E}(\mathfrak{P})) \Rightarrow \mathbf{e} \in \mathbb{E}(\mathfrak{X}) - U_a^2(\mathbb{E}(\mathfrak{X}) - \mathbb{E}(\mathfrak{P}))$, thus $L_a^2(\mathbb{E}(\mathfrak{P})) \subseteq \mathbb{E}(\mathfrak{X}) - U_a^2(\mathbb{E}(\mathfrak{X}) - \mathbb{E}(\mathfrak{P}))$ — — — (1). $\mathbf{e} \in \mathbb{E}(\mathfrak{X}) - U_a^2(\mathbb{E}(\mathfrak{X}) - \mathbb{E}(\mathfrak{P})) \Rightarrow \mathbf{e} \in \mathbb{E}(\mathfrak{X})$ and $\mathbf{e} \notin U_a^2(\mathbb{E}(\mathfrak{X}) - \mathbb{E}(\mathfrak{P})) \Rightarrow$ there exist $\mathbf{i}$ incidence on $\mathbf{e}$ such that $\text{AVE}(\mathbf{i}) \cap [\mathbb{E}(\mathfrak{X}) - \mathbb{E}(\mathfrak{P})] = \emptyset$ thus $\exists \text{AVE}(\mathbf{i}) \subseteq \mathbb{E}(\mathfrak{P}) \Rightarrow \mathbf{e} \in L_a^2(\mathbb{E}(\mathfrak{P}))$, hence $\Rightarrow \mathbb{E}(\mathfrak{X}) - U_a^2(\mathbb{E}(\mathfrak{X}) - \mathbb{E}(\mathfrak{P})) \subseteq L_a^2(\mathbb{E}(\mathfrak{P}))$ — — — (2). From (1) and (2) we have $L_a^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{X}) - [U_a^2(\mathbb{E}(\mathfrak{X}) - \mathbb{E}(\mathfrak{P}))]$.

Proposition 2.8. (Upper ² properties)

Let $\mathcal{F} = (\mathbb{V}(\mathfrak{X}), \mathbb{E}(\mathfrak{X}))$ be a generalized approximation space and $\mathfrak{P}, k \subseteq \mathfrak{X}$. Then:

$$(U_1^2) U_a^2(\emptyset) = \emptyset.$$

$$(U_2^2) \text{ If } \mathbb{E}(\mathfrak{P}) \subseteq \mathbb{E}(k), \text{ then } U_a^2(\mathbb{E}(\mathfrak{P})) \subseteq U_a^2(\mathbb{E}(k)).$$

$$(U_3^2) U_a^2(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(k)) \subseteq U_a^2(\mathbb{E}(\mathfrak{P})) \cap U_a^2(\mathbb{E}(k)).$$

$$(U_4^2) U_a^2(\mathbb{E}(\mathfrak{P})) \cup U_a^2(\mathbb{E}(k)) \subseteq U_a^2(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(k)).$$

$$(U_5^2) U_a^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{X}) - [L_a^2(\mathbb{E}(\mathfrak{X}) - \mathbb{E}(\mathfrak{P}))].$$

Proof:

$$(U_1^2) \text{ Clear from definition (2.1.(3)).}$$

(U₂²) Let $\mathbb{E}(\mathfrak{P}) \subseteq \mathbb{E}(k)$ and $\mathbf{e} \in U_a^2(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathbf{e} \in \mathbb{E}(\mathfrak{X})$ where $\mathbf{i}$ incidence on $\mathbf{e}$ for all $\text{AVE}(\mathbf{i}) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset$ and since $\mathbb{E}(\mathfrak{P}) \subseteq \mathbb{E}(k)$ then $\text{AVE}(\mathbf{i}) \cap \mathbb{E}(k) \neq \emptyset$ thus $\mathbf{e} \in U_a^2(\mathbb{E}(k))$ and thus $U_a^2(\mathbb{E}(\mathfrak{P})) \subseteq U_a^2(\mathbb{E}(k))$.

(U₃²) Since $(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(k)) \subseteq \mathbb{E}(\mathfrak{P})$ by (U₂²) above we get $U_a^2(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(k)) \subseteq U_a^2(\mathbb{E}(\mathfrak{P}))$ — — — (1). And since $(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(k)) \subseteq \mathbb{E}(k)$ by (U₂²) above we get $U_a^2(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(k)) \subseteq U_a^2(\mathbb{E}(k))$ — — — (2). From (1) and (2) we have $U_a^2(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(k)) \subseteq U_a^2(\mathbb{E}(\mathfrak{P})) \cap U_a^2(\mathbb{E}(k))$.

(U₄²) Since $\mathbb{E}(\mathfrak{P}) \subseteq (\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(k))$ by (U₂²) above we get $U_a^2(\mathbb{E}(\mathfrak{P})) \subseteq U_a^2(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(k))$ — — — (1). And since $\mathbb{E}(k) \subseteq (\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(k))$ by (U₂²) above we get $U_a^2(\mathbb{E}(k)) \subseteq U_a^2(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(k))$ — — — (2). From (1) and (2) we have $U_a^2(\mathbb{E}(\mathfrak{P})) \cup U_a^2(\mathbb{E}(k)) \subseteq U_a^2(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(k))$.

(U₅²) By Lower ² properties (L₅²) $L_a^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{X}) - [U_a^2(\mathbb{E}(\mathfrak{X}) - \mathbb{E}(\mathfrak{P}))] \Rightarrow \mathbb{E}(\mathfrak{X}) - L_a^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{X}) -$

$(\mathbb{E}(\xi) - [U_a^2(\mathbb{E}(\xi) - \mathbb{E}(\mathfrak{P}))]) \Rightarrow U_a^2(\mathbb{E}(\xi) - \mathbb{E}(\mathfrak{P})) = \mathbb{E}(\xi) - L_a^2(\mathbb{E}(\mathfrak{P}))$. Now we replace $\mathbb{E}(\xi) - \mathbb{E}(\mathfrak{P})$ for $\mathbb{E}(\mathfrak{P})$ we get $U_a^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\xi) - L_a^2(\mathbb{E}(\xi) - \mathbb{E}(\mathfrak{P}))$.

Remark 2.9. The following relations $L_a^1(\mathbb{E}(\mathfrak{P})) \subseteq \mathbb{E}(\mathfrak{P})$, $L_a^1(\emptyset) = \emptyset$, $\mathbb{E}(\mathfrak{P}) \subseteq U_a^1(\mathbb{E}(\mathfrak{P}))$ and $U_a^1(\mathbb{E}(\xi)) = \mathbb{E}(\xi)$. Are always satisfied in first approximation operators but are not necessarily satisfied in second approximation operators. The following example will illustrate this.

Example 2.10. In example (2.5) let's take $\mathbb{E}(\mathfrak{P}) = \{e_1, e_3\}$ and $L_a^2(\mathbb{E}(\mathfrak{P})) = \{e_2, e_4, e_5\}$ thus we note that $L_a^2(\mathbb{E}(\mathfrak{P})) \not\subseteq \mathbb{E}(\mathfrak{P})$, and let's take $\mathbb{E}(\mathfrak{P}) = \{e_2, e_4, e_5\}$ and $U_a^2(\mathbb{E}(\mathfrak{P})) = \{e_1, e_3\}$ thus $\mathbb{E}(\mathfrak{P}) \not\subseteq U_a^2(\mathbb{E}(\mathfrak{P}))$. Now, let's clarify $L_a^2(\emptyset) \neq \emptyset$ we take $\xi = S_n$ is star graph and let $\mathfrak{P} = \emptyset$ is sub graph of ξ then for all $e \in \mathbb{E}(\xi)$ there exist i incidence on e such that $NINVE(i) \subseteq \mathbb{E}(\mathfrak{P})$ this means there exist $AVE(i) \subseteq \mathbb{E}(\mathfrak{P})$ thus $L_a^2(\emptyset) = \mathbb{E}(\xi)$. And let's clarify $U_a^2(\mathbb{E}(\xi)) \neq \mathbb{E}(\xi)$ we take $\xi = S_n$ is star graph for all $e \in \mathbb{E}(\xi)$ where i incidence on e there exist $AVE(i) \cap \mathbb{E}(\mathfrak{P}) = \emptyset$ such that $AVE(i) = NINVE(i)$ thus $U_a^2(\mathbb{E}(\xi)) = \emptyset$.

Proposition 2.10. Let $\Gamma = (\mathbb{V}(\xi), \mathbb{E}(\xi))$ be a generalized approximation space and $\mathfrak{P}, K \subseteq \xi$. Then:

- (1) $NEG_a^2(\emptyset) = \mathbb{E}(\xi)$.
- (2) $NEG_a^2(\mathbb{E}(\mathfrak{P})) \cap NEG_a^2(\mathbb{E}(K)) \subseteq NEG_a^2(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K))$.
- (3) $NEG_a^2(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) \subseteq NEG_a^2(\mathbb{E}(\mathfrak{P})) \cup NEG_a^2(\mathbb{E}(K))$.
- (4) $Bd_a^2(\emptyset) = \emptyset$.

Proof:

(1) By definition (2.2. (3)) we get $NEG_a^1(\emptyset) = \mathbb{E}(\xi) - U_a^2(\emptyset)$ and from Upper 2 properties (U_1^2) we get $NEG_a^1(\emptyset) = \mathbb{E}(\xi)$.

(2) Let $e \in [NEG_a^2(\mathbb{E}(\mathfrak{P})) \cap NEG_a^2(\mathbb{E}(K))]$

$\Rightarrow e \in NEG_a^2(\mathbb{E}(\mathfrak{P})) \wedge e \in NEG_a^2(\mathbb{E}(K)) \Rightarrow e \notin$

$U_a^2(\mathbb{E}(\mathfrak{P})) \wedge e \notin U_a^2(\mathbb{E}(K))$

$\Rightarrow$ there exist i incidence on e such that $AVE(i) \cap \mathbb{E}(\mathfrak{P}) = \emptyset$ and $AVE(i) \cap \mathbb{E}(K) = \emptyset$, since $(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) \subseteq \mathbb{E}(\mathfrak{P}) \Rightarrow AVE(i) \cap (\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) = \emptyset$, hence $e \notin U_a^2(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K)) \Rightarrow e \in NEG_a^2(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K))$, thus

$NEG_a^2(\mathbb{E}(\mathfrak{P})) \cap NEG_a^2(\mathbb{E}(K)) \subseteq NEG_a^2(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K))$.

(3) Let $e \in NEG_a^2(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) \Rightarrow e \notin U_a^2(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K))$

$\Rightarrow$ there exist i incidence on e such that $AVE(i) \cap (\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) = \emptyset$, since $\mathbb{E}(\mathfrak{P}) \subseteq (\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) \Rightarrow AVE(i) \cap \mathbb{E}(\mathfrak{P}) = \emptyset \Rightarrow e \notin U_a^2(\mathbb{E}(\mathfrak{P})) \Rightarrow e \in$

$NEG_a^2(\mathbb{E}(\mathfrak{P}))$, since $\mathbb{E}(K) \subseteq (\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) \Rightarrow AVE(i) \cap \mathbb{E}(K) = \emptyset \Rightarrow e \notin U_a^2(\mathbb{E}(K)) \Rightarrow$

$e \in NEG_a^2(\mathbb{E}(K))$. Thus $e \in NEG_a^2(\mathbb{E}(\mathfrak{P})) \cup NEG_a^2(\mathbb{E}(K))$. So, $NEG_a^2(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) \subseteq NEG_a^2(\mathbb{E}(\mathfrak{P})) \cup NEG_a^2(\mathbb{E}(K))$.

(4) Since $Bd_a^2(\emptyset) = U_a^2(\emptyset) - L_a^2(\emptyset)$ from Upper² properties (U_1^2) we get $Bd_a^2(\emptyset) = \emptyset$.

Proposition 2.11. Let $\Gamma = (\mathbb{V}(\xi), \mathbb{E}(\xi))$ be a generalized approximation space, if Γ induced of antisymmetric graph ξ and $\mathfrak{P}, K \subseteq \xi$. Then:

(1) $L_a^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{P})$ or $L_a^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\xi)$ when $\mathbb{E}(\mathfrak{P}) = \mathbb{E}(\xi) - \{e\}$.

(2) $U_a^2(\mathbb{E}(\mathfrak{P})) \subseteq \mathbb{E}(\mathfrak{P})$.

(3) $L_a^2(\mathbb{E}(\mathfrak{P})) = L_a^2(L_a^2(\mathbb{E}(\mathfrak{P})))$.

(4) $U_a^2(\mathbb{E}(\mathfrak{P})) = U_a^2(U_a^2(\mathbb{E}(\mathfrak{P})))$.

(5) $U_a^2(\mathbb{E}(\mathfrak{P})) \subseteq L_a^2(\mathbb{E}(\mathfrak{P}))$.

(6) $Bd_a^2(\mathbb{E}(\mathfrak{P})) = \emptyset$.

(7) $Bd_a^2(\mathbb{E}(\mathfrak{P})) = Bd_a^2(Bd_a^2(\mathbb{E}(\mathfrak{P})))$.

(8) $Bd_a^2(\mathbb{E}(\mathfrak{P})) \subseteq \mathbb{E}(\mathfrak{P})$.

(9) $Bd_a^2(\mathbb{E}(\mathfrak{P})) \cap Bd_a^2(\mathbb{E}(K)) = Bd_a^2(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K))$.

(10) $Bd_a^2(\mathbb{E}(\mathfrak{P})) \cup Bd_a^2(\mathbb{E}(K)) = Bd_a^2(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K))$.

Proof:

Let ξ be an antisymmetric graph and $\mathfrak{P}, K \subseteq \xi$:

(1) When $\mathbb{E}(\mathfrak{P}) \neq \mathbb{E}(\xi) - \{e\}$ to prove that $L_a^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{P})$. Let $e \in \mathbb{E}(\mathfrak{P})$ and $e \notin L_a^2(\mathbb{E}(\mathfrak{P})) \Rightarrow \forall AVE(i) \not\subseteq \mathbb{E}(\mathfrak{P})$ where i incidence on e and this contradiction, since for all $e \in \mathbb{E}(\mathfrak{P}) \Rightarrow INVE(i) = \{e\} \subseteq \mathbb{E}(\mathfrak{P})$ because ξ is antisymmetric and thus $e \in L_a^2(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathbb{E}(\mathfrak{P}) \subseteq L_a^2(\mathbb{E}(\mathfrak{P}))$ --- (1).

Let $e \in L_a^2(\mathbb{E}(\mathfrak{P})) \Rightarrow \exists AVE(i) \subseteq \mathbb{E}(\mathfrak{P})$ where i incidence on e since $\mathbb{E}(\mathfrak{P}) \neq \mathbb{E}(\xi) - \{e\}$ so $AVE(i) = INVE(i) = \{e\}$ thus $e \in \mathbb{E}(\mathfrak{P}) \Rightarrow L_a^2(\mathbb{E}(\mathfrak{P})) \subseteq \mathbb{E}(\mathfrak{P})$ --- (2). From (1) and (2) we have $L_a^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{P})$. When $\mathbb{E}(\mathfrak{P}) = \mathbb{E}(\xi) - \{e\}$. Let $e \in \mathbb{E}(\xi)$, either $e \in \mathbb{E}(\mathfrak{P})$ then $INVE(i) = \{e\} \subseteq \mathbb{E}(\mathfrak{P})$ where i incidence on e thus $e \in L_a^2(\mathbb{E}(\mathfrak{P}))$. Or $e \notin \mathbb{E}(\mathfrak{P})$ so $e = e$ then $NINVE(i^*) \subseteq \mathbb{E}(\mathfrak{P})$ such that i^* incidence on e thus $e \in L_a^2(\mathbb{E}(\mathfrak{P}))$. Hence $\mathbb{E}(\xi) \subseteq L_a^2(\mathbb{E}(\mathfrak{P}))$ and $L_a^2(\mathbb{E}(\mathfrak{P})) \subseteq \mathbb{E}(\xi)$. So $L_a^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\xi)$.

(2) Let $e \in U_a^2(\mathbb{E}(\mathfrak{P}))$ and $e \notin \mathbb{E}(\mathfrak{P}) \Rightarrow \forall AVE(i) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset$ where i incidence on e , since ξ is antisymmetric graph then $INVE(i) = \{e\}$ and since $e \notin \mathbb{E}(\mathfrak{P})$ hence $INVE(i) \cap \mathbb{E}(\mathfrak{P}) = \emptyset$ let's take $AVE(i) = INVE(i)$ thus $AVE(i) \cap \mathbb{E}(\mathfrak{P}) = \emptyset$ this contradiction. Thus if $e \in U_a^2(\mathbb{E}(\mathfrak{P}))$ must $e \in \mathbb{E}(\mathfrak{P}) \Rightarrow U_a^2(\mathbb{E}(\mathfrak{P})) \subseteq \mathbb{E}(\mathfrak{P})$.

(3) When $\mathbb{E}(\mathfrak{P}) \neq \mathbb{E}(\xi) - \{e\}$ by (1) above $L_a^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\mathfrak{P})$ and by Lower² properties (L_2^2) we get $L_a^2(L_a^2(\mathbb{E}(\mathfrak{P}))) = L_a^2(\mathbb{E}(\mathfrak{P}))$, and when $\mathbb{E}(\mathfrak{P}) = \mathbb{E}(\xi) - \{e\}$ by (1) above $L_a^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\xi)$ --- (1) and by Lower² properties (L_2^2), so $L_a^2(L_a^2(\mathbb{E}(\mathfrak{P}))) = L_a^2(\mathbb{E}(\xi))$ --- (2) from Lower² properties (L_1^2) then $L_a^2(\mathbb{E}(\xi)) = \mathbb{E}(\xi)$ in (2) we get $L_a^2(L_a^2(\mathbb{E}(\mathfrak{P}))) = \mathbb{E}(\xi)$ in (1) we have $L_a^2(L_a^2(\mathbb{E}(\mathfrak{P}))) = L_a^2(\mathbb{E}(\mathfrak{P}))$.

(4) By (2) above we get $U_a^2(\mathbb{E}(\mathfrak{P})) \subseteq \mathbb{E}(\mathfrak{P})$ and by using Upper² properties (U_2^2) we get $U_a^2(U_a^2(\mathbb{E}(\mathfrak{P}))) \subseteq$

$U_a^2(\mathbb{E}(\mathfrak{P})) - \dots - (1)$. Now to prove $U_a^2(\mathbb{E}(\mathfrak{P})) \subseteq U_a^2(U_a^2(\mathbb{E}(\mathfrak{P})))$. Let $\mathbf{e} \in U_a^2(\mathbb{E}(\mathfrak{P})) \Rightarrow \forall \text{ AVE}(\mathbf{i}) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset$ where $\mathbf{i}$ incidence on $\mathbf{e}$, since $U_a^2(\mathbb{E}(\mathfrak{P})) \subseteq \mathbb{E}(\mathfrak{P}) \Rightarrow \forall \text{ AVE}(\mathbf{i}) \cap U_a^2(\mathbb{E}(\mathfrak{P})) \neq \emptyset \Rightarrow \mathbf{e} \in U_a^2(U_a^2(\mathbb{E}(\mathfrak{P})))$. Thus $U_a^2(\mathbb{E}(\mathfrak{P})) \subseteq U_a^2(U_a^2(\mathbb{E}(\mathfrak{P}))) - \dots - (2)$. From (1) and (2) we have $U_a^2(\mathbb{E}(\mathfrak{P})) = U_a^2(U_a^2(\mathbb{E}(\mathfrak{P})))$.

(5) By (2) above we get $U_a^2(\mathbb{E}(\mathfrak{P})) \subseteq \mathbb{E}(\mathfrak{P})$ and by (1) above we get $\mathbb{E}(\mathfrak{P}) \subseteq L_a^2(\mathbb{E}(\mathfrak{P}))$ and hence $U_a^2(\mathbb{E}(\mathfrak{P})) \subseteq L_a^2(\mathbb{E}(\mathfrak{P}))$.

(6) Suppose that $\text{Bd}_a^2(\mathbb{E}(\mathfrak{P})) \neq \emptyset \Rightarrow \exists \mathbf{e} \in \text{Bd}_a^2(\mathbb{E}(\mathfrak{P})) \Rightarrow \mathbf{e} \in U_a^2(\mathbb{E}(\mathfrak{P})) \wedge \mathbf{e} \notin L_a^2(\mathbb{E}(\mathfrak{P}))$, since $\mathbf{e} \in U_a^2(\mathbb{E}(\mathfrak{P})) \Rightarrow \forall \text{ AVE}(\mathbf{i}) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset$ where $\mathbf{i}$ incidence on $\mathbf{e}$, so $\text{INVE}(\mathbf{i}) \cap \mathbb{E}(\mathfrak{P}) \neq \emptyset$ and $\text{INVE}(\mathbf{i}) = \{\mathbf{e}\}$ because ξ is antisymmetric graph thus $\mathbf{e} \in \mathbb{E}(\mathfrak{P})$ by (1) above this contradiction with $\mathbf{e} \notin L_a^2(\mathbb{E}(\mathfrak{P}))$. Hence $\text{Bd}_a^2(\mathbb{E}(\mathfrak{P})) = \emptyset$. The proof of paragraphs (7), (8), (9) and (10) can be derived directly from (6).

Proposition 2.12. Let $\Gamma = (\mathbb{V}(\xi), \mathbb{E}(\xi))$ be a generalized approximation space, if Γ induced of antisymmetric graph ξ then for each $\mathfrak{P} \subseteq \xi$ is $(\mathfrak{P}$ -exact).

Proof: Let ξ be an antisymmetric graph and $\mathfrak{P} \subseteq \xi$. By proposition (2.10(6)) we get $\text{Bd}_a^2(\mathbb{E}(\mathfrak{P})) = \emptyset \Rightarrow |\text{Bd}_a^2(\mathbb{E}(\mathfrak{P}))| = 0$ and by definition (2.3) we get $f_a^2(\mathbb{E}(\mathfrak{P})) = 1 - \frac{|\text{Bd}_a^2(\mathbb{E}(\mathfrak{P}))|}{|\mathbb{E}(\xi)|} = 1 - \frac{0}{|\mathbb{E}(\xi)|} = 1$ thus $\mathfrak{P}$ -exact.

Proposition 2.13. Let $\Gamma = (\mathbb{V}(\xi), \mathbb{E}(\xi))$ be a generalized approximation space, if Γ induced of star graph ξ and $\mathfrak{P}, K \subseteq \xi$. Then:

- (1) $L_a^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\xi)$.
- (2) $U_a^2(\mathbb{E}(\mathfrak{P})) = \emptyset$.
- (3) $\text{Bd}_a^2(\mathbb{E}(\mathfrak{P})) = \emptyset$.
- (4) $\text{NEG}_a^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\xi)$.
- (5) $\text{Bd}_a^2(\mathbb{E}(\mathfrak{P})) = \text{Bd}_a^2(\text{Bd}_a^2(\mathbb{E}(\mathfrak{P})))$.
- (6) $\text{Bd}_a^2(\mathbb{E}(\mathfrak{P})) \subseteq \mathbb{E}(\mathfrak{P})$.
- (7) $\text{Bd}_a^2(\mathbb{E}(\mathfrak{P})) \cap \text{Bd}_a^2(\mathbb{E}(K)) = \text{Bd}_a^2(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K))$.
- (8) $\text{Bd}_a^2(\mathbb{E}(\mathfrak{P})) \cup \text{Bd}_a^2(\mathbb{E}(K)) = \text{Bd}_a^2(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K))$.
- (9) $\text{NEG}_a^2(\mathbb{E}(\mathfrak{P})) = \text{NEG}_a^2(\text{NEG}_a^2(\mathbb{E}(\mathfrak{P})))$.
- (10) $\text{NEG}_a^2(\mathbb{E}(\mathfrak{P})) \cap \text{NEG}_a^2(\mathbb{E}(K)) = \text{NEG}_a^2(\mathbb{E}(\mathfrak{P}) \cap \mathbb{E}(K))$.
- (11) $\text{NEG}_a^2(\mathbb{E}(\mathfrak{P})) \cup \text{NEG}_a^2(\mathbb{E}(K)) = \text{NEG}_a^2(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K))$.
- (12) $L_a^2(\mathbb{E}(\mathfrak{P})) = L_a^2(L_a^2(\mathbb{E}(\mathfrak{P})))$.
- (13) $\mathbb{E}(\mathfrak{P}) \subseteq L_a^2(U_a^2(\mathbb{E}(\mathfrak{P})))$.
- (14) $L_a^2(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) = L_a^2(\mathbb{E}(\mathfrak{P})) \cup L_a^2(\mathbb{E}(K))$.
- (15) $U_a^2(\mathbb{E}(\mathfrak{P})) = U_a^2(U_a^2(\mathbb{E}(\mathfrak{P})))$.
- (16) $U_a^2(L_a^2(\mathbb{E}(\mathfrak{P}))) \subseteq \mathbb{E}(\mathfrak{P})$.

$$(17) U_a^2(\mathbb{E}(\mathfrak{P}) \cup \mathbb{E}(K)) = U_a^2(\mathbb{E}(\mathfrak{P})) \cup U_a^2(\mathbb{E}(K)).$$

Proof:

Let ξ be a star graph and $\mathfrak{P}, K \subseteq \xi$:

(1) Since ξ is star then $\forall \mathbf{e} \in \mathbb{E}(\xi)$ and $\mathbf{i}$ incidence on $\mathbf{e}$ we have $\text{NINVE}(\mathbf{i}) = \emptyset$ and $\text{NINVE}(\mathbf{i}) \subseteq \mathbb{E}(\mathfrak{P}) \Rightarrow \exists \text{ AVE}(\mathbf{i})$ such that $\text{AVE}(\mathbf{i}) \subseteq \mathbb{E}(\mathfrak{P}) \Rightarrow \mathbf{e} \in L_a^2(\mathbb{E}(\mathfrak{P}))$, thus for all $\mathbf{e} \in \mathbb{E}(\xi)$ we get $\mathbf{e} \in L_a^2(\mathbb{E}(\mathfrak{P}))$. Hence $L_a^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\xi)$.

(2) Since ξ is star then $\forall \mathbf{e} \in \mathbb{E}(\xi)$ and $\mathbf{i}$ incidence on $\mathbf{e}$ we have $\text{NINVE}(\mathbf{i}) = \emptyset$ and $\text{NINVE}(\mathbf{i}) \cap \mathbb{E}(\mathfrak{P}) = \emptyset \Rightarrow \exists \text{ AVE}(\mathbf{i})$ such that $\text{AVE}(\mathbf{i}) \cap \mathbb{E}(\mathfrak{P}) = \emptyset \Rightarrow \mathbf{e} \notin U_a^2(\mathbb{E}(\mathfrak{P}))$ thus for all $\mathbf{e} \in \mathbb{E}(\xi)$ we get $\mathbf{e} \notin U_a^2(\mathbb{E}(\mathfrak{P}))$. Hence $U_a^2(\mathbb{E}(\mathfrak{P})) = \emptyset$.

(3) Since $\text{Bd}_a^2(\mathbb{E}(\mathfrak{P})) = U_a^2(\mathbb{E}(\mathfrak{P})) - L_a^2(\mathbb{E}(\mathfrak{P}))$ and by (1), (2) above we get $\text{Bd}_a^2(\mathbb{E}(\mathfrak{P})) = \emptyset$.

(4) Since $\text{NEG}_a^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\xi) - U_a^2(\mathbb{E}(\mathfrak{P}))$ and by (2) above we get $\text{NEG}_a^2(\mathbb{E}(\mathfrak{P})) = \mathbb{E}(\xi)$.

The proof of paragraphs (5), (6), (7), (8), (9), (10), and (11) can be derived directly from (3) and (4). And the proof of paragraphs (12), (13), (14), (15) (16) and (17) can be derived directly from (1) and (2).

Proposition 2.14. Let $\Gamma = (\mathbb{V}(\xi), \mathbb{E}(\xi))$ be a generalized approximation space, if Γ induced of star graph ξ then for all $\mathfrak{P} \subseteq \xi$ is $(\mathfrak{P}$ -exact).

Proof: Let ξ be a star graph and $\mathfrak{P} \subseteq \xi$. By proposition(4.2.12) we get $\text{Bd}_a^2(\mathbb{E}(\mathfrak{P})) = \emptyset \Rightarrow |\text{Bd}_a^2(\mathbb{E}(\mathfrak{P}))| = 0$ and by definition (2.3) we have

$$f_a^2(\mathbb{E}(\mathfrak{P})) = 1 - \frac{|\text{Bd}_a^2(\mathbb{E}(\mathfrak{P}))|}{|\mathbb{E}(\xi)|} = 1 - \frac{0}{|\mathbb{E}(\xi)|} = 1 \text{ thus } \mathfrak{P}\text{-exact.}$$

We will provide a practical real-life example and calculate the accuracy using operators L_a^2 and U_a^2 .

If the vertices in the graph represent cities and the edges represent the roads connecting these cities, with traffic congestion on these roads depending on the number of cities and the roads linking them, the practical objective of this study is to assess the extent of traffic congestion within the sub graph H. This approach is grounded in real-world applications, where managing traffic flow is critical to urban planning and efficiency. When accuracy is high, it indicates a high or good state of traffic flow. This occurs because an increase in the number of edges and a decrease in the number of vertices in H lead to a rise in the elements of $\text{La}(\text{H})$, which in turn reduces $\text{Bda}(\text{H})$ and increases the accuracy $\text{fa}(\text{H})$. Conversely, when the number of vertices is high and the number of edges is low in H, $\text{La}(\text{H})$ decreases, causing $\text{Bda}(\text{H})$ to increase and $\text{fa}(\text{H})$ to decline. This aligns with real-world road networks: when the number of cities (vertices) is low and the number of roads (edges) is high, traffic flow improves. In contrast, a high number of cities with few roads leads to traffic congestion. This practical framework provides a vital solution to a key issue faced by urban planners: optimizing road networks to reduce congestion and enhance the efficiency of transportation

systems. The reason $La(H)$ increases when edges increase and vertices decrease is,

$L_i^2(\mathbb{E}(\mathfrak{P}))$: denotes the increase in the number of roads in $\mathfrak{P}$, as per the definition $\mathbf{e} \in L_i^2(\mathbb{E}(\mathfrak{P}))$ then there exist $\mathbf{i}$ incidence on $\mathbf{e}$ such that $INVE(\mathbf{i}) \subseteq \mathbb{E}(\mathfrak{P})$, thus $L_i^2(\mathbb{E}(\mathfrak{P}))$ it's concerned with the number of roads in $\mathfrak{P}$.

$L_n^2(\mathbb{E}(\mathfrak{P}))$: represents the decrease in the number of cities in $\mathfrak{P}$, because $\mathbf{e} \in L_n^2(\mathbb{E}(\mathfrak{P}))$ there exist $\mathbf{i}$ incidence on $\mathbf{e}$ such that $NINVE(\mathbf{i}) \subseteq \mathbb{E}(\mathfrak{P})$ and this means $\mathbf{i} \notin \mathbb{V}(\mathfrak{P})$, thus $L_n^2(\mathbb{E}(\mathfrak{P}))$ it's concerned with the number of cities in $\mathfrak{P}$.

$L_a^2(\mathbb{E}(\mathfrak{P}))$: indicates a high number of roads in $\mathfrak{P}$ and a low number of cities, since $L_a^2(\mathbb{E}(\mathfrak{P})) = L_i^2(\mathbb{E}(\mathfrak{P})) \cup L_n^2(\mathbb{E}(\mathfrak{P}))$ thus $L_a^2(\mathbb{E}(\mathfrak{P}))$ it's concerned with the number of roads and cities in $\mathfrak{P}$.

Therefore, as $L_a^2(\mathbb{E}(\mathfrak{P}))$ increases, $Bd_a^2(\mathbb{E}(\mathfrak{P}))$ decreases, leading to an improvement in the accuracy of $\mathfrak{P}$. When the accuracy increases, this indicates that traffic congestion in $\mathfrak{P}$ is minimal, and conversely, lower accuracy signals greater congestion. This provides a practical and scientifically precise application for addressing traffic congestion problems, offering a clear and effective solution to enhance urban mobility.

Example 2.15. Let $\xi = (\mathbb{V}(\xi), \mathbb{E}(\xi))$ be a network of cities, and $\mathbb{V}(\xi) = \{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3, \mathbf{i}_4, \mathbf{i}_5, \mathbf{i}_6\}$ represents the set of cities, and

$$\mathbb{E}(\xi) = \left\{ \begin{array}{l} \mathbf{e}_1 = (\mathbf{i}_1, \mathbf{i}_2), \mathbf{e}_2 = (\mathbf{i}_1, \mathbf{i}_6), \mathbf{e}_3 = (\mathbf{i}_2, \mathbf{i}_4), \\ \mathbf{e}_4 = (\mathbf{i}_2, \mathbf{i}_5), \mathbf{e}_5 = (\mathbf{i}_3, \mathbf{i}_5), \\ \mathbf{e}_6 = (\mathbf{i}_3, \mathbf{i}_6), \mathbf{e}_7 = (\mathbf{i}_1, \mathbf{i}_5) \end{array} \right\}$$

represents the set of roads connecting the cities in $\mathbb{V}(\xi)$.

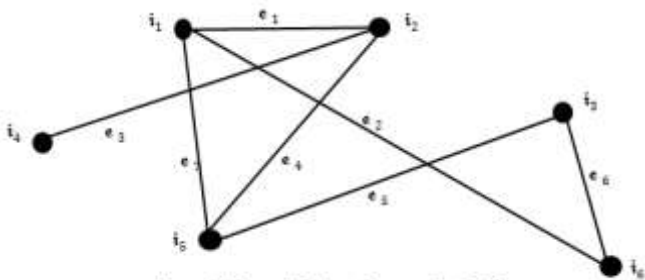


Figure 2.2: graph ξ given in example (2.15).

Then the incidence vertex edges systems are given by :

$$INVE(\mathbf{i}_1) = \{\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_7\}\}, INVE(\mathbf{i}_2) = \{\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4\}\},$$

$$INVE(\mathbf{i}_3) = \{\{\mathbf{e}_5, \mathbf{e}_6\}\}, INVE(\mathbf{i}_4) = \{\{\mathbf{e}_3\}\},$$

$$INVE(\mathbf{i}_5) = \{\{\mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_7\}\}, INVE(\mathbf{i}_6) = \{\{\mathbf{e}_2, \mathbf{e}_6\}\}.$$

And the non-incidence vertex edges systems are given by:

$$NINVE(\mathbf{i}_1) = \{\{\mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6\}\}, NINVE(\mathbf{i}_2) = \{\{\mathbf{e}_2, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\}\}, NINVE(\mathbf{i}_3) = \{\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_7\}\},$$

$$NINVE(\mathbf{i}_4) = \{\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_6, \mathbf{e}_7\}\},$$

$$NINVE(\mathbf{i}_5) = \{\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_6\}\},$$

$$NINVE(\mathbf{i}_6) = \{\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_7\}\}.$$

Also, we have the admixture vertex edges systems are given by: $AVES(\mathbf{i}_1) = \{\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_7\}, \{\mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6\}\},$

$$AVES(\mathbf{i}_2) = \{\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4\}, \{\mathbf{e}_2, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\}\},$$

$$AVES(\mathbf{i}_3) = \{\{\mathbf{e}_5, \mathbf{e}_6\}, \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_7\}\},$$

$$AVES(\mathbf{i}_4) = \{\{\mathbf{e}_3\}, \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_6, \mathbf{e}_7\}\},$$

$$AVES(\mathbf{i}_5) = \{\{\mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_7\}, \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_6\}\},$$

$$AVES(\mathbf{i}_6) = \{\{\mathbf{e}_2, \mathbf{e}_6\}, \{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_7\}\}.$$

The objective of this practical example is to select specific cities from ξ and determine a sub network of roads that provides the best traffic flow between these cities. This is achieved by calculating the accuracy.

Let the selected cities be $\mathbb{V}(\mathfrak{P}_1) = \{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_4, \mathbf{i}_5, \mathbf{i}_6\}$

If we choose the set of roads connecting these cities, it's $\mathbb{E}(\mathfrak{P}_1) = \{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_7\}$ then second lower of $\mathfrak{P}_1$ given by:

$$L_i^2(\mathbb{E}(\mathfrak{P}_1)) = \{\mathbf{e}_3\}, L_n^2(\mathbb{E}(\mathfrak{P}_1)) = \emptyset \text{ and } L_a^2(\mathbb{E}(\mathfrak{P}_1)) = \{\mathbf{e}_3\}. \text{ The second upper of } \mathfrak{P}_1 \text{ given by: } U_i^2(\mathbb{E}(\mathfrak{P}_1)) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_7\}, U_n^2(\mathbb{E}(\mathfrak{P}_1)) = \mathbb{E}(\xi) \text{ and } U_a^2(\mathbb{E}(\mathfrak{P}_1)) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_7\}, \text{ thus } Bd_i^2(\mathbb{E}(\mathfrak{P}_1)) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_7\},$$

$$Bd_n^2(\mathbb{E}(\mathfrak{P}_1)) = \mathbb{E}(\xi) \text{ and } Bd_a^2(\mathbb{E}(\mathfrak{P}_1)) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_7\},$$

$$\text{so } f_i^2(\mathbb{E}(\mathfrak{P}_1)) = \frac{3}{7}, f_n^2(\mathbb{E}(\mathfrak{P}_1)) = 0 \text{ and } f_a^2(\mathbb{E}(\mathfrak{P}_1)) = \frac{3}{7}.$$

such that $f_i^2(\mathbb{E}(\mathfrak{P}_1)) = \frac{3}{7}$ this indicates that the number of roads is low compared to the number of cities, $f_n^2(\mathbb{E}(\mathfrak{P}_1)) = 0$ this indicates that the number of cities is high compared to the number of roads. And $f_a^2(\mathbb{E}(\mathfrak{P}_1)) = \frac{3}{7}$ this indicates a low number of roads and a high number of cities, which means there is traffic congestion in the flow through the roads of $\mathfrak{P}_1$ connecting the cities $\mathbb{V}(\mathfrak{P}_1)$. To improve traffic flow in $\mathbb{V}(\mathfrak{P}_1)$, accuracy must be enhanced by adding new connecting roads. If we choose the set of roads $\mathbb{E}(\mathfrak{P}_2) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_7\}$ then second lower of $\mathfrak{P}_2$ given by: $L_i^2(\mathbb{E}(\mathfrak{P}_2)) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_7\}, L_n^2(\mathbb{E}(\mathfrak{P}_2)) = \{\mathbf{e}_5, \mathbf{e}_6\}$ and $L_a^2(\mathbb{E}(\mathfrak{P}_2)) = \mathbb{E}(\xi)$. The second upper of $\mathfrak{P}_2$ given by: $U_i^2(\mathbb{E}(\mathfrak{P}_2)) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_7\}, U_n^2(\mathbb{E}(\mathfrak{P}_2)) = \mathbb{E}(\xi) \text{ and } U_a^2(\mathbb{E}(\mathfrak{P}_2)) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_7\}$ thus $Bd_i^2(\mathbb{E}(\mathfrak{P}_2)) = \emptyset, Bd_n^2(\mathbb{E}(\mathfrak{P}_2)) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_7\}$, and $Bd_a^2(\mathbb{E}(\mathfrak{P}_2)) = \emptyset$, so $f_i^2(\mathbb{E}(\mathfrak{P}_2)) = 1, f_n^2(\mathbb{E}(\mathfrak{P}_2)) = \frac{2}{7}$ and $f_a^2(\mathbb{E}(\mathfrak{P}_2)) = 1$. such that $f_i^2(\mathbb{E}(\mathfrak{P}_2)) = 1$ this indicates a high or suitable number of roads compared to the cities, $f_n^2(\mathbb{E}(\mathfrak{P}_2)) = \frac{2}{7}$ this indicates a low or suitable number of cities compared to the roads. And $f_a^2(\mathbb{E}(\mathfrak{P}_2)) = 1$ this indicates an optimal road-to-city ratio, ensuring excellent traffic flow if these edges $\mathbb{E}(\mathfrak{P}_2)$ are chosen to connect the cities $\mathbb{V}(\mathfrak{P}_1)$.

3. REFERENCES:

1. K.S. Al'Dzhabri, A. Hamzha. and Y.S. Essa. On DG-topological spaces associated with directed graphs. Journal of Discrete Mathematical Sciences & Cryptography, 32(5), (2020) 1039-1046. Doi.org/10.1080/09720529.2020.1714886
2. K.S. Al'Dzhabri and M. F. Hani, On Certain Types of Topological Spaces Associated with Digraphs. Journal of Physics: Conference Series 1591(2020)012055.
3. K.S. Al'Dzhabri and et al, DG-domination topology in Digraph. Journal of Prime Research in Mathematics 2021, 17(2), 93–100. <http://jprm.sms.edu.pk/>
4. K.S. Al'Dzhabri, Enumeration of connected components of acyclic digraph. Journal of Discrete Mathematical Sciences and Cryptography, 2021, 24(7), 2047–2058. DOI: 10.1080/09720529.2021.1965299
5. L. W. Beineke and R. J. Wilson, Topics in topological graph theory, Cambridge University Press, 2009
6. F. Harary, Graph Theory (Addison-Wesley, Reading Mass, 1969).
7. M. Moller, General Topology, Authors, Notes, (2007).
8. M. Shokry, Approximations structures generated by trees vertices, International Journal of Engineering Science and Technology (IJEST), Vol.4 (2), February (2012), pp.406-418. 10.
9. M. Shokry, Closure concepts generated by directed trees; Journal of Mathematics & Computer Sciences, Vol.21 (1), (2010), pp.45-51.
10. S.S. Ray, Graph Theory with Algorithms and its Applications: In Applied Sciences and Technology, Springer, New Delhi, (2013).