

Exploring Klotski: An Investigation of the Minimum Number of Moves in a Special Case of Slide Puzzles

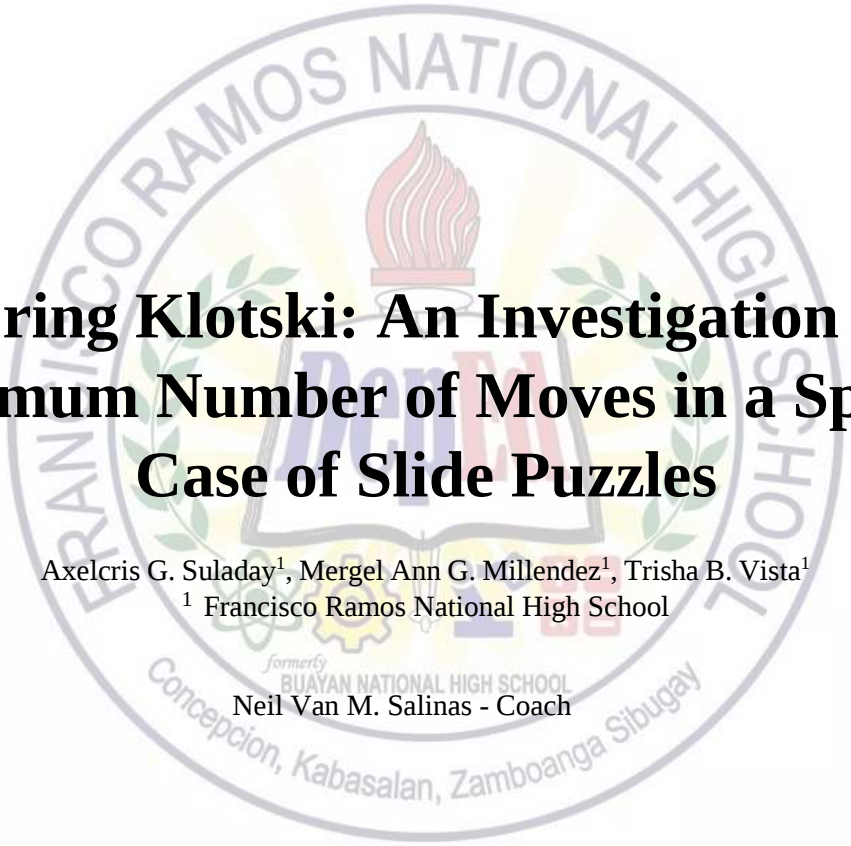
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December 2024

Abstract: Klotski is a special type of sliding puzzle that first emerged in the early 20th century. It refers to a variety of sliding block or tile games with the goal of moving a specific block to a predetermined spot. By changing the game's rule that requires a player to move a certain tile or block from one corner of the puzzle to its opposite corner, providing that it contains spaces more than or equal to one, this inquiry sought to ascertain the minimum number of moves. It was also recognized that the Klotski puzzle's dimension was directly influenced by the number of spaces. The results of the investigation proposed the use of these formulae in determining the minimum number of moves (M) given the number of columns/rows of a square-shaped Klotski: a. $M = 8s \sim 11$ if the number of spaces (S) is equal to 1, b. $M = 8s \sim 3S \sim 8$ if the number of spaces (S) is greater than or equal to 1 but less than or equal to 4, c. $M = 8s \sim S \sim 16$ if the number of spaces (S) is greater than or equal to 4 but less than or equal to infinity and the number of rows/columns (s) is greater than or equal to the number of spaces (S). Alternatively, 3 formulae were proposed to determine the minimum number of moves (M) of a rectangular-shaped Klotski: a. $M = 6l + 2w - 13$ if the number of spaces is equal to 1, b. $M = 4 \cdot 1 + 4lw - 14$ if the product of the dimension is the quantity of n plus 1 multiplied by the quantity of n plus 2, and c. $M = 6l \cdot 2w \sim S \sim 16$ if the product of the dimension is the quantity of n plus f multiplied by the quantity of n plus k , where f is greater than or equal to 1, k is greater than 2, k is greater than f , f is less than the number of spaces (S), and lastly if the number of spaces is greater than or equal to 2 but less than or equal to infinity. In the case where only the product (P) and difference (d) of the dimensions are given, the formula $M_{Pd} = 4 \cdot d^2 + 4P + 2d - 13$ may be used to determine the minimum number of moves if the number of spaces is equal to 1, while $M_{Pd} = 4 \cdot 1 + 4P - 14$ is used only if the product of the dimension is the quantity of n plus 1 multiplied by the quantity of n plus 2, and $M_{Pd} = 4 \cdot d^2 + 4P + 2d - S - 16$ will be utilized if the product of the dimension is the quantity of n plus f multiplied by the quantity of n plus k , where f is greater than or equal to 1, k is greater than 2, k is greater than f , f is less than the number of spaces (S), and the number of spaces is greater than or equal to 2 but less than or equal to infinity. It is recommended that additional investigations must be done for other cases of the puzzle, such as with triangles and other shaped Klotski puzzles, with consideration of the movement of more than 1 block and to determine the other possible real-life application of the concepts and result in this study.

Arithmetic, Klotski, Minimum Number of Moves, Pattern, Sliding Puzzle, Quadratic Formula, Piecewise function



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Arithmetic, Klotski, Minimum Number of Moves, Pattern, Sliding Puzzle, Quadratic Formula, Piecewise function

1 Chapter I

Introduction

Klotski is a type of a sliding puzzle that have started and gained popularity in the early 20th century. It refers to a whole group of similar sliding-block puzzles with a same objective of moving a certain block to a predetermined location [3]. Within the frame, the blocks can be moved anywhere by sliding—not turning, lifting or jumping. To complete the sliding puzzle, the starting and finishing positions are typically provided [2].

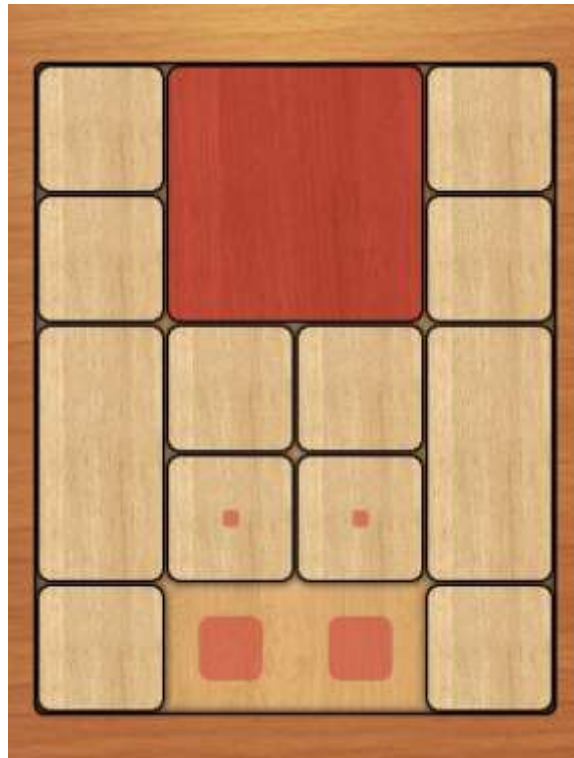


Figure 1: Sample of a Mobile Game Klotski

Upon watching a player do the Klotski puzzle by moving a tile from one place to another, the investigators observed some patterns that are related and was tackled in their Mathematics class in the previous school year. This urges them to perform a short experimentation of moving a tile in a diagonal direction in order to find the minimum number of moves and to formulate a mathematical investigation about the Klotski puzzle in hopes of discovering concepts that may contribute in the study of mathematics.

Thus, this study focuses on exploring the Klotski puzzle and to generate formulas out of the number of moves it needed to complete the puzzle. This investigation also aims to determine the minimum possible number of moves in solving a square or rectangular-shaped Klotski with the aid of the different mathematical concepts, particularly the arithmetic sequence.

Like other varieties of sliding puzzle, Klotski is also known as a challenging game that brings entertainment to people in all ages. This game also enhances an individual's problem solving skills, critical thinking skills and other mathematical related skills. Moreover, this game might not just be a "simple game" for it has the potential to unfold mathematical concepts that have a significant contribution to mathematical research.

Significance of the Study

This study adds to the corpus of information for everyone, especially for gamers, and it is noteworthy since the idea might be used in other games. This investigation makes fresh contribution to the field of games and geometry, due to its usage and relation to the concept of sequence and patterns. With the success of this study, not only students but everyone else may be able to benefit from this study by using the concepts in this investigation in the future and pass this knowledge to the next generation. Lastly, the investigators of this study are hoping that other investigators would find the relevance of this study into its real life application.

Statement of the Problem

The aim of the game under investigation is to slide a corner tile/block from its starting position (upper left corner) to its opposite corner (bottom right) in the least possible number of moves. In addition, each block/tile can only be moved up, down, left, right and one at a time. Diagonal movement and lifting the blocks are not permitted. There can be 1 or more spaces and the starting free space of the puzzle is located in the bottom right corner. If there are 2 or more spaces, the spaces should be vertically aligned above the predetermined spot.

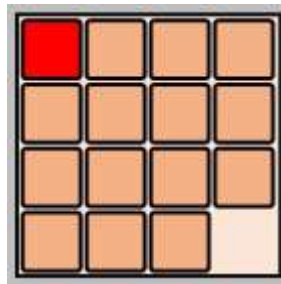


Figure 2: Sample Predetermined State of a Square Klotski

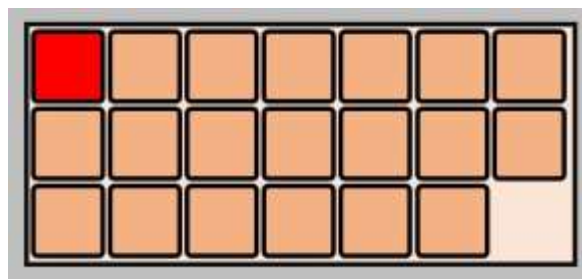


Figure 3: Sample Predetermined State of a Rectangular Klotski

Based on the aforementioned rule of the game, the following objectives were identified to facilitate the conduct of this investigation:

- To determine the minimum number of moves in the Klotski game
- To determine whether a pattern is formed by the minimum number of moves
- To create a formula in calculating the minimum number of moves for all square Klotski

- To create a formula in calculating the minimum number of moves for all rectangular Klotski
- To determine the relationship between the number of spaces and the number of rows and columns
- To create a formula in calculating the minimum number of moves given the number of spaces for all square Klotski
- To create a formula in calculating the minimum number of moves given the number of spaces for all rectangular Klotski

Review of Related Literature

Klotski and other Sliding Puzzles

A frame enclosed collection of forms makes up a sliding puzzle. The only way to move the shapes inside that frame is to slide; turning, lifting, or jumping are not permitted. The 15 puzzle, created in 1878 in the USA (by an unidentified person), was the one that actually kicked off the genre of puzzles.



Figure 4: Sample of a 3×3 Mobile Game Sliding Puzzle



Figure 5: Sample of a 5×5 Mobile Game Sliding Puzzle

The traditional Klotski puzzle is one of the notable sliding puzzle that was known as "Hua Rong Dao" in China and "Dad's Puzzler" of USA. In China, this traditional Klotski gained popularity in the 1930's based on the warrior from the Eastern Han Dynasty and well-known Chinese strategist Cao Cao. The Cao Cao

piece serves as the main header in sliding block puzzles, and the other nine pieces represent armed guards and other barriers. The player's task is to overcome these challenges and deliver Cao Cao to freedom at the bottom of the board [8]. On the other hand, Dad's puzzler was invented by one of America's leading mathematician, Olive C. Hazlett during the 1920s. There is no pictures to put together or number to get in order in this puzzle, it's just simply uses pieces of wood. To play this puzzle, the large square must be moved from position A to position C in order to complete the puzzle. Also, no pieces may be turned, raised off the bottom of the frame, or jumped [7].

The frame and base of the Klotski puzzle and Dad's Puzzler serves as the foundation upon which the puzzle is built, it also have the crucial role to constraints within which blocks can be shifted. It is commonly made out of wood or durable plastic. China's Klotski puzzle have one large block(2×2), representing the protagonist. It also have 4 medium rectangles(2×1) and 4 small squares(1×1) that can be slid in a specific direction to help the protagonist reach the exit. While Dad's puzzler's blocks have one 2×2 square, four 2×1 rectangles with the long side horizontal, two 2×1 rectangles with the long side vertical, and two 1×1 squares [6].

The Klotski puzzle has changed over time, giving rise to a variety of versions and adaptations that continue to intrigue and test puzzle enthusiasts. Some example of these are Klotski puzzle with Animated images, Klotski puzzle with a hexagonal frame, and a Klotski puzzle that holds more than 10 blocks that the traditional one have. The well-known Klotski puzzle game was also included in the improvement of technology after becoming part of Windows 3.1's Microsoft entertainment pack and has since become a classic [6].

A puzzle is a type of educational game that brings entertainment and challenges everyone to solve it. To find the right and enjoyable solution to a puzzle it requires the development of certain skill that may not only help solve the puzzle but also enhances one's cognitive abilities. This will put the observation skill, planning skill, critical thinking skill and patience up to a test. Playing the puzzle will help us develop our skills that we can also apply in our daily lives without taking away the fun [6].

According to the recent article of Zhong (2023), studying new algorithms to solve sliding puzzles has become increasingly important as sliding puzzle sports have developed. Zhong along with other investigators hope to develop an additive approximation method because it is difficult to solve the puzzle optimally. The length of the answer produced by this algorithm should only be a low-order term longer than the ideal solution [15].

Arithmetic Sequence

Carl Friedrich Gauss is a German Mathematician who contributed in many fields of Math and Science. When he was a young boy he was tasked to add the integers 1 to 100 by his teacher. As a Math prodigy he is, he was able to create his own formula which leads to the discovery of arithmetic sequence. It is an ordered set of numbers that have a common difference between each consecutive term [1].

Mathematicians and others who work with numbers can solve difficult mathematical problems by using arithmetic sequences, which are employed in algebra and geometry. Arithmetic sequences can be used to solve simple or complex problems but it's necessary to have the basic understanding about it to ensure that it is applied correctly. Finding patterns, such as a specific number of extra seats each row, is how this is done [4].

In the study of Oded Goldreich in 2011, They take into account a game in which players move various stones along an undirected graph's edges. Only one pebble may be present in each vertex at any one moment, and only one pebble may be transferred at a time (i.e., the pebble must be moved to an empty vertex). They demonstrate that it is NP-Hard to determine the shortest sequence of moves between two given "pebble configurations" [10].

Piecewise Function

A piecewise function is a function in which the outcome is defined by more than one formulas. It describes situations where a rule or relationship changes as the input value passes certain “boundaries” [11]. The earliest known use of the word piecewise is in the late 1600’s in the writing of Nathaniel Fairfax, a physician and antiquary. Since then, it has been significant and useful in different fields as long as there is a change based on its domain value; specifically, in mathematics, this function is employed when different expressions or formulas apply to various parts of the domain [5].

In the study of Nagayama, Sasao, and Butler (2014), to represent a numeric function compactly, they partition the domain of the function into uniform segments and transform the sub-function in each segment into an arithmetic spectrum. An arithmetic expression is derived from this, and a piecewise arithmetic expression for the function is obtained. This is later on widely used in the different parts of their study [13].

Quadratic Formula

The development or derivation of mathematical ideas is similarly as continuous as the historical evolution, as one mathematician picks up an idea where another mathematician leaves it. One of these is the quadratic formula, which is used to find the roots of a quadratic equation [12]. The Quadratic Formula uses the “ a ”, “ b ”, and “ c ” from “ $ax^2 + bx + c$ ”, where “ a ”, “ b ”, and “ c ” are just numbers; they are the “numerical coefficients” of the quadratic equation that is given to solve [14].

The quadratic formula is the easiest method to determine the roots of a quadratic equation. It provides a straightforward way to solve the equation, offering an alternative to factorization. In cases where a quadratic equation is difficult to factor, the quadratic formula is a convenient and efficient tool for quickly finding the roots [9].

Scope and Delimitations

This investigation aims to determine the minimum number of moves required to move a block in a Klotski puzzle from its starting location to its designated spot. The predetermined spot occupies one or more than spaces that is vertically aligned. A block or tile must be moved diagonally from the upper-left corner to the opposite bottom-right corner. The legal moves are limited to: (a) sliding the block/tile up, down, left, or right, and (b) only one block can be moved at a time. All blocks are of equal size, but the puzzle’s dimensions differ as the number of columns should be greater than the number of rows, it also vary depending on the number of spaces in the predetermined spot, where the puzzle’s dimension must be equal to or greater than the number of spaces in the predetermined spot disregarding the rows and column that is less than the chosen dimension. Additionally, this study aims to derive or create formulae based on observed patterns or sequences. Finally, the scope of this study is not limited to square Klotski puzzles; it also considers rectangular Klotski puzzles with varying dimensions and number of spaces in the predetermined spot.

2 Chapter II

2.1 Definition

Arithmetic Sequence- a sequence where the next terms after the first term is obtained by adding a constant number. The difference between each consecutive terms is called the common difference.

Klotski- comes from the polish word "klocki" which means wooden blocks. It is a sliding puzzle that aims to move the red block to exit at the bottom of the board or to move the red block from and into a specified location.

Minimum moves- the least quantity, amount possible, assignable, allowable, or attainable in moving a block from a Klotski puzzle.

Pattern- is a repeated arrangement of number or the set of numbers that are related to each other by following a specific rule.

Sliding puzzle- is a type of simple combination puzzle. It challenges the player to slide pieces along a certain routes on a board to reach a certain end arrangement.

Piecewise Function- is a function defined by multiple sub-functions each applying to a specific intervals of its domain.

Quadratic Formula- is a mathematical rule that helps find the solutions to equations of the form $ax^2 + bx + c = 0$, to figure out the value "x" that makes the equation true.

2.2 Known Results or Conjecture

The following are the conjectures generated by the investigators in the conduct of the investigation:

- The minimum number of moves for square and rectangular Klotski with varying dimensions forms an arithmetic sequence

- The dimension of a Klotski puzzle are directly influenced by the number of spaces it contains

- The formula to find the minimum number of moves (M) of a square Klotski is $M = 8s - 11$, where s is the number of columns or rows given that the number of space is equal to 1

- The formula to find the minimum number of moves of a rectangular Klotski are:

a. Given the difference and the product:

$$1) M_{Pd} = 4 \sqrt{d^2 + 4P} + 2d - 13, \text{ where } d \text{ is the positive difference and } P \text{ is the product between the number of rows and columns}$$

b. Given the number of rows or columns:

$$2) M = 6l + 2w - 13, \text{ where } l \text{ and } w \text{ are the number of rows and columns, and } l > w$$

- The formula to find the minimum number of moves (M) of a square Klotski given the spaces are greater than or equal to 2 are:

1. $M = 8s - 3S - 8$, this formula can be use if $1 \leq S \leq 4$ where s is the number of rows or columns and S is the number of spaces, and $s \geq S$

2. $M = 8s - S - 16$, this formula can be use if $4 \leq S \leq \infty$ where s is the number of rows or columns and S is the number of spaces, and $s \geq S$

. The formula to find the minimum number of moves (M) of a rectangular Klotski given the spaces are greater than or equal to 2 are:

a. Given the difference and the product

1) $M_{Pd} = 4 \sqrt{1 + 4P} - 14$, this formula can be used if the product of the dimension is $(n + 1)(n + 2)$

2) $M_{Pd} = 4 \sqrt{d^2 + 4P + 2d - S - 16}$, this formula can be used if the product of the dimension is $(n + f)(n + k)$, where $f \geq 1, k > 2, k > f, f < S$ and $2 \leq S \leq \infty$

b. Given the number of rows and columns

3) $M = 4 \sqrt{1 + 4lw} - 14$, this formula can be use if $(n + 1)(n + 2)$ where l is the number of columns and w is the number of rows, $l > w$

4) $M = 6l + 2w - S - 16$, this formula can be use if $(n + f)(n + k)$ where $f \geq 1, k > 2, k > f, f < S, 2 \leq S \leq \infty$, where l is the number of columns, w is the number of rows, S is the number of spaces, and $l > w$.

3 Chapter III

3.1 Researchers' Generated Data and Results

This section presents the data that the investigators gathered, the pattern uncovered and the formulation and derivation of the formula to find the minimum number of moves on the Klotski Puzzle. The following data and results are organized and presented based on the order of the specific objectives identified in section 1.3

Data for the Minimum Number of Moves

Table 1: Minimum Number of Moves based on the Number of Rows and Columns

x	1	2	3	4	5	6	7	...
1	0/	1	0/	0/	0/	0/	0/	...
2	1	5	9	15	21	27	33	...
3	0/	9	13	17	23	29	35	...
4	0/	15	17	21	25	31	37	...
5	0/	21	23	25	29	33	39	...
6	0/	27	29	31	33	37	41	...
7	0/	33	35	37	39	41	45	...
...	...	...	...	...	...	...	...	...

The minimum number of moves in a square Klotski is always obtained by traveling the main block/tile around the diagonal path of the puzzle or it follows a stair-like path. For the rectangular Klotski, a similar strategy can be used were the main block/tile should travel also around the diagonal path of the puzzle.

Pattern for the Minimum Number of Moves

As shown in Table 2, the minimum number of moves of a square Klotski (in an increasing number of sides) forms an Arithmetic Sequence whose common difference is 8.

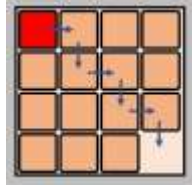


Figure 6: Path to obtain the Minimum Number of Moves of a Square Klotski

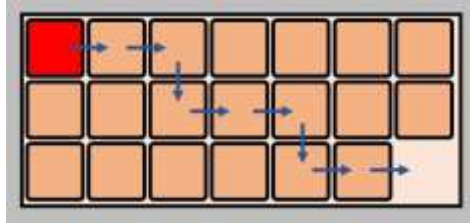


Figure 7: Path to obtain the Minimum Number of Moves of a Rectangular Klotski

Table 2: Pattern for the Minimum Number of Moves for a Square Klotski

x	1	2	3	4	5	6	7	...
1	0/	1	0/	0/	0/	0/	0/	...
2	1	5	9	15	21	27	33	...
3	0/	9	13	17	23	29	35	...
4	0/	15	17	21	25	31	37	...
5	0/	21	23	25	29	33	39	...
6	0/	27	29	31	33	37	41	...
7	0/	33	35	37	39	41	45	...
...	...	...	...	...	...	...	...	...

Table 3: Pattern for the Minimum Number of Moves for a Rectangular Klotski

x	1	2	3	4	5	6	7	...
1	0/	1	0/	0/	0/	0/	0/	...
2	1	5	9	15	21	27	33	...
3	0/	9	13	17	23	29	35	...
4	0/	15	17	21	25	31	37	...
5	0/	21	23	25	29	33	39	...
6	0/	27	29	31	33	37	41	...
7	0/	33	35	37	39	41	45	...
...	...	...	...	...	...	...	...	...

As shown in Table 3, the minimum number of moves of a rectangular Klotski (in an increasing number of sides) forms also an Arithmetic Sequence whose common difference is 8 just like the pattern of the square Klotski. Furthermore, the first term of each sequence in the rectangular Klotski also forms an arithmetic sequence whose common difference is 6.

Formula for the Minimum Number of Moves of a Square Klotski;

$$(n+1)(n+1) \text{ or } (n+1)^2 \quad \{n|n \in \mathbb{Z}^+\}$$

Product(P)	Minimum No. of Moves (M)
4	5
9	13
16	21
25	29
36	37
49	45
64	53
81	61
$P = (n+1)^2$	$M = 8n - 3$

Formulation of $M = 8n - 3$

$$M = 5 + (n-1)8$$

$$M = 5 + 8n - 8$$

$$M = 8n - 3 \quad (1)$$

Derivation of n

$$P = (n+1)^2$$

$$\sqrt{P} = n+1$$

$$\sqrt{P} - 1 = n$$

$$n = \sqrt{P} - 1 \quad (2)$$

Minimum number of moves in terms of the product (M_P)

$$M = 8n - 3$$

$$M_P = 8(\sqrt{P} - 1) - 3$$

$$M_P = 8\sqrt{P} - 8 - 3$$

$$M_P = 8\sqrt{P} - 11 \quad (3)$$

Since $\sqrt{P}$ is just equal to the number of rows/columns (s) of the square Klotski, then the minimum number of moves (M) is

$$M = 8s - 11 \quad (4)$$

Formula for the Minimum Number of Moves of a Rectangular Klotski;

$$(n+1)(n+k) \quad \{n|n \in \mathbb{Z}^+\} \quad \{k|k \in \mathbb{Z}^+, k > 1\}$$

The arithmetic sequence formula of the different minimum number of moves (M) for each $(n+1)(n+k)$

Minimum no. of moves for $(n+1)(n+2)$		Minimum no. of moves for $(n+1)(n+3)$	
$P_{(n+1)(n+2)}$	$M_{(n+1)(n+2)}$	$P_{(n+1)(n+3)}$	$M_{(n+1)(n+3)}$
6	9	8	15
12	17	15	23
20	25	24	31
30	33	35	39
...	...	...	...
$(n+1)(n+2)$	$8n+1$	$(n+1)(n+3)$	$8n+7$

Minimum no. of moves for $(n+1)(n+4)$		Minimum no. of moves for $(n+1)(n+5)$	
$P_{(n+1)(n+4)}$	$M_{(n+1)(n+4)}$	$P_{(n+1)(n+5)}$	$M_{(n+1)(n+5)}$
10	21	12	27
18	29	21	35
28	37	32	43
40	45	45	51
...	...	...	...
$(n+1)(n+4)$	$8n+13$	$(n+1)(n+5)$	$8n+19$

The following table summarizes the product and the minimum number of moves for $(n+1)(n+k)$ Klotski:

Product (P)	Minimum no. of moves (M)
$(n+1)(n+2)$	$8n+1$
$(n+1)(n+3)$	$8n+7$
$(n+1)(n+4)$	$8n+13$
$(n+1)(n+5)$	$8n+19$
$(n+1)(n+6)$	$8n+25$
$(n+1)(n+7)$	$8n+31$
...	...
$P = (n+1)(n+k)$	$M = 8n+6k-11$

Formulation of $M = 8n + 6k - 11$

$$\begin{aligned}
 M &= 8n + (1 + [k-2]6) \\
 M &= 8n + (1 + 6k - 12) \\
 M &= 8n + 1 + 6k - 12 \\
 M &= 8n + 6k - 11
 \end{aligned} \tag{5}$$

Derivation of n

$$\begin{aligned}
 P &= (n+1)(n+k) \\
 P &= n^2 + kn + n + k \\
 P &= n^2 + (k+1)n + k \\
 0 &= n^2 + (k+1)n + k - P
 \end{aligned} \tag{6}$$

using quadratic formula to find n

$$\begin{aligned}
 n &= \frac{-(k+1) \pm \sqrt{(k+1)^2 - 4(1)(k-P)}}{2(1)} \\
 n &= \frac{-k-1 \pm \sqrt{k^2 + 2k + 1 - 4k + 4P}}{2} \\
 n &= \frac{-k-1 \pm \sqrt{k^2 - 2k + 1 + 4P}}{2} \\
 n &= \frac{-k-1 \pm \sqrt{(k-1)^2 + 4P}}{2}
 \end{aligned} \tag{7}$$

Minimum number of moves in terms of the product (M_P)

$$\begin{aligned}
 M &= 8n + 6k - 11 \\
 M_P &= 8\left(\frac{-k-1 \pm \sqrt{(k-1)^2 + 4P}}{2}\right) + 6k - 11 \\
 M_P &= 4(-k-1 \pm \sqrt{(k-1)^2 + 4P}) + 6k - 11 \\
 M_P &= -4k - 4 \pm 4\sqrt{(k-1)^2 + 4P} + 6k - 11 \\
 M_P &= 4\sqrt{(k-1)^2 + 4P} + 2k - 15
 \end{aligned} \tag{8}$$

Since k is the difference (d) of the two sides +1,

$$\begin{aligned}
 d &= (n+k) - (n+1) \\
 d &= n+k-n-1 \\
 d &= k-1 \\
 d+1 &= k \\
 k &= d+1
 \end{aligned} \tag{9}$$

hence the minimum number of moves given the product and difference (M_{Pd}) is

$$\begin{aligned}
 M_{Pd} &= 4\sqrt{(d+1-1)^2 + 4P} + 2(d+1) - 15 \\
 M_{Pd} &= 4\sqrt{(d+1-1)^2 + 4P} + 2d + 2 - 15 \\
 M_{Pd} &= 4\sqrt{(d)^2 + 4P} + 2d - 13 \\
 M_{Pd} &= 4\sqrt{d^2 + 4P} + 2d - 13
 \end{aligned} \tag{10}$$

Furthermore, if we let l and w be the number of columns and rows and $l > w$, then the minimum number

of moves of a rectangular Klotski (M) is

$$\begin{aligned}
 M_{Pd} &= 4 \sqrt{d^2 + 4P} + 2d - 13 \\
 M &= 4 \sqrt{(l-w)^2 + 4(l)(w) + 2(l-w) - 13} \\
 M &= 4 \sqrt{l^2 - 2lw + w^2 + 4lw + 2l - 2w - 13} \\
 M &= 4 \sqrt{l^2 + 2lw + w^2 + 2l - 2w - 13} \\
 M &= 4 \sqrt{(l+w)^2 + 2l - 2w - 13} \\
 M &= 4(l+w) + 2l - 2w - 13 \\
 M &= 4l + 4w + 2l - 2w - 13 \\
 \mathbf{M} &= \mathbf{6l + 2w - 13}
 \end{aligned} \tag{11}$$

Formula for the Minimum Number of Moves of a Square Klotski given the number of spaces;

Table 4: Data of the Minimum number of moves for Square and Rectangular Klotski with 2 Spaces

x	1	2	3	4	5	6	7	...
1	0/	0/	0/	0/	0/	0/	0/	...
2	0/	2	6	10	16	22	28	...
3	0/		10	14	18	24	30	...
4	0/			18	22	26	32	...
5	0/				26	30	34	...
6	0/					34	38	...
7	0/						42	...
...	...	...	...	...	...	...	...	...

$$(n+1)(n+1) \text{ or } (n+1)^2 \{n|n \in \mathbb{Z}^+\}$$

Product(P)	Minimum No. of Moves (M)
4	2
9	10
16	18
25	26
36	34
49	42
$P = (n+1)^2$	$M = 8n - 6$

Formulation of $M = 8n - 6$

$$M = 2 + (n-1)8$$

$$M = 2 + 8n - 8$$

$$M = 8n - 6 \tag{12}$$

Derivation of n

$$P = (n + 1)^2$$

$$\sqrt{P} = \sqrt{(n + 1)^2}$$

$$\sqrt{P} = n + 1$$

$$\sqrt{P - 1} = n$$

$$n = \sqrt{P - 1} \quad (13)$$

Minimum number of moves in terms of the product (M_P)

$$M_P = 8(\sqrt{P} - 1) - 6$$

$$M_P = 8\sqrt{P} - 8 - 6$$

$$M_P = 8\sqrt{P} - 14 \quad (14)$$

Since $\sqrt{P}$ is just equal to the number of rows/columns (s) of the square Klotski, then the minimum number of moves (M) given the 2 spaces is

$$M = 8s - 14 \quad (15)$$

Table 5: Data of the Minimum number of moves for Square and Rectangular Klotski with 3 Spaces

x	1	2	3	4	5	6	7	...
1	0/	0/	0/	0/	0/	0/	0/	...
2	0/	2	3	7	13	19	25	...
3	0/		7	11	17	23	29	...
4	0/			15	19	25	31	...
5	0/				23	27	33	...
6	0/					31	35	...
7	0/						39	...
...	...							...

$$(n + 2)(n + 2) \text{ or } (n + 2)^2 \{n|n \in \mathbb{Z}^+\}$$

Formulation of $M = 8n - 1$

$$M = 7 + (n - 1)8$$

$$M = 7 + 8n - 8$$

$$M = 8n - 1 \quad (16)$$

Derivation of n

$$P = (n + 2)^2$$

$$\sqrt{P} = \sqrt{(n + 2)^2}$$

$$\sqrt{P} = n + 2$$

$$\sqrt{P - 2} = n$$

$$n = \sqrt{P - 2} \quad (17)$$

Minimum number of moves in terms of the product (M_P)

$$\begin{aligned}
 M_P &= 8(\sqrt{P-2}) - 1 \\
 M_P &= 8\sqrt{P-16} - 1 \\
 M_P &= 8\sqrt{P-17}
 \end{aligned} \tag{18}$$

Since $\sqrt{P}$ is just equal to the number of rows/columns (s) of the square Klotski, then the minimum number of moves (M) given the 3 spaces is

$$M = 8s - 17 \tag{19}$$

Table 6: Data of the Minimum number of moves for Square and Rectangular Klotski with 4 Spaces

x	1	2	3	4	5	6	7	...
1	0/	0/	0/	0/	0/	0/	0/	...
2	0/	0/	3	8	10	17	20	...
3	0/	3	7	8	15	16	22	...
4	0/	8	8	12	18	24	30	...
5	0/	8	15		20	26	32	...
6	0/	14	16			28	34	...
7	0/	20	22				36	...
...	...	...	...					...

$$(n+3)(n+3) \text{ or } (n+3)^2 \{n|n \in \mathbb{Z}^+\}$$

Formulation of $M = 8n - 1$

$$\begin{aligned}
 M &= 12 + (n-1)8 \\
 M &= 12 + 8n - 8 \\
 M &= 8n + 4
 \end{aligned} \tag{20}$$

Derivation of n

$$\begin{aligned}
 P &= (n+3)^2 \\
 \sqrt{P} &= n+3 \\
 \sqrt{P} - 3 &= n \\
 n &= \sqrt{P-9}
 \end{aligned} \tag{21}$$

Minimum number of moves in terms of the product (M_P)

$$\begin{aligned}
 M_P &= 8(\sqrt{P-9}) + 4 \\
 M_P &= 8\sqrt{P-24} + 4 \\
 M_P &= 8\sqrt{P-20}
 \end{aligned} \tag{22}$$

Since $\sqrt{P}$ is just equal to the number of rows/columns (s) of the square Klotski, then the minimum number of moves (M) given the 4 spaces is

$$M = 8s - 20 \quad (23)$$

The following table summarizes the number of spaces and the minimum number of moves for square Klotski whose number of sides is greater than or equal to the number of spaces:

No. of Spaces (S)	Minimum no. of moves (M)
1	$8s - 11$
2	$8s - 14$
3	$8s - 17$
4	$8s - 20$

$$M = 8s - 3S - 8$$

Formulation of $M = 8s - 3S - 8$

$$M = (8s - 11) + (S - 1)(-3)$$

$$M = 8s - 11 - 3S + 3$$

$$M = 8s - 3S - 8 \quad (24)$$

The above-mentioned formula (Eq. 24) calculates the minimum number of moves in a square Klotski which has spaces ranging from 1 to 4, where s is the number of rows or columns and S is the number of spaces in the puzzle.

Square Klotski with spaces ranging from 5 to ∞

Using the data from the following tables and following the same process in deriving the formula for the square Klotski that contains spaces ranging from 1 to 4, the investigators was able to derived a general formula that calculates the minimum number of moves for square Klotski that contains spaces ranging from 5 to infinity.

Table 7: Data of the Minimum number of moves for Square and Rectangular Klotski with 5 Spaces

x	4	5	6	7	8	9	10	...
4	0/	0/	0/	0/	0/	0/	0/	...
5	0/	19	25	31	37	43	49	...
6	0/		27	33	39	45	51	...
7	0/			35	41	47	53	...
8	0/				43	49	55	...
9	0/					51	57	...
10	0/						59	...
...	...							...

Table 8: Data of the Minimum number of moves for Square and Rectangular Klotski with 6 Spaces

x	5	6	7	8	9	10	11	12	...
5	0/	0/	0/	0/	0/	0/	0/	0/	...
6	0/	26	32	38	44	50	56	62	...
7	0/		34	40	46	52	58	64	...
8	0/			42	48	54	60	66	...
9	0/				50	56	62	68	...
10	0/					58	64	70	...
11	0/						66	72	...
12	0/							74	...
...	...								...

Table 9: Data of the Minimum number of moves for Square and Rectangular Klotski with 7 Spaces

x	6	7	8	9	10	11	12	13	...
6	0/	0/	0/	0/	0/	0/	0/	0/	...
7	0/	33	39	45	51	57	63	69	...
8	0/		41	47	53	59	65	71	...
9	0/			49	55	61	67	73	...
10	0/				57	63	69	75	...
11	0/					65	71	77	...
12	0/						73	79	...
13	0/							81	...
...	...								...

No. of Spaces (S)	Minimum no. of moves (M)
5	$8s - 21$
6	$8s - 22$
7	$8s - 23$
8	$8s - 24$
9	$8s - 25$
10	$8s - 26$
11	$8s - 27$
...	...
$M = 8s - S - 16$	

Formulation of $M = 8s - S - 16$

$$M = (8s - 21) + (S - 5)(-1)$$

$$M = 8s - 21 - S + 5$$

$$\boxed{M = 8s - S - 16} \quad (25)$$

The above-mention formula (Eq. 25) calculates the minimum number of moves for a square Klotski given the number of rows or columns (s) and the number of spaces (S) and $4 \leq S \leq \infty$

Formula for the Minimum Number of Moves of a Rectangular Klotski given the number of spaces;

Table 10: Data of the Minimum number of moves for Square and Rectangular Klotski with 8 Spaces

x	7	8	9	10	11	12	13	14	...
7	0/	0/	0/	0/	0/	0/	0/	0/	...
8	0/	40	46	52	58	64	70	76	...
9	0/		48	54	60	66	72	78	...
10	0/			56	62	68	74	80	...
11	0/				64	70	76	82	...
12	0/					72	78	84	...
13	0/						80	86	...
14	0/							88	...
...	...								...

Table 11: Data of the Minimum number of moves for Square and Rectangular Klotski with 9 Spaces

x	8	9	10	11	12	13	14	15	...
8	0/	0/	0/	0/	0/	0/	0/	0/	...
9	0/	47	53	59	65	71	77	83	...
10	0/		55	61	67	73	79	85	...
11	0/			63	69	75	81	87	...
12	0/				71	77	83	89	...
13	0/					79	85	91	...
14	0/						87	93	...
15	0/							95	...
...	...								...

2 Spaces

Using the data in table 4 in formulating the formula for the minimum number of moves for rectangular Klotski containing 2 spaces, the investigators was able to derived the formula through the same process as equation 5, 6, 7, 8, 9, 10, and 11.

$$(n + 1)(n + k) \{n|n \in \mathbb{Z}^+\} \{k|k \in \mathbb{Z}^+, k > 1\}$$

Product (P)	Minimum no. of moves (M)
$(n + 1)(n + 2)$	$8n - 2$
$(n + 1)(n + 3)$	$8n + 2$
$(n + 1)(n + 4)$	$8n + 8$
$(n + 1)(n + 5)$	$8n + 14$
$(n + 1)(n + 6)$	$8n + 20$
$(n + 1)(n + 7)$	$8n + 26$
...	...
$P = (n + 1)(n + k)$	

In the above-shown table, the row that is highlighted in red has a separated formula. Hence, only calculates the minimum number of moves in a rectangular Klotski that has the dimension of $(n + 1)(n + 2)$. This is due to the constant of the explicit formula $8n - 2$ since it does not coincide with the common difference of 6 if it is included in the sequence.

Table 12: Data of the Minimum number of moves for Square and Rectangular Klotski with 10 Spaces

x	9	10	11	12	13	14	15	16	...
9	0/	0/	0/	0/	0/	0/	0/	0/	...
10	0/	54	60	66	72	78	84	90	...
11	0/		62	68	74	80	86	92	...
12	0/			70	76	82	88	94	...
13	0/				78	84	90	96	...
14	0/					86	92	98	...
15	0/						94	100	...
16	0/							102	...
...	...								...

Table 13: Data of the Minimum number of moves for Square and Rectangular Klotski with 11 Spaces

x	10	11	12	13	14	15	16	17	...
10	0/	0/	0/	0/	0/	0/	0/	0/	...
11	0/	61	67	73	79	85	91	97	...
12	0/		69	75	81	87	93	99	...
13	0/			77	83	89	95	101	...
14	0/				85	91	97	103	...
15	0/					93	99	105	...
16	0/						101	107	...
17	0/							109	...
...	...								...

Formulation of the Formula of $8n - 2$

$$P = (n + 1)(n + k)$$

$$P = n^2 + kn + n + k$$

$$P = n^2 + (k + 1)n + k$$

$$0 = n^2 + (k + 1)n + k - P \quad (26)$$

using the quadratic equation to find n

$$n = \frac{-(k + 1) \pm \sqrt{(k + 1)^2 + 4(1)(k - P)}}{2(1)}$$

$$n = \frac{-k - 1 \pm \sqrt{k^2 + 2k + 1 - 4k + 4P}}{2}$$

$$n = \frac{-k - 1 \pm \sqrt{k^2 - 2k + 1 + 4P}}{2}$$

$$n = \frac{-k - 1 \pm \sqrt{(k - 1)^2 + 4P}}{2} \quad (27)$$

Minimum number of moves in terms of the product (M_P)

$$\begin{aligned}
 M_P &= 8 \left(\frac{-k-1 \pm \sqrt{(k-1)^2 + 4P}}{2} \right) - 2 \\
 M_P &= 4(-k-1 \pm \sqrt{(k-1)^2 + 4P}) - 2 \\
 M_P &= -4k - 4 \pm 4 \sqrt{(k-1)^2 + 4P} - 2 \\
 M_P &= 4 \sqrt{(k-1)^2 + 4P} - 4k - 6
 \end{aligned} \tag{28}$$

since, $k = 2$. then,

$$\begin{aligned}
 M_P &= 4 \sqrt{(2-1)^2 + 4P} - 4(2) - 6 \\
 M_P &= 4 \sqrt{1 + 4P} - 8 - 6 \\
 M_P &= 4 \sqrt{1 + 4P} - 14
 \end{aligned} \tag{29}$$

Furthermore, if we let l and w the number of columns and rows and $l > w$, then

$$\begin{aligned}
 M &= 4 \sqrt{1 + 4P} - 14 \\
 M &= 4 \sqrt{1 + 4(lw)} - 14 \\
 M &= 4 \sqrt{1 + 4lw} - 14
 \end{aligned} \tag{30}$$

Derivation of the formula for the sequence starting from $8n + 2$;

$$(n+1)(n+k) \{n|n \in \mathbb{Z}^+\} \{k|k \in \mathbb{Z}^+, k > 2\}$$

The following table summarizes the product and the minimum number of moves for a $(n+1)(n+k)$ Klotski, where $k > 2$:

Product (P)	Minimum no. of moves (M)
$(n+1)(n+3)$	$8n+2$
$(n+1)(n+4)$	$8n+8$
$(n+1)(n+5)$	$8n+14$
$(n+1)(n+6)$	$8n+20$
$(n+1)(n+7)$	$8n+26$
...	...
$P = (n+1)(n+k)$	$M = 8n+6k-16$

Formulation of $M = 8n + 6k - 16$

$$\begin{aligned}
 M &= (8n+2) + (k-3)6 \\
 M &= 8n+2+6k-18 \\
 M &= 8n+6k-16
 \end{aligned} \tag{31}$$

using the quadratic equation to find n (See Eq. 27 for the solution)

$$n = \frac{-k-1 \pm \sqrt{(k-1)^2 + 4P}}{2}$$

Minimum number of moves in terms of the product (M_P)

$$\begin{aligned} M_P &= 8 \left(\frac{-k-1 \pm \sqrt{(k-1)^2 + 4P}}{2} \right) + 6k - 16 \\ M_P &= 4(-k-1 \pm \sqrt{(k-1)^2 + 4P}) + 6k - 16 \\ M_P &= -4k - 4 \pm 4\sqrt{(k-1)^2 + 4P} + 6k - 16 \\ M_P &= 4\sqrt{(k-1)^2 + 4P} + 2k - 20 \end{aligned} \quad (32)$$

the difference of the two sides $+1$ (See Eq. 9 for the solution),

$$k = d + 1$$

hence the minimum number of moves given the product and difference (M_{Pd}) is

$$\begin{aligned} M_{Pd} &= 4\sqrt{(d+1-1)^2 + 4P} + 2(d+1) - 20 \\ M_{Pd} &= 4\sqrt{(d+1-1)^2 + 4P} + 2d + 2 - 20 \\ M_{Pd} &= 4\sqrt{(d)^2 + 4P} + 2d - 18 \\ M_{Pd} &= 4\sqrt{d^2 + 4P} + 2d - 18 \end{aligned} \quad (33)$$

Furthermore, if we let l and w be the number of columns and rows and $l > w$, then the minimum number of moves of a rectangular Klotski (M) is

$$\begin{aligned} M_{Pd} &= 4\sqrt{d^2 + 4P} + 2d - 18 \\ M &= 4\sqrt{(l-w)^2 + 4(l)(w)} + 2(l-w) - 18 \\ M &= 4\sqrt{l^2 - 2lw + w^2 + 4lw} + 2l - 2w - 18 \\ M &= 4\sqrt{l^2 + 2lw + w^2} + 2l - 2w - 18 \\ M &= 4(l+w)^2 + 2l - 2w - 18 \\ M &= 4(l+w) + 2l - 2w - 18 \\ M &= 4l + 4w + 2l - 2w - 18 \\ M &= 6l + 2w - 18 \end{aligned} \quad (34)$$

3 Spaces

Using the data in table 5 and following the same solutions, here are the derived formulae;

$$(n+1)(n+k) \{n|n \in \mathbb{Z}^+\} \{k|k \in \mathbb{Z}^+, k > 2\}$$

The following table summarizes the product and the minimum number of moves in a $(n + 2)(n + k)$ Klotski:

Product (P)	Minimum no. of moves (M)
$(n+2)(n+3)$	$8n+3$
$(n+2)(n+4)$	$8n+9$
$(n+2)(n+5)$	$8n+15$
$(n+2)(n+6)$	$8n+21$
$(n+2)(n+7)$	$8n+27$
$(n+2)(n+8)$	$8n+33$
...	...
$P = (n+2)(n+k)$	$M = 8n+6k-15$

Formulation of $M = 8n + 6k - 15$

$$\begin{aligned}
M &= 8n + (3 + [k-3]6) \\
M &= 8n + (3 + 6k - 18) \\
M &= 8n + 3 + 6k - 18 \\
M &= 8n + 6k - 15
\end{aligned} \tag{35}$$

Derivation of n

$$\begin{aligned}
P &= (n+2)(n+k) \\
P &= n^2 + kn + 2n + 2k \\
P &= n^2 + (k+2)n + 2k \\
0 &= n^2 + (k+2)n + 2k - P
\end{aligned} \tag{36}$$

using the quadratic equation to find n

$$\begin{aligned}
n &= \frac{-(k+2) \pm \sqrt{(k+2)^2 + 4(1)(2k-P)}}{2(1)} \\
n &= \frac{-k-2 \pm \sqrt{k^2 + 4k + 4 - 8k + 4P}}{2} \\
n &= \frac{-k-2 \pm \sqrt{k^2 - 4k + 4 + 4P}}{2} \\
n &= \frac{-k-2 \pm \sqrt{(k-2)^2 + 4P}}{2}
\end{aligned} \tag{37}$$

Minimum number of moves in terms of the product (M_P)

$$\begin{aligned}
M_P &= 8\left(\frac{-k-2 \pm \sqrt{(k-2)^2 + 4P}}{2}\right) + 6k - 15 \\
M_P &= 4(-k-2 \pm \sqrt{(k-2)^2 + 4P}) + 6k - 15 \\
M_P &= -4k - 8 \pm 4\sqrt{(k-2)^2 + 4P} + 6k - 15
\end{aligned}$$

$$M_P = 4 \frac{q}{(k-2)^2 + 4P + 2k - 23} \tag{38}$$

the difference of the two sides +2

$$\begin{aligned}
 d &= (n + k) - (n + 2) \\
 d &= n + k - n - 2 \\
 d &= k - 2 \\
 k &= d + 2
 \end{aligned} \tag{39}$$

hence the minimum number of moves given the product and difference (M_{Pd}) is

$$\begin{aligned}
 M_{Pd} &= 4 \sqrt{(d + 2)^2 + 4P + 2(d + 2) - 23} \\
 M_{Pd} &= 4 \sqrt{d^2 + 4P + 2d + 4 - 23} \\
 M_{Pd} &= 4 \sqrt{d^2 + 4P + 2d - 19}
 \end{aligned} \tag{40}$$

Furthermore, if we let l and w be the number of columns and rows and $l > w$, then the minimum number of moves of a rectangular Klotski (M) is

$$\begin{aligned}
 M_{Pd} &= 4 \sqrt{d^2 + 4P + 2d + 4 - 23} \\
 M &= 4 \sqrt{(l - w)^2 + 4(l)(w) + 2(l - w) - 19} \\
 M &= 4 \sqrt{l^2 + 2lw + w^2 + 4lw + 2l - 2w - 19} \\
 M &= 4 \sqrt{l^2 + 2lw + w^2 + 2l - 2w - 19} \\
 M &= 4 \sqrt{(l + w)^2 + 2l - 2w - 19} \\
 M &= 4(l + w) + 2l - 2w - 19 \\
 M &= 4l + 4w + 2l - 2w - 19 \\
 M &= 6l + 2w - 19
 \end{aligned} \tag{41}$$

4 Spaces

Using the data from table 6, the derived formulae are;

$$(n + 1)(n + k) \{n | n \in \mathbb{Z}^+\} \{k | k \in \mathbb{Z}^+, k > 3\}$$

The following tables summarizes the product and the minimum number of moves for a $(n + 3)(n + k)$ Klotski:

Product (P)	Minimum no. of moves (M)
$(n + 3)(n + 4)$	$8n + 10$
$(n + 3)(n + 5)$	$8n + 16$
$(n + 3)(n + 6)$	$8n + 22$
$(n + 3)(n + 7)$	$8n + 28$
$(n + 3)(n + 8)$	$8n + 34$
$(n + 3)(n + 9)$	$8n + 40$

...	...
$P = (n + 3)(n + k)$	$M = 8n + 6k - 14$

Formulation of $M = 8n + 6k - 14$

$$\begin{aligned}
 M &= 8n + (10 + [k - 4]6) \\
 M &= 8n + (10 + 6k - 24) \\
 M &= 8n + 10 + 6k - 24 \\
 M &= 8n + 6k - 14
 \end{aligned} \tag{42}$$

Derivation of n

$$\begin{aligned}
 P &= (n + 3)(n + k) \\
 P &= n^2 + kn + 3n + 3k \\
 P &= n^2 + (k + 3)n + 3k \\
 0 &= n^2 + (k + 3)n + 3k - P
 \end{aligned} \tag{43}$$

using the quadratic equation to find n

$$\begin{aligned}
 n &= \frac{-(k + 3) \pm \sqrt{(k + 3)^2 + 4(1)(3k - P)}}{2(1)} \\
 n &= \frac{-k - 3 \pm \sqrt{k^2 + 6k + 9 - 12k + 4P}}{2} \\
 n &= \frac{-k - 3 \pm \sqrt{k^2 - 6k + 9 + 4P}}{2} \\
 n &= \frac{-k - 3 \pm \sqrt{(k - 3)^2 + 4P}}{2}
 \end{aligned} \tag{44}$$

Minimum number of moves in terms of the product (M_P)

$$\begin{aligned}
 M_P &= 8\left(\frac{-k - 3 \pm \sqrt{(k - 3)^2 + 4P}}{2}\right) + 6k - 14 \\
 M_P &= 4(-k - 3 \pm \sqrt{(k - 3)^2 + 4P}) + 6k - 14 \\
 M_P &= -4k - 12 \pm 4\sqrt{(k - 3)^2 + 4P} + 6k - 14 \\
 M_P &= 4\sqrt{(k - 3)^2 + 4P} + 2k - 26
 \end{aligned} \tag{45}$$

the difference of the two sides +3

$$\begin{aligned}
 d &= (n + k) - (n + 3) \\
 d &= n + k - n - 3 \\
 d &= k - 3 \\
 k &= d + 3
 \end{aligned} \tag{46}$$

hence the minimum number of moves given the product and difference (M_{Pd}) is

$$\begin{aligned}
 M_{Pd} &= 4\sqrt{((d + 3) - 3)^2 + 4P} + 2(d + 3) - 26 \\
 M_{Pd} &= 4\sqrt{d^2 + 4P} + 2d + 6 - 26 \\
 M_{Pd} &= 4\sqrt{d^2 + 4P} + 2d - 20
 \end{aligned} \tag{47}$$

Furthermore, if we let l and w be the number of columns and rows and $l > w$, then the minimum number of moves of a rectangular Klotski (M) is

$$\begin{aligned}
 M_{Pd} &= 4 \sqrt{d^2 + 4P} + 2d - 20 \\
 M &= 4 \sqrt{(l-w)^2 + 4(l)(w) + 2(l-w) - 20} \\
 M &= 4 \sqrt{l^2 + 2lw + w^2 + 4lw + 2l - 2w - 20} \\
 M &= 4 \sqrt{l^2 + 2lw + w^2 + 2l - 2w - 20} \\
 M &= 4 \sqrt{(l+w)^2 + 2l - 2w - 20} \\
 M &= 4(l+w) + 2l - 2w - 20 \\
 M &= 4l + 4w + 2l - 2w - 20 \\
 \mathbf{M} &= \mathbf{6l + 2w - 20}
 \end{aligned} \tag{48}$$

5 Spaces

$$(n+1)(n+k) \{n|n \in \mathbb{Z}^+\} \{k|k \in \mathbb{Z}^+, k > 4\}$$

The following tables summarizes the product and the minimum number of moves in a $(n+4)(n+k)$ Klotski:

Product (P)	Minimum no. of moves (M)
$(n+4)(n+5)$	$8n+17$
$(n+4)(n+6)$	$8n+23$
$(n+4)(n+7)$	$8n+29$
$(n+4)(n+8)$	$8n+35$
$(n+4)(n+9)$	$8n+41$
$(n+4)(n+10)$	$8n+47$
...	...
$P = (n+4)(n+k)$	$M = 8n+6k-13$

Formulation of $M = 8n + 6k - 13$

$$\begin{aligned}
 M &= 8n + (17 + [k-5]6) \\
 M &= 8n + (17 + 6k - 30) \\
 M &= 8n + 17 + 6k - 30 \\
 M &= 8n + 6k - 13
 \end{aligned} \tag{49}$$

Derivation of n

$$\begin{aligned}
 P &= (n+4)(n+k) \\
 P &= n^2 + kn + 4n + 4k \\
 P &= n^2 + (k+4)n + 4k
 \end{aligned}$$

$$0 = n^2 + (k + 4)n + 4k - P$$

(50)

using the quadratic equation to find n

$$\begin{aligned}
 n &= \frac{-(k+4) \pm \sqrt{(k+4)^2 + 4(1)(4k-P)}}{2(1)} \\
 n &= \frac{-k-4 \pm \sqrt{k^2 + 8k + 16 - 16k + 4P}}{2} \\
 n &= \frac{-k-4 \pm \sqrt{k^2 - 8k + 16 + 4P}}{2} \\
 n &= \frac{-k-4 \pm \sqrt{(k-4)^2 + 4P}}{2}
 \end{aligned} \tag{51}$$

Minimum number of moves in terms of the product (M_P)

$$\begin{aligned}
 M_P &= 8\left(\frac{-k-4 \pm \sqrt{(k-4)^2 + 4P}}{2}\right) + 6k - 13 \\
 M_P &= 4(-k-4 \pm \sqrt{(k-4)^2 + 4P}) + 6k - 13 \\
 M_P &= -4k - 16 \pm 4\sqrt{(k-4)^2 + 4P} + 6k - 13 \\
 M_P &= 4\sqrt{(k-4)^2 + 4P} + 2k - 29
 \end{aligned} \tag{52}$$

the difference of the two sides $+4$

$$\begin{aligned}
 d &= (n+k) - (n+4) \\
 d &= n+k-n-4 \\
 d &= k-4 \\
 k &= d+4
 \end{aligned} \tag{53}$$

hence the minimum number of moves given the product and difference (M_{Pd}) is

$$\begin{aligned}
 M_{Pd} &= 4\sqrt{(d+4-4)^2 + 4P} + 2(d+4) - 29 \\
 M_{Pd} &= 4\sqrt{d^2 + 4P} + 2d + 8 - 29 \\
 M_{Pd} &= 4\sqrt{d^2 + 4P} + 2d - 21
 \end{aligned} \tag{54}$$

Furthermore, if we let l and w be the number of columns and rows and $l > w$, then the minimum number of moves of a rectangular Klotski (M) is

$$\begin{aligned}
 M_{Pd} &= 4\sqrt{d^2 + 4P} + 2d - 21 \\
 M &= 4\sqrt{(l-w)^2 + 4(l)(w)} + 2(l-w) - 21 \\
 M &= 4\sqrt{l^2 + 2lw + w^2 + 4lw} + 2l - 2w - 21 \\
 M &= 4\sqrt{l^2 + 2lw + w^2} + 2l - 2w - 21 \\
 M &= 4\sqrt{(l+w)^2} + 2l - 2w - 21 \\
 M &= 4(l+w) + 2l - 2w - 21 \\
 M &= 4l + 4w + 2l - 2w - 21 \\
 M &= 6l + 2w - 21
 \end{aligned}$$

The following table summarizes the product and the minimum number of moves given the product and the difference for a $(n + f)(n + k)$ rectangular Klotski, where $f \geq 1, k > 2, k > f, f < S$, and $2 \leq S \leq \infty$:

No. of Spaces (S)	Minimum no. of moves (M_{Pd})
2	$4\sqrt{d^2 + 4P} + 2d - 18$
3	$4\sqrt{d^2 + 4P} + 2d - 19$
4	$4\sqrt{d^2 + 4P} + 2d - 20$
5	$4\sqrt{d^2 + 4P} + 2d - 21$
6	$4\sqrt{d^2 + 4P} + 2d - 22$
...	...

Formulation of $M_{Pd} = 4\sqrt{d^2 + 4P} + 2d - S - 16$

$$M_{Pd} = (4\sqrt{d^2 + 4P} + 2d - 18) + (S - 2)(-1)$$

$$M_{Pd} = 4\sqrt{d^2 + 4P} + 2d - 18 - S + 2$$

$$M_{Pd} = 4\sqrt{d^2 + 4P} + 2d - S - 16 \quad (56)$$

The above-mentioned formula (Eq. 56) calculates the minimum number of moves required to solve the rectangular Klotski puzzle whose product is $(n + f)(n + k)$, where $f \geq 1, k > 2, k > f, f < S$, and $2 \leq S \leq \infty$ and if given the product and the difference of the sides.

The following table summarizes the product and the minimum number of moves in a $(n + f)(n + k)$ rectangular Klotski, where $f \geq 1, k > 2, k > f, f < S$, and $2 \leq S \leq \infty$:

No. of Spaces (S)	Minimum no. of moves (M)
2	$6l + 2w - 18$
3	$6l + 2w - 19$
4	$6l + 2w - 20$
5	$6l + 2w - 21$
6	$6l + 2w - 22$
...	...

Formulation of $M = 6l + 2w - S - 16$

$$M = (6l + 2w - 18) + (S - 2)(-1)$$

$$M = 6l + 2w - 18 - S + 2$$

$$M = 6l + 2w - S - 16 \quad (57)$$

The above-mentioned formula (Eq. 57) calculates the minimum number of moves required to solve the rectangular Klotski puzzle whose product is $(n + f)(n + k)$, where $f \geq 1, k > 2, k > f, f < S$, and $2 \leq S \leq \infty$.

General Formulae

General Formula for Square Klotski:

$$M = \begin{cases} 8s - 3S - 8 & \text{if } 1 \leq S \leq 4, s \geq S \\ 8s - S - 16 & \text{if } 4 \leq S \leq \infty, s \geq S \end{cases} \quad (58)$$

General Formula for Rectangular Klotski:

$$M = \begin{cases} \sqrt{4l^2 + 4w - 14} & \text{if } (n+1)(n+2) \\ 6l + 2w - 13 & \text{if } S = 1 \\ 4l + 2w - S - 16 & \text{if } (n+f)(n+k) \end{cases} \quad (59)$$

where $f \geq 1$,
 $k > 2, k > f$,
 $f < S$,
 $2 \leq S \leq \infty$

General Formula for Rectangular Klotski given the difference (d) and the product (P):

$$M_{Pd} = \begin{cases} \sqrt{4d^2 + 4P - 14} & \text{if } (n+1)(n+2) \\ 4d^2 + 4P + 2d - 13 & \text{if } S = 1 \\ 4d^2 + 4P + 2d - S - 16 & \text{if } (n+f)(n+k) \end{cases} \quad (60)$$

where $f \geq 1$,
 $k > 2, k > f$,
 $f < S$,
 $2 \leq S \leq \infty$

M - Minimum number of Moves

P - Product ($l \times w$)

d - Difference ($l - w$)

s - Number of rows or columns

l - Number of Columns

w - Number of Rows

S - Number of Spaces

Examples

For Formula $M = 8s - 3S - 8$ (Eq. 58):

In a square Klotski with a 5×5 dimension with 4 spaces, by applying the formula $M = 8s - 3S - 8$, the minimum number of moves is

$$M = 8(5) - 3(4) - 8$$

$$M = 40 - 12 - 8$$

$$M = 20$$

For Formula $M = 8s - S - 16$ (Eq. 58):

In a square Klotski with a 8×8 dimension with 6 spaces, by applying the formula $M = 8s - S - 16$, the minimum number of moves is

$$M = 8(8) - 6 - 16$$

$$M = 64 - 6 - 16$$

$$M = 42$$

For Formula $M = 4\sqrt{1 + 4lw} - 14$ (Eq. 59):

In a rectangular Klotski with a 3×4 dimension, by applying the formula $M = 4\sqrt{1 + 4lw} - 14$, the minimum number of moves is

$$M = 4\sqrt{1 + 4(4)(3)} - 14$$

$$M = 4\sqrt{1 + 48} - 14$$

$$M = 4\sqrt{49} - 14$$

$$M = 4(7) - 14$$

$$M = 28 - 14$$

$$M = 14$$

For Formula $M = 6l + 2w - 13$ (Eq. 59):

In a rectangular Klotski with a 3×7 dimension, by applying the formula $M = 6l + 2w - 13$, the minimum number of moves is

$$M = 6(7) + 2(3) - 13$$

$$M = 42 + 6 - 13$$

$$M = 35$$

For Formula $M = 6l + 2w - S - 16$ (Eq. 59):

In a rectangular Klotski with a 9×12 dimension with 8 spaces, by applying the formula $M = 6l + 2w - S - 16$, the minimum number of moves is

$$M = 6(12) + 2(9) - 8 - 16$$

$$M = 72 + 18 - 8 - 16$$

$$M = 66$$

For Formula $M_{Pd} = 4\sqrt{1 + 4P} - 14$ (Eq. 60)

In a rectangular Klotski that has the product of 12, by applying the formula $M_{Pd} = 4\sqrt{1 + 4P} - 14$, the

minimum number of moves is

$$\begin{aligned}
 M_{Pd} &= 4 \sqrt{1 + 4(12)} - 14 \\
 M_{Pd} &= 4 \sqrt{49} - 14 \\
 M_{Pd} &= 4(7) - 14 \\
 M_{Pd} &= 28 - 14 \\
 M_{Pd} &= 14
 \end{aligned}$$

For Formula $M_{Pd} = 4 \sqrt{d^2 + 4P + 2d} - 13$ (Eq. 60)

In a rectangular Klotski whose product and difference of the dimensions are 21 and 4, respectively, by applying the formula for the product and difference, the minimum number of moves is

$$\begin{aligned}
 M_{Pd} &= 4 \sqrt{4^2 + 4(21) + 2(4)} - 13 \\
 M_{Pd} &= 4 \sqrt{16 + 84 + 8} - 13 \\
 M_{Pd} &= 4 \sqrt{100} - 5 \\
 M_{Pd} &= 4(10) - 5 \\
 M_{Pd} &= 40 - 5 \\
 M_{Pd} &= 35
 \end{aligned}$$

For Formula $M_{Pd} = 4 \sqrt{d^2 + 4P + 2d - S} - 16$ (Eq. 60)

In a rectangular Klotski with a product and a difference of 108 and 3 respectively and a spaces of 8, by applying the formula $M_{Pd} = 4 \sqrt{d^2 + 4P + 2d - S} - 16$, the minimum number of moves is

$$\begin{aligned}
 M_{Pd} &= 4 \sqrt{(3)^2 + 4(108) + 2(3) - (8) - 16} \\
 M_{Pd} &= 4 \sqrt{9 + 432 + 6 - 8 - 16} \\
 M_{Pd} &= 4 \sqrt{441 + 6 - 8 - 16} \\
 M_{Pd} &= 4(21) + 6 - 8 - 16 \\
 M_{Pd} &= 66
 \end{aligned}$$

4 Chapter IV

4.1 Summary of the Study

Therefore, the investigators came up with a technique in the game Klotski Puzzle where the player must move the tile diagonally from its starting point to its predetermined spot in order to obtain the minimum number of moves. Varied formulae were derived and formulated to determine the minimum number of moves based on the dimensions and number of spaces in the predetermined spot of the Klotski puzzle. The investigators used the piecewise function, the quadratic formula, and the arithmetic sequence formed by the pattern whose common difference are 8 and 6 as their basis in developing these formulae.

Thus, this study shows potential not limiting only in the field of games but also in some real life situation that involves algorithm that calculate the minimum number of moves needed to reach a specific goal such

as parking assistance system, traffic management, cargo handling, space optimization, and other situations that deals with constraints and involves the process of optimizing movement.

4.2 Recommendation

The investigators recommend to study further not just on square and rectangles but also on triangle and other shaped Klotskis. The movement of more than 1 block can also be taken into consideration in the future conduct of related investigation. Also, blocks with different shapes and sizes will be a good topic for investigation. Lastly, the investigators of this study recommends future researchers to investigate the relevance of the results of this study and to assess the feasibility of the suggested potential real life application.

5 Appendices

5.1 Source Code

```
1      \documentclass [11 pt]{ article }
2      \usepackage [margin =1 in]{ geometry }
3      \usepackage { amsmath , amssymb , bbm }
4      \usepackage { graphicx }
5      \usepackage { xcolor , soul }
6      \usepackage { multicol }
7      \usepackage { colortbl , booktabs }
8      \sethlcolor{ yellow }
9      \usepackage { calc }
10     \usepackage { mathtools }
11     \usepackage { mathptmx }
12     \usepackage { color }
13     \usepackage { amsfonts }
14     \title{\textbf{Exploring Klotski: An Investigation of the Minimum
15           Number of Moves in a Special Case of Slide Puzzles}}
16     \author{Axelcris G. Suladay \\\ Mergel Ann G. Millendez \\\ Trisha B.
17           Vista }
18     \date{December 2024}
19     \usepackage [ firstpage = true ]{ background }
20     \backgroundsetup {
21       scale =3 ,
22       angle =0 ,
23       firstpage = false ,
24       opacity =0.20 ,
25       contents ={\begin{tikzpicture}[remember picture ,overlay]
26         \node at ([yshift=0pt,xshift=0pt]current page.center)
27         {\includegraphics [ width =1.5 in ]{" FRNHS _LOGO "}};
28       \end{tikzpicture} } }
29     \usepackage { xcolor }
30     \usepackage { listings }
31     \lstset
32     {
33       language = {[ LaTeX ]TeX},
34       %alsolanguage = { PGF / TikZ },
35       frame = single ,
36       framesep = \fboxsep ,
37       framerule = \fboxrule ,
38       rulecolor = \color{ red },
39       xleftmargin = \dimexpr \fboxsep + \fboxrule ,
40       xrightmargin = \dimexpr \fboxsep + \fboxrule ,
41       breaklines = true ,
42       basicstyle = \small \tt ,
43       keywordstyle = \color{ red } \sf ,
44       identifierstyle = \color{ magenta },
45       commentstyle = \color{ gray },
46       %backgroundcolor = \color{ white },
```

```

47         \tabsize=2,
48         \columns = flexible ,
49     }
50
51     \DeclareUnicodeCharacter {2205}{ $\emptyset$ }
52
53     \newlength \dlf
54     \newcommand \alignedbox [2]{
55         % #1 = before alignment
56         % #2 = after alignment
57         &
58         \begin{group}
59         \settowidth \dlf{$\displaystyle #1$}
60         \addtolength \dlf{\fboxsep + \fboxrule}
61         \hspace{-\dlf}
62         \fcolorbox{white}{yellow}{\Large$\displaystyle #1 #2$}
63         \end{group}
64     }
65
66     \newcommand \alignedboxn [2]{
67         % #1 = before alignment
68         % #2 = after alignment
69         &
70         \begin{group}
71         \settowidth \dlf{$\displaystyle #1$}
72         \addtolength \dlf{\fboxsep + \fboxrule}
73         \hspace{-\dlf}
74         \fcolorbox{white}{yellow}{$\displaystyle #1 #2$}
75         \end{group}
76     }
77
78     \definecolor{RED}{HTML}{FFC7CE}
79
80     \begin{document}
81         \onecolumn
82         \thispagestyle{empty} \setcounter{page}{0}%
83         \textcolor{white}{hgghjg}
84         \vfill
85         \begin{center}
86             \Huge \textbf{Exploring Klotski: An Investigation of the Minimum
87             Number of Moves in a Special Case of Slide Puzzles}
88             \vspace*{0.25 in} \\\
89             \normalsize Axelcris G. Suladay\textsuperscript{1}, Mergel Ann G.
90             Millendez\textsuperscript{1}, Trisha B.
91             Vista\textsuperscript{1} \\\
92             \textsuperscript{1} Francisco Ramos National High School
93         \end{center}
94         \vfill
95         \newpage
96         \maketitle
97         \begin{abstract}
98             Klotski is a special type of sliding puzzle that first emerged in
99             the early 20th century. It refers to a variety of sliding
100             block or tile games with the goal of moving a specific block

```

to a predetermined spot. By changing the game's rule that requires a player to move a certain tile or block from one corner of the puzzle to its opposite corner, providing that it contains spaces more than or equal to one, this inquiry sought to ascertain the minimum number of moves. It was also recognized that the Klotski puzzle's dimension was directly influenced by the number of spaces.

The results of the investigation proposed the use of these formulae in determining the minimum number of moves (M) given the number of columns/rows of a square-shaped Klotski:

- $M = 8 - S$ if the number of spaces (S) is equal to 1,
- $M = 8 - S$ if the number of spaces (S) is greater than or equal to 1 but less than or equal to 4,
- $M = 8 - S$ if the number of spaces (S) is greater than or equal to 4 but less than or equal to infinity and the number of rows/columns (S) is greater than or equal to the number of spaces (S). Alternatively, 3 formulae were proposed to determine the minimum number of moves (M) of a rectangular-shaped Klotski: a. $M = 6l + 2w - 13$ if the number of spaces is equal to 1, b. $M = 4\sqrt{1 + 4lw} - 14$ if the product of the dimension is the quantity of n plus 1 multiplied by the quantity of n plus 2, and c. $M = 6 - l + 2w - 16$ if the product of the dimension is the quantity of n plus f multiplied by the quantity of n plus k , where f is greater than or equal to 1, k is greater than 2, k is greater than f , f is less than the number of spaces (S), and lastly if the number of spaces is greater than or equal to 2 but less than or equal to infinity. In the case where only the product (P) and difference (d) of the dimensions are given, the formula $M_{\{Pd\}} = 4\sqrt{d^2 + 4P} + 2d - 13$ may be used to determine the minimum number of moves if the number of spaces is equal to 1, while $M_{\{Pd\}} = 4\sqrt{1 + 4P} - 14$ is used only if the product of the dimension is the quantity of n plus 1 multiplied by the quantity of n plus 2, and $M_{\{Pd\}} = 4\sqrt{d^2 + 4P} + 2d - S - 16$ will be utilized if the product of the dimension is the quantity of n plus f multiplied by the quantity of n plus k , where f is greater than or equal to 1, k is greater than 2, k is greater than f , f is less than the number of spaces (S), and the number of spaces is greater than or equal to 2 but less than or equal to infinity. It is recommended that additional investigations must be done for other cases of the puzzle, such as with triangles and other shaped Klotski puzzles, with consideration of the movement of more than 1 block and to determine the other possible real-life application of the concepts and result in this study.

\end{abstract}

\begin{center}

\textit{Arithmetic, Klotski, Minimum Number of Moves, Pattern, Sliding Puzzle, Quadratic Formula, Piecewise function}

\end{center}

\newpage

```

103 \section{Chapter I}
104 \subsection{Introduction}
105 \paragraph{}
106 Klotski is a type of a sliding puzzle that have started and gained
    popularity in the early 20th century. It refers to a whole
    group of similar sliding-block puzzles with a same objective of
    moving a certain block to a predetermined location
    \cite{sliding}. Within the frame, the blocks can be moved
    anywhere by —slidingnot turning, lifting or jumping. To complete
    the sliding puzzle, the starting and finishing positions are
    typically provided \cite{kiddle}.
107 \begin{figure}[h!]
108 \centering
109 \includegraphics[width=3in]{Figures/Klotski_Puzzle}
110 \caption{Sample of a Mobile Game Klotski}
111 \label{fig:klotskipuzzle}
112 \end{figure}
113

```

114 Upon watching a player **do** the Klotski puzzle by moving a tile from
 one place to another, the investigators observed some patterns
 that are related and was tackled **in** their Mathematics class **in**
 the previous school year. This urges them to perform a short
 experimentation of moving a tile **in** a diagonal direction **in**
 order to find the minimum number of moves and to formulate a
 mathematical investigation about the Klotski puzzle **in** hopes of
 discovering concepts that may contribute **in** the study of
 mathematics .

115
 116 Thus, this study focuses on exploring the Klotski puzzle and to
 generate formulas out of the number of moves it needed to
complete the puzzle. This investigation also aims to determine
 the minimum possible number of moves **in** solving a square or
 rectangular-shaped Klotski with the aid of the different
 mathematical concepts, particularly the arithmetic sequence .

117
 118 Like other varieties of sliding puzzle, Klotski is also known as a
 challenging game that brings entertainment to people **in** all
 ages. This game also enhances an individual's problem solving
 skills ,critical thinking skills and other mathematical related
 skills .Moreover ,this game might not just be a " simple game "
 for it has the potential to unfold mathematical concepts that
 have a significant contribution to mathematical research .

119 \subsubsection{Significance of the Study}

120 \paragraph{}

121 This study adds to the corpus of information for everyone ,

especially for gamers ,and it is noteworthy since the idea might be used
 in other games .This investigation makes fresh contribution to the field
 of games and geometry ,due to its usage and relation to the concept of
 sequence and patterns .With the success of this study ,not only students
 but everyone else may be able to benefit from this study by using the
 concepts in this investigation in the future and pass this knowledge to
 the next generation .Lastly ,the investigators of this study are hoping
 that other investigators would find the relevance of this study into its
 real life application .

```

122 \ subsection{ Statement_of_the_Problem }
123 \ paragraph{}
124 The aim of the game under investigation is to slide a corner
    tile / block from its starting position (upper left corner) to its opposite
    corner (bottom right) in the least possible number of moves. In addition,
    each block / tile can only be moved up, down, left, right and one at a
    time. Diagonal movement and lifting the blocks are not permitted. There
    can be 1 or more spaces and the starting free space of the puzzle is
    located in the bottom right corner. If there are 2 or more spaces, the
    spaces should be vertically aligned above the predetermined spot.
125 \ begin{figure}[h]
126 \ centering
127 \ includegraphics [ width = 1.5 in ] { Figures / Slide1 }
128 \ caption { Sample_Predetermined_State_of_a_Square_Klotski }
129 \ label { fig : slide1 }
130 \ end{figure}
131 \ begin{figure}[h]
132 \ centering
133 \ includegraphics [ width = 3 in ] { Figures / Slide2 }
134 \ caption { Sample_Predetermined_State_of_a_Rectangular_Klotski }
135 \ label { fig : slide2 }
136 \ end{figure}
137
138 Based on the aforementioned rule of the game, the following
    objectives were identified to facilitate the conduct of this
    investigation : \ \
139
140 $ \ bullet$ To determine the minimum number of moves in the Klotski
    game \ \
141
142 $ \ bullet$ To determine whether a pattern is formed by the minimum
    number of moves \ \
143
144 $ \ bullet$ To create a formula in calculating the minimum number of
    moves for all square Klotski \ \
145
146 $ \ bullet$ To create a formula in calculating the minimum number of
    moves for all rectangular Klotski \ \
147
148 $ \ bullet$ To determine the relationship between the number of
    spaces and the number of rows and columns \ \
149
150 $ \ bullet$ To create a formula in calculating the minimum number of
    moves given the number of spaces for all square Klotski \ \
151
152 $ \ bullet$ To create a formula in calculating the minimum number of
    moves given the number of spaces for all rectangular Klotski
153 \ subsection{ Review_of_Related_Literature }
154 \ subsubsection{ Klotski_and_other_Sliding_Puzzles }
155 \ paragraph{}
156 A frame enclosed collection of forms makes up a sliding puzzle. The
    only way to move the shapes inside that frame is to slide; turning,
    lifting, or jumping are not permitted. The 15 puzzle, created in 1878 in
    the USA (by an unidentified person), was the one that actually kicked off

```

```

157 the_genre_of_puzzles .\\
158 \ begin{figure}[h]
159 \ centering
160 \ includegraphics [ width =4 in]{ Figures / Slide_Puzzle_ 3 x 3 }
161 \ caption { Sample_of_a_ $3 \ times_ 3 $ _Mobile _Game _Sliding _Puzzle }
162 \ label{ fig : slidepuzzle 3x3 }
163 \ end{figure }
164 \ begin{figure}[h]
165 \ centering
166 \ includegraphics [ width =4 in]{ Figures / Slide_Puzzle_ 5 x 5 }
167 \ caption { Sample_of_a_ $5 \ times_ 5 $ _Mobile _Game _Sliding _Puzzle }
168 \ label{ fig : slidepuzzle 5x5 }
169 \ end{figure }
170 \begin{document}
The traditional Klotski puzzle is one of the notable sliding puzzle
that was known as "Hua Rong Dao" in China and "Dad's Puzzler" of USA. In
China ,this traditional Klotski gained popularity in the 1930 's based on
the warrior from the Eastern Han Dynasty and well –known Chinese
strategist Cao Cao .The Cao Cao piece serves as the main header in
sliding block puzzles ,and the other nine pieces represent armed guards
and other barriers .The player 's task is to overcome these challenges and
deliver Cao Cao to freedom at the bottom of the board
\ cite{admin_ 2021 _sequential} .On the other hand ,Dad 's puzzler was
invented by one of America 's leading mathematician ,Olive C. Hazlett
during the 1920 s. There is no pictures to put together or number to get
in order in this puzzle ,it 's just simply uses pieces of wood .To play
this puzzle ,the large square must be moved from position A to position C
in order to complete the puzzle .Also ,no pieces may be turned ,raised
off the bottom of the frame ,or jumped \ cite{ _dads }.
171 \begin{document}
172 \begin{document}
The frame and base of the Klotski puzzle and Dad 's Puzzler serves
as the foundation upon which the puzzle is built ,it also have the
crucial role to constraints within which blocks can be shifted .It is
commonly made out of wood or durable plastic .China 's Klotski puzzle have
one large block ( $2 \ times_ 2 $ ) ,representing the protagonist .It also have
4 medium rectangles ( $2 \ times_ 1 $ ) and 4 small squares ( $1 \ times_ 1 $ ) that
can be slid in a specific direction to help the protagonist reach the
exit .While Dad 's puzzler 's blocks have one $2 \ times_ 2 $ square ,four
$2 \ times_ 1 $ rectangles with the long side horizontal ,two $2 \ times_ 1 $
rectangles with the long side vertical ,and two $1 \ times_ 1 $ squares
\ cite{ _2023 _the }.
173 \begin{document}
174 \begin{document}
The Klotski puzzle has changed over time ,giving rise to a variety
of versions and adaptations that continue to intrigue and test puzzle
enthusiasts .Some example of these are Klotski puzzle with Animated
images ,Klotski puzzle with a hexagonal frame ,and a Klotski puzzle that
holds more than 10 blocks that the traditional one have .The well –known
Klotski puzzle game was also included in the improvement of technology
after becoming part of Windows 3.1 's Microsoft entertainment pack and has
since become a classic \ cite{ _2023 _the }.
175 \begin{document}
176 \begin{document}
A puzzle is a type of educational game that brings entertainment
and challenges everyone to solve it .To find the right and enjoyable
solution to a puzzle it requires the development of certain skill that

```

may not only help solve the puzzle but also enhances one's cognitive abilities. This will put the observation skill, planning skill, critical thinking skill and patience up to a test. Playing the puzzle will help us develop our skills that we can also apply in our daily lives without taking away the fun \ cite{ _2023 _the }.

According to the recent article of Zhong (2023), studying new algorithms to solve sliding puzzles has become increasingly important as sliding puzzle sports have developed. Zhong along with other investigators hope to develop an additive approximation method because it is difficult to solve the puzzle optimally. The length of the answer produced by this algorithm should only be a low – order term longer than the ideal solution \ cite{ zhong_2023 _additive }.

\\
\\ subsection{ Arithmetic Sequence }
\\ paragraph{ }

Carl Friedrich Gauss is a German Mathematician who contributed in many fields of Math and Science. When he was a young boy he was tasked to add the integers 1 to 100 by his teacher. As a Math prodigy he is, he was able to create his own formula which leads to the discovery of arithmetic sequence. It is an ordered set of numbers that have a common difference between each consecutive term \ cite{ gauss }.

Mathematicians and others who work with numbers can solve difficult mathematical problems by using arithmetic sequences, which are employed in algebra and geometry. Arithmetic sequences can be used to solve simple or complex problems but it's necessary to have the basic understanding about it to ensure that it is applied correctly. Finding patterns, such as a specific number of extra seats each row, is how this is done \ cite{ a2017 _how }.

In the study of Oded Goldreich in 2011, They take into account a game in which players move various stones along an undirected graph's edges. Only one pebble may be present in each vertex at any one moment, and only one pebble may be transferred at a time (i.e., the pebble must be moved to an empty vertex). They demonstrate that it is NP – Hard to determine the shortest sequence of moves between two given "pebble configurations" \ cite{ odedgoldreich_2011 _finding }.

\\
\\ subsection{ Piecewise Function }
\\ paragraph{ }

A piecewise function is a function in which the outcome is defined by more than one formulas. It describes situations where a rule or relationship changes as the input value passes certain "boundaries" \ cite{ lumen _define }. The earliest known use of the word piecewise is in the late 1600's in the writing of Nathaniel Fairfax, a physician and antiquary. Since then, it has been significant and useful in different fields as long as there is a change based on its domain value; specifically, in mathematics, this function is employed when different expressions or formulas apply to various parts of the domain \ cite{ a2023 _piecewise }.

In the study of Nagayama, Sasao, and Butler (2014), to represent a numeric function compactly, they partition the domain of the function

into uniform segments and transform the sub-function in each segment into an arithmetic spectrum. An arithmetic expression is derived from this, and a piecewise arithmetic expression for the function is obtained. This is later on widely used in the different parts of their study \cite{nagayama_piecewise}.

\\
\subsubsection{Quadratic Formula}
\paragraph{}

The development or derivation of mathematical ideas is similarly as continuous as the historical evolution, as one mathematician picks up an idea where another mathematician leaves it. One of these is the quadratic formula, which is used to find the roots of a quadratic equation \cite{mathnasium_2017_the}. The Quadratic Formula uses the " a ", " b ", and " c " from " $ax^2 + bx + c$ ", where " a ", " b ", and " c " are just numbers; they are the "numerical coefficients" of the quadratic equation that is given to solve \cite{stapel_2019_the}.

The quadratic formula is the easiest method to determine the roots of a quadratic equation. It provides a straightforward way to solve the equation, offering an alternative to factorization. In cases where a quadratic equation is difficult to factor, the quadratic formula is a convenient and efficient tool for quickly finding the roots \cite{cuemath_2024_quadratic}.

\\
\subsection{Scope and Delimitations}
\paragraph{}

This investigation aims to determine the minimum number of moves required to move a block in a Klotski puzzle from its starting location to its designated spot. The predetermined spot occupies one or more than spaces that is vertically aligned. A block or tile must be moved diagonally from the upper-left corner to the opposite bottom-right corner. The legal moves are limited to: (a) sliding the block / tile up, down, left, or right, and (b) only one block can be moved at a time. All blocks are of equal size, but the 'puzzles' dimensions differ as the number of columns should be greater than the number of rows, it also vary depending on the number of spaces in the predetermined spot, where the 'puzzles' dimension must be equal to or greater than the number of spaces in the predetermined spot disregarding the rows and column that is less than the chosen dimension. Additionally, this study aims to derive or create formulae based on observed patterns or sequences. Finally, the scope of this study is not limited to square Klotski puzzles; it also considers rectangular Klotski puzzles with varying dimensions and number of spaces in the predetermined spot.

\\
\section{Chapter II}
\subsection{Definition}
\noindent

\textbf{Arithmetic Sequence} – a sequence where the next terms after the first term is obtained by adding a constant number. The difference between each consecutive terms is called the common difference. \\ \\

\textbf{Klotski} – comes from the polish word "klocki" which means wooden blocks. It is a sliding puzzle that aims to move the red block to exit at the bottom of the board or to move the red block from and into a specified location. \\ \\

209 \textbf{ Minimum moves }–the least quantity ,amount possible ,
 assignable ,allowable ,or attainable in moving a block from a Klotski
 puzzle .\\ \\
 210 \textbf{ Pattern }– is a repeated arrangement of number or the set of
 numbers that are related to each other by following a specific rule .\\ \\
 211 \textbf{ Sliding puzzle }– is a type of simple combination puzzle .It
 challenges the player to slide pieces along a certain routes on a board
 to reach a certain end arrangement .\\ \\
 212 \textbf{ Piecewise Function }– is a function defined by multiple
 sub – functions each applying to a specific intervals of its domain .\\ \\
 213 \textbf{ Quadratic Formula }– is a mathematical rule that helps find
 the solutions to equations of the form $sax^2 + bx + c = 0$,to figure out the
 value " x " that makes the equation true .

214 \subsection{ Known Results or Conjecture }

215 \paragraph{}

216 The following are the conjectures generated by the investigators in
 the conduct of the investigation :\\
 217

218 \$ \bullet\$ The minimum number of moves for square and rectangular
 Klotski with varying dimensions forms an arithmetic sequence \\
 219

220 \$ \bullet\$ The dimension of a Klotski puzzle are directly influenced
 by the number of spaces it contains \\
 221

222 \$ \bullet\$ The formula to find the minimum number of moves (M) of
 a square Klotski is $M = 8s - 11$,where s is the number of columns or rows
 given that the number of space is equal to 1 \\
 223

224 \$ \bullet\$ The formula to find the minimum number of moves of a
 rectangular Klotski are :

225 \begin{itemize}

226 \item [a.] Given the difference and the product:

227 \begin{itemize}

228 \item [1)] $M_{Pd} = 4 \sqrt{d^2 + 4P} + 2d - 13$,where d is the
 positive difference and P is the product between the number of rows and
 columns

229 \end{itemize}

230 \item [b.] Given the number of rows or columns :

231 \begin{itemize}

232 \item [2)] $M = 6l + 2w - 13$,where l and w are the number of
 rows and columns ,and $l > w$

233 \end{itemize}

234 \end{itemize}

235

236 \$ \bullet\$ The formula to find the minimum number of moves (M) of
 a square Klotski given the spaces are greater than or equal to 2 are :

237 \begin{enumerate}

238 \item $M = 8s - 3S - 8$,this formula can be use if $1 \leq S \leq 4$
 where s is the number of rows or columns and S is the number of
 spaces ,and $s \geq S$

239 \item $M = 8s - S - 16$,this formula can be use if $4 \leq S \leq$
 \infty where s is the number of rows or columns and S is the number
 of spaces ,and $s \geq S$

240 \end{enumerate}

```

241 .....
242 .....$ \bullet$ The formula to find the minimum number of moves ($M$) of
    a rectangular Klotski given the spaces are greater than or equal to $2$
    are :
243 ..... \begin{itemize}
244 ..... \item [a.] Given the difference and the product
245 ..... \begin{itemize}
246 ..... \item [1)] $M_{Pd}=4\sqrt{1+4P}-14$, this formula can be used
    if the product of the dimension is $(n+1)(n+2)$
247 ..... \item [2)] $M_{Pd}=4\sqrt{d^2+4P}+2d-16$, this formula can be
    used if the product of the dimension is $(n+f)(n+k)$, where $f\geq 1$,
    $k>2$, $k>f$, $f<S$ and $2\leq S\leq\infty$
248 ..... \end{itemize}
249 ..... \item [b.] Given the number of rows and columns
250 ..... \begin{itemize}
251 ..... \item [3)] $M=4\sqrt{1+4lw}-14$, this formula can be use if
    $(n+1)(n+2)$ where $l$ is the number of columns and $w$ is the number of
    rows, $l>w$
252 ..... \item [4)] $M=6l+2w-16$, this formula can be use if
    $(n+f)(n+k)$ where $f\geq 1$, $k>2$, $k>f$, $f<S$, $2\leq S\leq\infty$,
    where $l$ is the number of columns, $w$ is the number of rows, $S$ is the
    number of spaces, and $l>w$.
253 ..... \end{itemize}
254 ..... \end{itemize}
255 .....
256 ..... \section{Chapter III}
257 ..... \subsection{Researchers' Generated Data and Results}
258 ..... \paragraph{}
259 ..... This section presents the data that the investigators gathered, the
    pattern uncovered and the formulation and derivation of the
    formula to find the minimum number of moves on the Klotski
    Puzzle. The following data and results are organized and
    presented based on the order of the specific objectives
    identified in section 1.3 \\
260 ..... \\
261 ..... \textbf{Data for the Minimum Number of Moves}
262 ..... \begin{table}[h]
263 ..... \centering
264 ..... \caption{Minimum Number of Moves based on the Number of Rows and
    Columns}
265 ..... \label{fig: slide6}
266 ..... % Table generated by Excel2LaTeX from sheet 'Sheet1'
267 ..... \begin{tabular}{|c|c|c|c|c|c|c|c|c|}
268 ..... \hline
269 ..... x & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots \\
270 ..... \hline
271 ..... 1 & 0/ & 1 & 0/ & 0/ & 0/ & 0/ & 0/ & \dots \\
272 ..... \hline
273 ..... 2 & 1 & 5 & 9 & 15 & 21 & 27 & 33 & \dots \\
274 ..... \hline
275 ..... 3 & 0/ & 9 & 13 & 17 & 23 & 29 & 35 & \dots \\
276 ..... \hline
277 ..... 4 & 0/ & 15 & 17 & 21 & 25 & 31 & 37 & \dots \\
278 ..... \hline

```

```

279         5 & 0/ & 21 & 23 & 25 & 29 & 33 & 39 & ...   \\
280         \hline
281         6 & 0/ & 27 & 29 & 31 & 33 & 37 & 41 & ...   \\
282         \hline
283         7 & 0/ & 33 & 35 & 37 & 39 & 41 & 45 & ...   \\
284         \hline...
285         & ... & ... & ... & ... & ... & ... & ... & ...   \\
286         \hline
287     \end{tabular}%
288
289 \end{table}
290 \begin{figure}[h!]
291
292 \end{figure}
293
294
295 The minimum number of moves in a square Klotski is always obtained
    by traveling the main block /tile around the diagonal path of
    the puzzle or it follows a stair-like path. For the rectangular
    Klotski, a similar strategy can be used were the main
    block /tile should travel also around the diagonal path of the
    puzzle. \\
296 \begin{figure}[h]
297     \centering
298     \includegraphics[width=1 in]{Figures /Slide3}
299     \caption{Path to obtain the Minimum Number of Moves of a Square
        Klotski}
300     \label{fig : slide3}
301 \end{figure}
302
303 \begin{figure}[h]
304     \centering
305     \includegraphics[width=2.5 in]{Figures /Slide4}
306     \caption{Path to obtain the Minimum Number of Moves of a
        Rectangular Klotski}
307     \label{fig : slide4}
308 \end{figure}
309 \noindent
310 \textbf{Pattern for the Minimum Number of Moves}
311
312 \begin{table}[h!]
313     \centering
314     \caption{Pattern for the Minimum Number of Moves for a Square
        Klotski}
315     \label{fig : slide7}
316     % Table generated by Excel2LaTeX from sheet 'Sheet1'
317     \begin{tabular}{|c|c|c|c|c|c|c|c|c|}
318         \hline
319         x & 1 & 2 & 3 & 4 & 5 & 6 & 7 & ...   \\
320         \hline
321         1 & \cellcolor[rgb]{1, .78, 0/.808} & \cellcolor[rgb]{1,
            .78, .808} & 1 & \cellcolor[rgb]{1, .78, 0/.808} &
            \cellcolor[rgb]{1, .78, 0/.808} & \cellcolor[rgb]{1,
            .78, 0/.808} & \cellcolor[rgb]{1, .78, 0/.808} &
            \cellcolor[rgb]{1, .78, 0/.808} &

```

```

322         \cellcolor[rgb]{ 1, .78, 0/.808} & \cellcolor[rgb]{ 1,
323         .78, ... .808} \\
324     \hline
325     2 & \cellcolor[rgb]{ 1, .78, .808}1 & \cellcolor[rgb]{ 1,
        .922, .612}5 & 9 & 15 & 21 & 27 & 33 & ... \\
326     \hline
327     3 & \cellcolor[rgb]{ 1, .78, 0/.808} & 9 & \cellcolor[rgb]{ 1,
        .922, .612}13 & 17 & 23 & 29 & 35 & ... \\
328     \hline
329     4 & \cellcolor[rgb]{ 1, .78, 0/.808} & 15 & 17 &
        \cellcolor[rgb]{ 1, .922, .612}21 & 25 & 31 & 37 & ... \\
330     \hline
331     5 & \cellcolor[rgb]{ 1, .78, 0/.808} & 21 & 23 & 25 &
        \cellcolor[rgb]{ 1, .922, .612}29 & 33 & 39 & ... \\
332     \hline
333     6 & \cellcolor[rgb]{ 1, .78, 0/.808} & 27 & 29 & 31 & 33 &
        \cellcolor[rgb]{ 1, .922, .612}37 & 41 & ... \\
334     \hline
335     7 & \cellcolor[rgb]{ 1, .78, 0/.808} & 33 & 35 & 37 & 39 & 41
        & \cellcolor[rgb]{ 1, .922, .612}45 & ... \\
336     \hline...
337     & \cellcolor[rgb]{ 1, .78, ... .808} & ... & ... & ... &
        ... & ... & ... & \cellcolor[rgb]{ 1, .922, .....612}
        \\
338     \hline
339     \end{tabular}%
340
341     \end{table}

```

As shown in Table \ref*{fig:slide7}, the minimum number of moves a square Klotski (in an increasing number of sides) forms an Arithmetic Sequence whose common difference is \$8\$.

```

342 \begin{table}[h!]
343 \centering
344 \caption{Pattern for the Minimum Number of Moves for a
345 Rectangular Klotski}
346 \label{tab:slide9}
347 % Table generated by Excel2LaTeX from sheet 'Sheet1'
348 \begin{tabular}{|c|c|c|c|c|c|c|c|c|}
349 \hline
350 x & 1 & 2 & 3 & 4 & 5 & 6 & 7 & ... \\
351 \hline
352 1 & \cellcolor[rgb]{ 1, .78, 0/.808} & \cellcolor[rgb]{ 1,
        .78, .808}1 & \cellcolor[rgb]{ 1, .78, 0/.808} &
        \cellcolor[rgb]{ 1, .78, 0/.808} & \cellcolor[rgb]{ 1,
        .78, 0/.808} & \cellcolor[rgb]{ 1, .78, 0/.808} &
        \cellcolor[rgb]{ 1, .78, 0/.808} &
        \cellcolor[rgb]{ 1, .78, 0/.808} & \cellcolor[rgb]{ 1,
        .78, ... .808} \\
353 \hline
354 2 & \cellcolor[rgb]{ 1, .78, .808}1 & \cellcolor[rgb]{ 1,
        .922, .612}5 & \cellcolor[rgb]{ .776, .937, .808}9 &
        \cellcolor[rgb]{ .753, .902, .961}15 & \cellcolor[rgb]{
        .776, .937, .808}21 & \cellcolor[rgb]{ .753, .902,

```

```

355         .961}27 & \cellcolor[rgb]{.776, .937, .808}33 &
356         \cellcolor[rgb]{.753, .902, .....961} \\
\hline
3 & \cellcolor[rgb]{1, .78, 0/.808} & 9 & \cellcolor[rgb]{1,
.922, .612}13 & \cellcolor[rgb]{.776, .937, .808}17 &
\cellcolor[rgb]{.753, .902, .961}23 & \cellcolor[rgb]{
.776, .937, .808}29 & \cellcolor[rgb]{.753, .902,
.961}35 & \cellcolor[rgb]{.776, .937, ... .808} \\
357 \hline
358 4 & \cellcolor[rgb]{1, .78, 0/.808} & 15 & 17 &
\cellcolor[rgb]{1, .922, .612}21 & \cellcolor[rgb]{
.776, .937, .808}25 & \cellcolor[rgb]{.753, .902,
.961}31 & \cellcolor[rgb]{.776, .937, .808}37 &
\cellcolor[rgb]{.753, .902, .....961} \\
359 \hline
360 5 & \cellcolor[rgb]{1, .78, 0/.808} & 21 & 23 & 25 &
\cellcolor[rgb]{1, .922, .612}29 & \cellcolor[rgb]{
.776, .937, .808}33 & \cellcolor[rgb]{.753, .902,
.961}39 & \cellcolor[rgb]{.776, .937, ... .808} \\
361 \hline
362 6 & \cellcolor[rgb]{1, .78, 0/.808} & 27 & 29 & 31 & 33 &
\cellcolor[rgb]{1, .922, .612}37 & \cellcolor[rgb]{
.776, .937, .808}41 & \cellcolor[rgb]{.753, .902, ...
.961} \\
363 \hline
364 7 & \cellcolor[rgb]{1, .78, 0/.808} & 33 & 35 & 37 & 39 & 41
& \cellcolor[rgb]{1, .922, .612}45 & \cellcolor[rgb]{
.776, .937, ... .808} \\
365 \hline...
366 & \cellcolor[rgb]{1, .78, ... .808} & ... & ... & ... &
... & ... & ... & \cellcolor[rgb]{1, .922, .....612}
\\
367 \hline
368 \end{tabular}%
369
370 \end{table}
371
372 As shown in Table \ref*{tab:slide9}, the minimum number of moves of
a rectangular Klotski (in an increasing number of sides) forms
also an Arithmetic Sequence whose common difference is $8$ just
like the pattern of the square Klotski. Furthermore, the first
term of each sequence in the rectangular Klotski also forms an
arithmetic sequence whose common difference is $6$. \\
373
374 \textbf{Formula for the Minimum Number of Moves of a Square}
Klotski; \[(n+1)(n+1) \text{ or } (n+1)^2 \text{ or } \{n|n \text{ in }
\mathbb{Z}^+\}]
375 \begin{center}
376 \begin{tabular}{cc}
377 Product($P$) & Minimum No. of Moves ($M$) \\
378 \hline
379 4 & 5 \\
380 9 & 13 \\
381 16 & 21

```

```

382         25 & 29 \\
383         36 & 37 \\
384         49 & 45 \\
385         64 & 53 \\
386         81 & 61 \\
387         \hline
388         $P=(n+1)^2$ & $M=8n-3$
389     \end{tabular}
390 \end{center}
391
392 \begin{center}
393     Formulation of $M=8n-3$
394     \begin{align}
395         \label{eqn:8n-3 1 space Square}
396         M=& 5+(n-1)8\ nonumber \\
397         M=& 5+8n-8\ nonumber \\
398         M=& 8n-3
399     \end{align}
400 \end{center}
401
402 \begin{center}
403     Derivation of $n$
404     \begin{align}
405         \label{eqn:N 1 spaces Square}
406         &P=(n+1)^2\ nonumber \\
407         &\sqrt{P}=\sqrt{(n+1)^2}\ nonumber \\
408         &\sqrt{P}=n+1\ nonumber \\
409         &\sqrt{P}-1=n\ nonumber \\
410         &n=\sqrt{P}-1
411     \end{align}
412 \end{center}
413
414 \begin{center}
415     Minimum number of moves in terms of the product ($M_P$)
416     \begin{align}
417         \label{eqn:MP 1 spaces Square}
418         &M=8n-3\ nonumber \\
419         &M_P=8(\sqrt{P}-1)-3\ nonumber \\
420         &M_P=8\sqrt{P}-8-3\ nonumber \\
421         &M_P=8\sqrt{P}-11
422     \end{align}
423 \end{center}
424
425 \begin{center}
426     Since $\sqrt{P}$ is just equal to the number of rows/columns
         ($s$) of the square Klotski, then the minimum number of moves
         ($M$) is
427     \begin{align}
428         \label{eqn:FF 1 spaces Square}
429         \alignedbox{ }{M=8s-11}
430     \end{align}
431 \end{center}
432 \noindent

```

**\textbf{Formula for the Minimum Number of Moves of a Rectangular
Klotski}; \[(n+1)(n+k) \ \ \ \{n|n \ \text{in} \ \mathbb{Z}^+\} \ \ \ \{k|k
 \ \text{in} \ \mathbb{Z}^+, k>1\} \]**

The arithmetic sequence formula of the different minimum number of
moves (M) for each $(n+1)(n+k)$

```
\begin{center}
\begin{tabular}{cccc}
\multicolumn{2}{c}{Minimum no. of moves for  $(n+1)(n+2)$ } & & 
\multicolumn{2}{c}{Minimum no. of moves for  $(n+1)(n+3)$ } \\
\cline{1-2}
\cline{4-5}
 $P_{(n+1)(n+2)}$  &  $M_{(n+1)(n+2)}$  & &  $P_{(n+1)(n+3)}$  & 
 $M_{(n+1)(n+3)}$  \\
\cline{1-2}
\cline{4-5}
6 & 9 & & 8 & 15 \\
12 & 17 & & 15 & 23 \\
20 & 25 & & 24 & 31 \\
30 & 33 & & 35 & 39 \\
... & ... & & ... & ... \\
\cline{1-2}
\cline{4-5}
 $(n+1)(n+2)$  &  $8n+1$  & &  $(n+1)(n+3)$  &  $8n+7$ 
\end{tabular}
\end{center}
```

```
\begin{center}
\begin{tabular}{cccc}
\multicolumn{2}{c}{Minimum no. of moves for  $(n+1)(n+4)$ } & & 
\multicolumn{2}{c}{Minimum no. of moves for  $(n+1)(n+5)$ } \\
\cline{1-2}
\cline{4-5}
 $P_{(n+1)(n+4)}$  &  $M_{(n+1)(n+4)}$  & &  $P_{(n+1)(n+5)}$  & 
 $M_{(n+1)(n+5)}$  \\
\cline{1-2}
\cline{4-5}
10 & 21 & & 12 & 27 \\
18 & 29 & & 21 & 35 \\
28 & 37 & & 32 & 43 \\
40 & 45 & & 45 & 51 \\
... & ... & & ... & ... \\
\cline{1-2}
\cline{4-5}
 $(n+1)(n+4)$  &  $8n+13$  & &  $(n+1)(n+5)$  &  $8n+19$ 
\end{tabular}
\end{center}
```

The following table summarizes the product and the minimum number
of moves for $(n+1)(n+k)$ Klotski:

```

479 \begin{center}
480 \begin{tabular}{cc}
481 \hline
482 Product ($P$) & Minimum no. of moves ($M$) \\
483 \hline
484 $(n+1)(n+2)$ & $8n+1$ \\
485 $(n+1)(n+3)$ & $8n+7$ \\
486 $(n+1)(n+4)$ & $8n+13$ \\
487 $(n+1)(n+5)$ & $8n+19$ \\
488 $(n+1)(n+6)$ & $8n+25$ \\
489 $(n+1)(n+7)$ & $8n+31$ \\
490 ... & ... \\
491 \hline
492 $P=(n+1)(n+k)$ & $M=8n+6k-11$ \\
493 \end{tabular}
494 \end{center}
495
496 \noindent
497 Formulation of $M=8n+6k-11$
498 \begin{align}
499 \label{eqn:1 Spaces Rec Explicit}
500 &M=8n+(1+[k-2]6) \nonumber \\
501 &M=8n+(1+6k-12) \nonumber \\
502 &M=8n+1+6k-12 \nonumber \\
503 &M=8n+6k-11 \\
504 \end{align}
505 Derivation of $n$
506 \begin{align}
507 \label{eqn:1 spaces Rec N}
508 P=&(n+1)(n+k) \nonumber \\
509 P=&n^2+kn+n+k \nonumber \\
510 P=&n^2+(k+1)n+k \nonumber \\
511 0=&n^2+(k+1)n+k-P \\
512 \end{align}
513 \begin{center}
514
515 using quadratic formula to find $n$
516 \begin{align}
517 \label{eqn:1 spaces rec quad}
518 n=&\frac{-(k+1) \pm \sqrt{(k+1)^2-4(1)(k-P)}}{2} \nonumber \\
519 n=&\frac{-k-1 \pm \sqrt{k^2+2k+1-4k+4P}}{2} \nonumber \\
520 n=&\frac{-k-1 \pm \sqrt{k^2-2k+1+4P}}{2} \nonumber \\
521 n=&\frac{-k-1 \pm \sqrt{(k-1)^2+4P}}{2} \\
522 \end{align}
523 \end{center}
524 \noindent
525 Minimum number of moves in terms of the product ($M_P$)
526 \begin{align}
527 \label{eqn:1 spaces rec MP}
528 M=&8n+6k-11 \nonumber \\
529 M_P=&8\left(\frac{-k-1 \pm \sqrt{(k-1)^2+4P}}{2}\right)+6k-11 \nonumber \\
530 M_P=&4(-k-1 \pm \sqrt{(k-1)^2+4P})+6k-11 \nonumber \\
531 M_P=&-4k-4 \pm 4\sqrt{(k-1)^2+4P}+6k-11 \\
532 \nonumber

```



```

\cellcolor[rgb]{ 1, .78, 0/.808} & \cellcolor[rgb]{ 1,
.78,.....808} \\
579 \hline
580 2 & \cellcolor[rgb]{ 1, .78, 0/.808} & \cellcolor[rgb]{ .776,
.937, .808}2 & \cellcolor[rgb]{ .776, .937, .808}6 &
\cellcolor[rgb]{ 1, .922, .612}10 & \cellcolor[rgb]{ 1,
.922, .612}16 & \cellcolor[rgb]{ 1, .922, .612}22 &
\cellcolor[rgb]{ 1, .922, .612}28 & \cellcolor[rgb]{ 1,
.922, ... .612} \\
581 \hline
582 3 & \cellcolor[rgb]{ 1, .78, 0/.808} & & \cellcolor[rgb]{
.776, .937, .808}10 & \cellcolor[rgb]{ .776, .937,
.808}14 & \cellcolor[rgb]{ 1, .922, .612}18 &
\cellcolor[rgb]{ 1, .922, .612}24 & \cellcolor[rgb]{ 1,
.922, .612}30 & \cellcolor[rgb]{ 1, .922, ... .612} \\
583 \hline
584 4 & \cellcolor[rgb]{ 1, .78, 0/.808} & & &
\cellcolor[rgb]{ .776, .937, .808}18 & \cellcolor[rgb]{
.776, .937, .808}22 & \cellcolor[rgb]{ 1, .922, .612}26
& \cellcolor[rgb]{ 1, .922, .612}32 & \cellcolor[rgb]{ 1,
.922, ... .612} \\
585 \hline
586 5 & \cellcolor[rgb]{ 1, .78, 0/.808} & & & &
\cellcolor[rgb]{ .776, .937, .808}26 & \cellcolor[rgb]{
.776, .937, .808}30 & \cellcolor[rgb]{ 1, .922, .612}34
& \cellcolor[rgb]{ 1, .922, ... .612} \\
587 \hline
588 6 & \cellcolor[rgb]{ 1, .78, 0/.808} & & & & &
\cellcolor[rgb]{ .776, .937, .808}34 & \cellcolor[rgb]{
.776, .937, .808}38 & \cellcolor[rgb]{ 1, .922, ...
.612} \\
589 \hline
590 7 & \cellcolor[rgb]{ 1, .78, 0/.808} & & & & & &
\cellcolor[rgb]{ .776, .937, .808}42 & \cellcolor[rgb]{
.776, .937,.....808} \\
591 \hline...
592 & \cellcolor[rgb]{ 1, .78, ... .808} & ... & ... & ... &
... & ... & ... & \cellcolor[rgb]{ .776, .937, ...
.808} \\
593 \hline
594 \end{tabular}%
595
596 \label{tab:2 spaces}
597 \end{table}
598 \begin{center}
599 \begin{tabular}{cc}
600 \multicolumn{2}{c}{$(n+1)(n+1)$\ or\ $(n+1)^2$ \ \{n\}n \ in
\mathbb{Z}^{+}} \\
601 & \\
602 Product($P$) & Minimum No. of Moves ($M$) \\
603 \hline
604 4 & 2 \\
605 9 & 10 \\
606 16 & 18

```

```

607         25 & 26 \\
608         36 & 34 \\
609         49 & 42 \\
610         \hline
611         $P=(n+1)^2$ & $M=8n-6$
612     \end{tabular}
613 \end{center}
614
615 \begin{center}
616     Formulation of $M=8n-6$
617     \begin{align}
618         \label{eqn:2 spaces Ex}
619         & M=2+(n-1)8 \nonumber \\
620         & M=2+8n-8 \nonumber \\
621         & M=8n-6
622     \end{align}
623 \end{center}
624
625 \begin{center}
626     Derivation of $n$
627     \begin{align}
628         \label{eqn:2 spaces N}
629         & P=(n+1)^2 \nonumber \\
630         & \sqrt{P}=\sqrt{(n+1)^2} \nonumber \\
631         & \sqrt{P}=n+1 \nonumber \\
632         & \sqrt{P}-1=n \nonumber \\
633         & n=\sqrt{P}-1
634     \end{align}
635 \end{center}
636
637 \noindent
638 Minimum number of moves in terms of the product ($M_P$)
639 \begin{align}
640     \label{eqn:2 spaces MP}
641     & M_P=8(\sqrt{P}-1)-6 \nonumber \\
642     & M_P=8\sqrt{P}-8-6 \nonumber \\
643     & M_P=8\sqrt{P}-14
644 \end{align}
645
646
647 Since $\sqrt{P}$ is just equal to the number of rows/columns ($s$)
        of the square Klotski, then the minimum number of moves ($M$)
        given the 2 spaces is
648 \begin{align}
649     \label{eqn:2 spaces FF}
650     \alignedbox{ }{M=8s-14}
651 \end{align}
652
653 \begin{table}[h]
654     \centering
655     \caption{Data of the Minimum number of moves for Square and
        Rectangular Klotski with 3 Spaces}
656     \begin{tabular}{|c|c|r|r|r|c|c|c|}
657         \hline

```

```

658 x & 1 & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} &
\multicolumn{1}{c}{4} & 5 & 6 & 7 & ... \\
659 \hline
660 1 & \cellcolor[rgb]{ 1, .78, 0 .808} &

& \multicolumn{1}{c}{\cellcolor[rgb]{ 1, .78, 0 .808}} &
& \multicolumn{1}{c}{\cellcolor[rgb]{ 1, .78, 0 .808}} &
& \multicolumn{1}{c}{\cellcolor[rgb]{ 1, .78, 0 .808}} &

& \cellcolor[rgb]{ 1, .78, 0 .808} & \cellcolor[rgb]{ 1,
.78, 0/ .808} & \cellcolor[rgb]{ 1, .78, 0 .808} &

& \cellcolor[rgb]{ 1, .78, ... .808} \\
661 \hline
662 2 & \cellcolor[rgb]{ 1, .78, 0 .808} &

& \multicolumn{1}{c}{\cellcolor[rgb]{ 1, .78, .808}2} &
& \multicolumn{1}{c}{\cellcolor[rgb]{ 1, .78, .808}3} &
& \multicolumn{1}{c}{\cellcolor[rgb]{ 1, .78, .808}7} &
& \cellcolor[rgb]{ 1, .78, .808}13 & \cellcolor[rgb]{ 1,
.78, .808}19 & \cellcolor[rgb]{ 1, .78, .808}25 &
& \cellcolor[rgb]{ 1, .78, ... .808} \\
663 \hline
664 3 & \cellcolor[rgb]{ 1, .78, 0 .808} & &

& \multicolumn{1}{c}{\cellcolor[rgb]{ .776, .937, .808}7}
& \multicolumn{1}{c}{\cellcolor[rgb]{ .776, .937,
.808}11} & \cellcolor[rgb]{ 1, .922, .612}17 &
& \cellcolor[rgb]{ 1, .922, .612}23 & \cellcolor[rgb]{ 1,
.922, .612}29 & \cellcolor[rgb]{ 1, .922, ... .612} \\
665 \hline
666 4 & \cellcolor[rgb]{ 1, .78, 0 .808} & & &

& \multicolumn{1}{c}{\cellcolor[rgb]{ .776, .937, .808}15}
& \cellcolor[rgb]{ .776, .937, .808}19 & \cellcolor[rgb]{
1, .922, .612}25 & \cellcolor[rgb]{ 1, .922, .612}31 &
& \cellcolor[rgb]{ 1, .922, ... .612} \\
667 \hline
668 5 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & &

& \cellcolor[rgb]{ .776, .937, .808}23 & \cellcolor[rgb]{
.776, .937, .808}27 & \cellcolor[rgb]{ 1, .922, .612}33
& \cellcolor[rgb]{ 1, .922, ... .612} \\
669 \hline
670 6 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & & &

& \cellcolor[rgb]{ .776, .937, .808}31 & \cellcolor[rgb]{
.776, .937, .808}35 & \cellcolor[rgb]{ 1, .922, ...
.612} \\
671 \hline
672 7 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & & &

& \cellcolor[rgb]{ .776, .937, .808}39 & \cellcolor[rgb]{
.776, .937, ... .808} \\
673 \hline...
674 & \cellcolor[rgb]{ 1, .78, ... .808} & & & & &

& \cellcolor[rgb]{ .776, .937, .....808} \\
675 \hline
676 \end{tabular}%

```

```

681 Formulation of  $M=8n-1$  $
682 \begin{align}
683 \quad \backslash label{eqn :3 spaces Ex}
684 \quad \&M=7+(n-1)8\ nonumber \ \backslash
685 \quad \&M=7+8n-8\ nonumber \ \backslash
686 \quad \&M=8n-1
687 \quad \end{align}
688 \end{center}
689
690 \begin{center}
691 Derivation of  $n$ $
692 \begin{align}
693 \quad \backslash label{eqn :3 spaces N}
694 \quad \&P=(n+2)^2\ nonumber \ \backslash
695 \quad \&\sqrt{P}=\sqrt{(n+2)^2}\ nonumber \ \backslash
696 \quad \&\sqrt{P}=n+2\ nonumber \ \backslash
697 \quad \&\sqrt{P}-2=n\ nonumber \ \backslash
698 \quad \&n=\sqrt{P}-2
699 \quad \end{align}
700 \end{center}
701
702 Minimum number of moves in terms of the product ( $M_P$ )
703 \begin{align}
704 \quad \backslash label{eqn :3 spaces MP}
705 \quad \&M_P=8(\sqrt{P}-2)-1\ nonumber \ \backslash
706 \quad \&M_P=8\sqrt{P}-16-1\ nonumber \ \backslash
707 \quad \&M_P=8\sqrt{P}-17
708 \quad \end{align}
709
710 Since  $\sqrt{P}$ $ is just equal to the number of rows/columns ( $s$ $)
of the square Klotski, then the minimum number of moves ( $M$ $)
given the 3 spaces is
711 \begin{align}
712 \quad \backslash label{eqn :3 spaces FF}
713 \quad \backslash alignedbox{ }{M=8s-17}
714 \quad \end{align}
715
716 \begin{table}[h]
717 \quad \backslash centering
718 \quad \backslash caption{Data of the Minimum number of moves for Square and
Rectangular Klotski with 4 Spaces}
719 \quad \backslash begin { tabular }{|c|c|c|c|c|c|c|c|}
720 \quad \backslash hline
721 \quad x \& 1 \& 2 \& 3 \& 4 \& 5 \& 6 \& 7 \& \dots \quad \backslash
722 \quad \backslash hline
723 \quad 1 \& \backslash cellcolor[rgb]{ 1, .78, 0/.808} \& \backslash cellcolor[rgb]{ 1,
.78, 0/.808} \& \backslash cellcolor[rgb]{ 1, .78, 0/.808} \&
\backslash cellcolor[rgb]{ 1, .78, 0/.808} \& \backslash cellcolor[rgb]{ 1,
.78, 0/.808} \& \backslash cellcolor[rgb]{ 1, .78, 0/.808} \&
\backslash cellcolor[rgb]{ 1, .78, 0/.808} \& \backslash cellcolor[rgb]{ 1,
.78, \dots .808} \quad \backslash
724 \quad \backslash hline
725 \quad 2 \& \backslash cellcolor[rgb]{ 1, .78, 0/.808} \& \backslash cellcolor[rgb]{ 1,
.78, 0/.808} \& \backslash cellcolor[rgb]{ 1, .78, .808}3 \&

```

```

\cellcolor[rgb]{ 1, .78, .808}8 & \cellcolor[rgb]{ 1,
.78, .808}10 & \cellcolor[rgb]{ 1, .78, .808}17 &
\cellcolor[rgb]{ 1, .78, .808}20 & \cellcolor[rgb]{ 1,
.78, ... .808} \\
726 \hline
727 3 & \cellcolor[rgb]{ 1, .78, 0 .808} & \cellcolor[rgb]{ 1,
.78, .808}3 & \cellcolor[rgb]{ 1, .78, .808}7 &
\cellcolor[rgb]{ 1, .78, .808}8 & \cellcolor[rgb]{ 1,
.78, .808}15 & \cellcolor[rgb]{ 1, .78, .808}16 &
\cellcolor[rgb]{ 1, .78, .808}22 & \cellcolor[rgb]{ 1,
.78, ... .808} \\
728 \hline
729 4 & \cellcolor[rgb]{ 1, .78, 0 .808} & \cellcolor[rgb]{ 1,
.78, .808}8 & \cellcolor[rgb]{ 1, .78, .808}8 &
\cellcolor[rgb]{ .776, .937, .808}12 & \cellcolor[rgb]{
.776, .937, .808}18 & \cellcolor[rgb]{ 1, .922, .612}24
& \cellcolor[rgb]{ 1, .922, .612}30 & \cellcolor[rgb]{ 1,
.922, ... .612} \\
730 \hline
731 5 & \cellcolor[rgb]{ 1, .78, 0 .808} & \cellcolor[rgb]{ 1,
.78, .808}8 & \cellcolor[rgb]{ 1, .78, .808}15 & &
\cellcolor[rgb]{ .776, .937, .808}20 & \cellcolor[rgb]{
.776, .937, .808}26 & \cellcolor[rgb]{ 1, .922, .612}32
& \cellcolor[rgb]{ 1, .922, ... .612} \\
732 \hline
733 6 & \cellcolor[rgb]{ 1, .78, 0 .808} & \cellcolor[rgb]{ 1,
.78, .808}14 & \cellcolor[rgb]{ 1, .78, .808}16 & &
& \cellcolor[rgb]{ .776, .937, .808}28 & \cellcolor[rgb]{
.776, .937, .808}34 & \cellcolor[rgb]{ 1, .922, ...
.612} \\
734 \hline
735 7 & \cellcolor[rgb]{ 1, .78, 0 .808} & \cellcolor[rgb]{ 1,
.78, .808}20 & \cellcolor[rgb]{ 1, .78, .808}22 & &
& & \cellcolor[rgb]{ .776, .937, .808}36 &
\cellcolor[rgb]{ .776, .937, ... .808} \\
736 \hline...
737 & \cellcolor[rgb]{ 1, .78, ... .808} & \cellcolor[rgb]{ 1,
.78, ... .808} & \cellcolor[rgb]{ 1, .78, .....808} &
& & & \cellcolor[rgb]{ .776, .937, .....808} \\
738 \hline
739 \end{tabular}%
740
741 \label{tab:4 spaces}
742 \end{table}
743 \[(n+3)(n+3)\ or\ (n+3)^2\ \{n|n\ in\ \mathbb{Z}^+\} \\\]
744 \begin{center}
745 Formulation of  $M=8n-1$ 
746 \begin{align}
747 \label{eqn:4 spaces Ex}
748 &M=12+(n-1)8\text{nonumber} \\
749 &M=12+8n-8\text{nonumber}

```

```

750      &M=8n+4
751      \end{align}
752 \end{center}

753
754 \begin{center}
755   Derivation of $n$
756   \begin{align}
757     \label{eqn:4 spaces N}
758     &P=(n+3)^2 \nonumber \\
759     &\sqrt{P}=\sqrt{(n+3)^2} \nonumber \\
760     &\sqrt{P}=n+3 \nonumber \\
761     &\sqrt{P}-3=n \nonumber \\
762     &n=\sqrt{P}-3
763   \end{align}
764 \end{center}

765
766 Minimum number of moves in terms of the product ($M_P$)
767 \begin{align}
768   \label{eqn:4 spaces MP}
769   &M_P=8(\sqrt{P}-3)+4 \nonumber \\
770   &M_P=8\sqrt{P}-24+4 \nonumber \\
771   &M_P=8\sqrt{P}-20
772 \end{align}
773
774 Since $\sqrt{P}$ is just equal to the number of rows/columns ($s$)
       of the square Klotski, then the minimum number of moves ($M$)
       given the 4 spaces is
775 \begin{align}
776   \label{eqn:4 spaces FF}
777   \alignedbox{ }{M=8s-20}
778 \end{align}
779
780 The following table summarizes the number of spaces and the minimum
       number of moves for square Klotski whose number of sides is
       greater than or equal to the number of spaces:
781 \begin{center}
782   \begin{tabular}{cc}
783     No. of Spaces ($S$) & Minimum no. of moves ($M$) \\
784     \hline
785     1 & $8s-11$ \\
786     2 & $8s-14$ \\
787     3 & $8s-17$ \\
788     4 & $8s-20$ \\
789     \hline
790     \multicolumn{2}{c}{ $M=8s-3S-8$ }
791   \end{tabular}
792 \end{center}
793
794 \begin{center}
795   Formulation of $M=8s-3S-8$
796   \begin{align}
797     \label{eqn:1 -4 spaces square formula}
798     M=&(8s-11)+(S-1)(-3) \nonumber \\
799     M=&8s-11-3S+3 \nonumber \\
800     \alignedbox{M=}{8s-3S-8}
801   \end{align}

```


803 The above-mentioned formula (Eq.
804 $\text{ref}\{eqn:1-4\text{spaces}\text{square}\text{gen}\text{formula}\}$) calculates the minimum
number of moves **in** a square Klotski which has spaces ranging
from 1 to 4, where s is the number of rows or columns and s
is the number of spaces **in** the puzzle.

805 $\backslash\backslash$
806 $\backslash\backslash$
807 $\backslash\text{noindent}$
808 $\backslash\text{textbf}\{\text{Square Klotski with spaces ranging from 5 to }\infty\}$
809 $\backslash\text{paragraph}\{\}$
810 Using the data from the following tables and following the same
process **in** deriving the formula **for** the square Klotski that
contains spaces ranging from 1 to 4, the investigators was able to
derived a general formula that calculates the minimum number of
moves **for** square Klotski that contains spaces ranging from 5 to
infinity.

811 $\backslash\text{begin}\{\text{table}\}\{\text{h}\}$
812 $\backslash\text{centering}$
813 $\backslash\text{caption}\{\text{Data of the Minimum number of moves for Square and}$
814 $\text{Rectangular Klotski with 5 Spaces}\}$
815 $\backslash\text{begin}\{\text{tabular}\}\{\text{c|c|c|c|c|c|c|c|c}\}$
816 $\backslash\text{hline}$
817 $x \ \& \ 4 \ \& \ 5 \ \& \ 6 \ \& \ 7 \ \& \ 8 \ \& \ 9 \ \& \ 10 \ \& \ \dots \quad \backslash\backslash$
818 $\backslash\text{hline}$
819 $4 \ \& \ \text{cellcolor}[rgb]\{1, .78, 0\} \ \& \ \text{cellcolor}[rgb]\{1,$
 $.78, 0\} \ \& \ \text{cellcolor}[rgb]\{1, .78, 0\} \ \&$
 $\text{cellcolor}[rgb]\{1, .78, 0\} \ \& \ \text{cellcolor}[rgb]\{1,$
 $.78, 0\} \ \& \ \text{cellcolor}[rgb]\{1, .78, 0\} \ \&$
 $\text{cellcolor}[rgb]\{1, .78, 0\} \ \& \ \text{cellcolor}[rgb]\{1,$
 $.78, \dots\} \quad \backslash\backslash$
820 $\backslash\text{hline}$
821 $5 \ \& \ \text{cellcolor}[rgb]\{1, .78, 0\} \ \& \ \text{cellcolor}[rgb]\{.776,$
 $.937, .808\} \ \& \ \text{cellcolor}[rgb]\{1, .922, .612\} \ \&$
 $\text{cellcolor}[rgb]\{1, .922, .612\} \ \& \ \text{cellcolor}[rgb]\{1,$
 $.922, .612\} \ \& \ \text{cellcolor}[rgb]\{1, .922, .612\} \ \&$
 $\text{cellcolor}[rgb]\{1, .922, .612\} \ \& \ \text{cellcolor}[rgb]\{1,$
 $.922, \dots, .612\} \quad \backslash\backslash$
822 $\backslash\text{hline}$
823 $6 \ \& \ \text{cellcolor}[rgb]\{1, .78, 0\} \ \& \ \text{cellcolor}[rgb]\{$
 $.776, .937, .808\} \ \& \ \text{cellcolor}[rgb]\{1, .922, .612\} \ \&$
 $\text{cellcolor}[rgb]\{1, .922, .612\} \ \& \ \text{cellcolor}[rgb]\{1,$
 $.922, .612\} \ \& \ \text{cellcolor}[rgb]\{1, .922, .612\} \ \&$
 $\text{cellcolor}[rgb]\{1, .922, \dots, .612\} \quad \backslash\backslash$
824 $\backslash\text{hline}$
825 $7 \ \& \ \text{cellcolor}[rgb]\{1, .78, 0\} \ \& \ \text{cellcolor}[rgb]\{$
 $\text{cellcolor}[rgb]\{.776, .937, .808\} \ \& \ \text{cellcolor}[rgb]\{$

826
827

```
1, .922, .612}41 & \cellcolor[rgb]{ 1, .922, .612}47 &  
\cellcolor[rgb]{ 1, .922, .612}53 & \cellcolor[rgb]{ 1,  
.922, ... .612} \\  
\hline  
8 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & &  
  
\cellcolor[rgb]{ .776, .937, .808}43 & \cellcolor[rgb]{
```

```

1, .922, .612}49 & \cellcolor[rgb]{ 1, .922, .612}55 &
\cellcolor[rgb]{ 1, .922, ... .612} \\
828 \hline
829 9 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & &

\cellcolor[rgb]{ .776, .937, .808}51 & \cellcolor[rgb]{
1, .922, .612}57 & \cellcolor[rgb]{ 1, .922, .....612}
\\
830 \hline
831 10 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & & &

\cellcolor[rgb]{ .776, .937, .808}59 & \cellcolor[rgb]{
1, .922, ... .612} \\
832 \hline...
833 & \cellcolor[rgb]{ 1, .78, ... .808} & & & & &
& \cellcolor[rgb]{ .776, .937, ... .808} \\
834 \hline
835 \end{tabular}%
836
837 \label{ tab :5 spaces }
838 \end{ table }
839
840 \begin{ table }[h]
841 \centering
842 \caption{Data of the Minimum number of moves for Square and
Rectangular Klotski with 6 Spaces}
843 % Table generated by Excel2LaTeX from sheet 'Sheet1 '
844 \ begin { tabular }{|c|c|c|c|c|c|c|c|c|c|}
845 \hline
846 x & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & ... \\
847 \hline
848 5 & \cellcolor[rgb]{ 1, .78, 0 .808} & \cellcolor[rgb]{ 1,
.78, 0/ .808} & \cellcolor[rgb]{ 1, .78, 0 .808} &
\cellcolor[rgb]{ 1, .78, 0 .808} & \cellcolor[rgb]{ 1,
.78, 0/ .808} & \cellcolor[rgb]{ 1, .78, 0 .808} &
\cellcolor[rgb]{ 1, .78, 0 .808} & \cellcolor[rgb]{ 1,
.78, 0 .808} & \cellcolor[rgb]{ 1, .78, ... .808} \\
849 \hline
850 6 & \cellcolor[rgb]{ 1, .78, 0 .808} & \cellcolor[rgb]{ .776,
.937, .808}26 & \cellcolor[rgb]{ 1, .922, .612}32 &
\cellcolor[rgb]{ 1, .922, .612}38 & \cellcolor[rgb]{ 1,
.922, .612}44 & \cellcolor[rgb]{ 1, .922, .612}50 &
\cellcolor[rgb]{ 1, .922, .612}56 & \cellcolor[rgb]{ 1,
.922, .612}62 & \cellcolor[rgb]{ 1, .922, ... .612} \\
851 \hline
852 7 & \cellcolor[rgb]{ 1, .78, 0 .808} & & \cellcolor[rgb]{
.776, .937, .808}34 & \cellcolor[rgb]{ 1, .922, .612}40

```

853
854

```
& \cellcolor[rgb]{ 1, .922, .612}46 & \cellcolor[rgb]{ 1,
.922, .612}52 & \cellcolor[rgb]{ 1, .922, .612}58 &
\cellcolor[rgb]{ 1, .922, .612}64 & \cellcolor[rgb]{ 1,
.922, ... .612} \\
\hline
8 & \cellcolor[rgb]{ 1, .78, 0 .808} & & &

\cellcolor[rgb]{ .776, .937, .808}42 & \cellcolor[rgb]{
1, .922, .612}48 & \cellcolor[rgb]{ 1, .922, .612}54 &
\cellcolor[rgb]{ 1, .922, .612}60 & \cellcolor[rgb]{ 1,
```

```

855         .922, .612}66 & \cellcolor[rgb]{ 1, .922, ... .612} \\
856 \hline
9 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & &
\\
\cellcolor[rgb]{ .776, .937, .808}50 & \cellcolor[rgb]{
1, .922, .612}56 & \cellcolor[rgb]{ 1, .922, .612}62 &
\cellcolor[rgb]{ 1, .922, .612}68 & \cellcolor[rgb]{ 1,
.922, ... .612} \\
857 \hline
858 10 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & &
\\
\cellcolor[rgb]{ .776, .937, .808}58 & \cellcolor[rgb]{
1, .922, .612}64 & \cellcolor[rgb]{ 1, .922, .612}70 &
\cellcolor[rgb]{ 1, .922, ... .612} \\
859 \hline
860 11 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & &
\\
\cellcolor[rgb]{ .776, .937, .808}66 & \cellcolor[rgb]{
1, .922, .612}72 & \cellcolor[rgb]{ 1, .922, .....612}
\\
861 \hline
862 12 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & &
\\
& \cellcolor[rgb]{ .776, .937, .808}74 & \cellcolor[rgb]{
1, .922, ... .612} \\
863 \hline...
864 & \cellcolor[rgb]{ 1, .78, ... .808} & & & &
& & \cellcolor[rgb]{ .776, .937, ... .808} \\
865 \hline
866 \end{tabular}%
867
868
869 \label{ tab :6 spaces }
870 \end{table}
871 \begin{table}[h]
872 \centering
873 \caption{Data of the Minimum number of moves for Square and
Rectangular Klotski with 7 Spaces}
874 % Table generated by Excel2LaTeX from sheet 'Sheet1'
875 \begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
876 \hline
877 x & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & ... \\
878 \hline
879 6 & \cellcolor[rgb]{ 1, .78, 0/ .808} & \cellcolor[rgb]{ 1,
.78, 0/ .808} & \cellcolor[rgb]{ 1, .78, 0/ .808} &
\cellcolor[rgb]{ 1, .78, 0/ .808} & \cellcolor[rgb]{ 1,
.78, 0/ .808} & \cellcolor[rgb]{ 1, .78, 0/ .808} &
\cellcolor[rgb]{ 1, .78, 0/ .808} & \cellcolor[rgb]{ 1,
.78, 0/ .808} & \cellcolor[rgb]{ 1, .78, ... .808} \\
880 \hline
881 7 & \cellcolor[rgb]{ 1, .78, 0/ .808} & \cellcolor[rgb]{ .776,

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```

.937, .808}33 & \cellcolor[rgb]{ 1, .922, .612}39 &
\cellcolor[rgb]{ 1, .922, .612}45 & \cellcolor[rgb]{ 1,
.922, .612}51 & \cellcolor[rgb]{ 1, .922, .612}57 &
\cellcolor[rgb]{ 1, .922, .612}63 & \cellcolor[rgb]{ 1,
.922, .612}69 & \cellcolor[rgb]{ 1, .922, ... .612} \\
\hline

```

```

883      8 & \cellcolor[rgb]{ 1, .78, 0 .808} & & \cellcolor[rgb]{
      .776, .937, .808}41 & \cellcolor[rgb]{ 1, .922, .612}47
      & \cellcolor[rgb]{ 1, .922, .612}53 & \cellcolor[rgb]{ 1,
      .922, .612}59 & \cellcolor[rgb]{ 1, .922, .612}65 &
      \cellcolor[rgb]{ 1, .922, .612}71 & \cellcolor[rgb]{ 1,
      .922, ... .612} \\
884 \hline
885      9 & \cellcolor[rgb]{ 1, .78, 0 .808} & & &
      \cellcolor[rgb]{ .776, .937, .808}49 & \cellcolor[rgb]{
      1, .922, .612}55 & \cellcolor[rgb]{ 1, .922, .612}61 &
      \cellcolor[rgb]{ 1, .922, .612}67 & \cellcolor[rgb]{ 1,
      .922, .612}73 & \cellcolor[rgb]{ 1, .922, ... .612} \\
886 \hline
887      10 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & &
      \cellcolor[rgb]{ .776, .937, .808}57 & \cellcolor[rgb]{
      1, .922, .612}63 & \cellcolor[rgb]{ 1, .922, .612}69 &
      \cellcolor[rgb]{ 1, .922, .612}75 & \cellcolor[rgb]{ 1,
      .922, ... .612} \\
888 \hline
889      11 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & & &
      \cellcolor[rgb]{ .776, .937, .808}65 & \cellcolor[rgb]{
      1, .922, .612}71 & \cellcolor[rgb]{ 1, .922, .612}77 &
      \cellcolor[rgb]{ 1, .922, ... .612} \\
890 \hline
891      12 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & & &
      \cellcolor[rgb]{ .776, .937, .808}73 & \cellcolor[rgb]{
      1, .922, .612}79 & \cellcolor[rgb]{ 1, .922, .....612}
      \\
892 \hline
893      13 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & & &
      & \cellcolor[rgb]{ .776, .937, .808}81 & \cellcolor[rgb]{
      1, .922, ... .612} \\
894 \hline...
895      & \cellcolor[rgb]{ 1, .78, ... .808} & & & & &
      & & \cellcolor[rgb]{ .776, .937, ... .808} \\
896 \hline
897 \end{tabular}%
898
899
900 \label{tab:7 spaces}
901 \end{table}
902
903 \begin{table}[h]
904 \centering
905 \caption{Data of the Minimum number of moves for Square and
      Rectangular Klotski with 8 Spaces}
906 % Table generated by Excel2LaTeX from sheet 'Sheet1'
907 \begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
908 \hline

```

```
909 x & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & ... \\
910 \hline
911 7 & \cellcolor[rgb]{ 1, .78, 0/.808} & \cellcolor[rgb]{ 1,
    .78, 0/.808} & \cellcolor[rgb]{ 1, .78, 0/.808} &
    \cellcolor[rgb]{ 1, .78, 0/.808} & \cellcolor[rgb]{ 1,
```

```

.78, 0/.808} & \cellcolor[rgb]{ 1, .78, 0 .808} &

\cellcolor[rgb]{ 1, .78, 0 .808} & \cellcolor[rgb]{ 1,

.78, 0 .808} & \cellcolor[rgb]{ 1, .78, ... .808} \\
912 \hline

8 & \cellcolor[rgb]{ 1, .78, 0 .808} & \cellcolor[rgb]{ .776,
913 .937, .808}40 & \cellcolor[rgb]{ 1, .922, .612}46 &
\cellcolor[rgb]{ 1, .922, .612}52 & \cellcolor[rgb]{ 1,
.922, .612}58 & \cellcolor[rgb]{ 1, .922, .612}64 &
\cellcolor[rgb]{ 1, .922, .612}70 & \cellcolor[rgb]{ 1,
.922, .612}76 & \cellcolor[rgb]{ 1, .922, ... .612} \\
914 \hline
915 9 & \cellcolor[rgb]{ 1, .78, 0 .808} & & \cellcolor[rgb]{

.776, .937, .808}48 & \cellcolor[rgb]{ 1, .922, .612}54
& \cellcolor[rgb]{ 1, .922, .612}60 & \cellcolor[rgb]{ 1,
.922, .612}66 & \cellcolor[rgb]{ 1, .922, .612}72 &
\cellcolor[rgb]{ 1, .922, .612}78 & \cellcolor[rgb]{ 1,
.922, ... .612} \\
916 \hline
917 10 & \cellcolor[rgb]{ 1, .78, 0 .808} & & &

\cellcolor[rgb]{ .776, .937, .808}56 & \cellcolor[rgb]{
1, .922, .612}62 & \cellcolor[rgb]{ 1, .922, .612}68 &
\cellcolor[rgb]{ 1, .922, .612}74 & \cellcolor[rgb]{ 1,
.922, .612}80 & \cellcolor[rgb]{ 1, .922, ... .612} \\
918 \hline
919 11 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & &

\cellcolor[rgb]{ .776, .937, .808}64 & \cellcolor[rgb]{
1, .922, .612}70 & \cellcolor[rgb]{ 1, .922, .612}76 &
\cellcolor[rgb]{ 1, .922, .612}82 & \cellcolor[rgb]{ 1,
.922, ... .612} \\
920 \hline
921 12 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & &

\cellcolor[rgb]{ .776, .937, .808}72 & \cellcolor[rgb]{
1, .922, .612}78 & \cellcolor[rgb]{ 1, .922, .612}84 &
\cellcolor[rgb]{ 1, .922, ... .612} \\
922 \hline
923 13 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & & &

\cellcolor[rgb]{ .776, .937, .808}80 & \cellcolor[rgb]{
1, .922, .612}86 & \cellcolor[rgb]{ 1, .922, .....612}
\\
924 \hline
925 14 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & & &

& \cellcolor[rgb]{ .776, .937, .808}88 & \cellcolor[rgb]{
1, .922, ... .612} \\
926 \hline...
```

```

927          & \cellcolor[rgb]{ 1, .78, ... .808} & & & & &
          & & \cellcolor[rgb]{ .776, .937, ... .808} \\
928      \hline
929      \end{tabular}%
930
931      \label{tab :8 spaces }
932  \end{ table }
933
934  \begin{ table }[h]
935      \centering

```

```

936 \caption{Data of the Minimum number of moves for Square and
      Rectangular Klotski with 9 Spaces}
937 % Table generated by Excel2LaTeX from sheet 'Sheet1'
938 \begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
939 \hline
940 x & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \dots \\
941 \hline
942 8 & \cellcolor[rgb]{1, .78, 0 .808} & \cellcolor[rgb]{1,
      .78, 0/ .808} & \cellcolor[rgb]{1, .78, 0 .808} &
      \cellcolor[rgb]{1, .78, 0 .808} & \cellcolor[rgb]{1,
      .78, 0/ .808} & \cellcolor[rgb]{1, .78, 0 .808} &
      \cellcolor[rgb]{1, .78, 0 .808} & \cellcolor[rgb]{1,
      .78, 0/ .808} & \cellcolor[rgb]{1, .78, \dots .808} \\
943 \hline
944 9 & \cellcolor[rgb]{1, .78, 0 .808} & \cellcolor[rgb]{.776,
      .937, .808} & 47 & \cellcolor[rgb]{1, .922, .612} & 53 &
      \cellcolor[rgb]{1, .922, .612} & 59 & \cellcolor[rgb]{1,
      .922, .612} & 65 & \cellcolor[rgb]{1, .922, .612} & 71 &
      \cellcolor[rgb]{1, .922, .612} & 77 & \cellcolor[rgb]{1,
      .922, .612} & 83 & \cellcolor[rgb]{1, .922, \dots .612} \\
945 \hline
946 10 & \cellcolor[rgb]{1, .78, 0 .808} & & \cellcolor[rgb]{
      .776, .937, .808} & 55 & \cellcolor[rgb]{1, .922, .612} & 61
      & \cellcolor[rgb]{1, .922, .612} & 67 & \cellcolor[rgb]{1,
      .922, .612} & 73 & \cellcolor[rgb]{1, .922, .612} & 79 &
      \cellcolor[rgb]{1, .922, .612} & 85 & \cellcolor[rgb]{1,
      .922, \dots .612} \\
947 \hline
948 11 & \cellcolor[rgb]{1, .78, 0 .808} & & & &
      \cellcolor[rgb]{.776, .937, .808} & 63 & \cellcolor[rgb]{
      1, .922, .612} & 69 & \cellcolor[rgb]{1, .922, .612} & 75 &
      \cellcolor[rgb]{1, .922, .612} & 81 & \cellcolor[rgb]{1,
      .922, .612} & 87 & \cellcolor[rgb]{1, .922, \dots .612} \\
949 \hline
950 12 & \cellcolor[rgb]{1, .78, 0 .808} & & & &
      \cellcolor[rgb]{.776, .937, .808} & 71 & \cellcolor[rgb]{
      1, .922, .612} & 77 & \cellcolor[rgb]{1, .922, .612} & 83 &
      \cellcolor[rgb]{1, .922, .612} & 89 & \cellcolor[rgb]{1,
      .922, \dots .612} \\
951 \hline
952 13 & \cellcolor[rgb]{1, .78, 0 .808} & & & &
      \cellcolor[rgb]{.776, .937, .808} & 79 & \cellcolor[rgb]{
      1, .922, .612} & 85 & \cellcolor[rgb]{1, .922, .612} & 91 &
      \cellcolor[rgb]{1, .922, \dots .612} \\

```

```

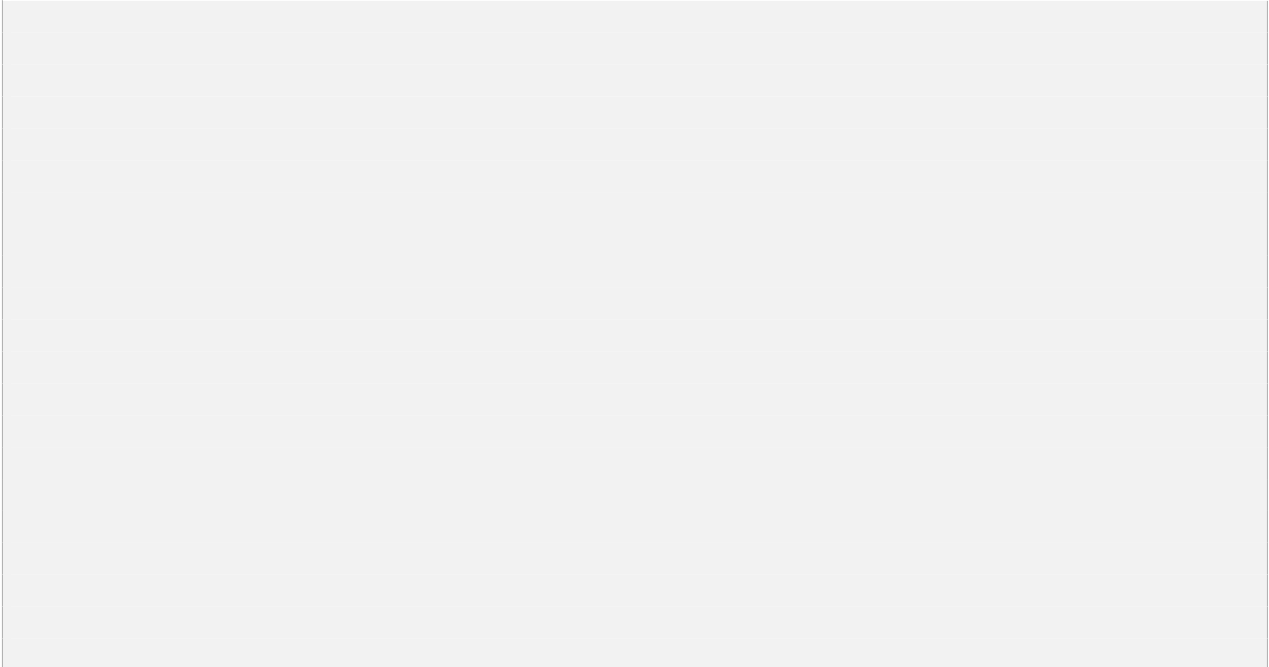
953 \hline
954 14 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & & &
      \cellcolor[rgb]{ .776, .937, .808}87 & \cellcolor[rgb]{
      1, .922, .612}93 & \cellcolor[rgb]{ 1, .922,.....612}
      \\
955 \hline
956 15 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & & &
      & \cellcolor[rgb]{ .776, .937, .808}95 & \cellcolor[rgb]{
      1, .922, ... .612} \\
957 \hline...
```

```

958      & \cellcolor[rgb]{ 1, .78, ... .808} & & & & & &
959      & & \cellcolor[rgb]{ .776, .937, ... .808} \\
960      \hline
961      \end{tabular}%
962
963      \label{tab:9 spaces}
964      \end{table}
965
966      \begin{table}[h]
967      \centering
968      \caption{Data of the Minimum number of moves for Square and
969              Rectangular Klotski with 10 Spaces}
970      % Table generated by Excel2LaTeX from sheet 'Sheet1'
971      \begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
972      \hline
973      x & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & ... \\
974      \hline
975      9 & \cellcolor[rgb]{ 1, .78, 0/.808} & \cellcolor[rgb]{ 1,
976      .78, 0/.808} & \cellcolor[rgb]{ 1, .78, 0/.808} & &
977      \cellcolor[rgb]{ 1, .78, 0/.808} & \cellcolor[rgb]{ 1,
978      .78, 0/.808} & \cellcolor[rgb]{ 1, .78, 0/.808} & &
979      \hline
980      10 & \cellcolor[rgb]{ 1, .78, 0/.808} & \cellcolor[rgb]{ .776,
981      .937, .808}54 & \cellcolor[rgb]{ 1, .922, .612}60 & &
982      \cellcolor[rgb]{ 1, .922, .612}66 & \cellcolor[rgb]{ 1,
983      .922, .612}72 & \cellcolor[rgb]{ 1, .922, .612}78 & &
984      \cellcolor[rgb]{ 1, .922, .612}84 & \cellcolor[rgb]{ 1,
985      .922, .612}90 & \cellcolor[rgb]{ 1, .922, ... .612} \\
986      \hline
987      11 & \cellcolor[rgb]{ 1, .78, 0/.808} & & \cellcolor[rgb]{
988      .776, .937, .808}62 & \cellcolor[rgb]{ 1, .922, .612}68
989      & \cellcolor[rgb]{ 1, .922, .612}74 & \cellcolor[rgb]{ 1,
990      .922, .612}80 & \cellcolor[rgb]{ 1, .922, .612}86 & &
991      \cellcolor[rgb]{ 1, .922, .612}92 & \cellcolor[rgb]{ 1,
992      .922, ... .612} \\
993      \hline
994      12 & \cellcolor[rgb]{ 1, .78, 0/.808} & & & &
995      \cellcolor[rgb]{ .776, .937, .808}70 & \cellcolor[rgb]{
996      1, .922, .612}76 & \cellcolor[rgb]{ 1, .922, .612}82 & &
997      \cellcolor[rgb]{ 1, .922, .612}88 & \cellcolor[rgb]{ 1,
998      .922, .612}94 & \cellcolor[rgb]{ 1, .922, ... .612} \\
999      \hline
1000      13 & \cellcolor[rgb]{ 1, .78, 0/.808} & & & &
1001      \cellcolor[rgb]{ .776, .937, .808}78 & \cellcolor[rgb]{
1002      1, .922, .612}84 & \cellcolor[rgb]{ 1, .922, .612}90 & &
1003      \cellcolor[rgb]{ 1, .922, .612}96 & \cellcolor[rgb]{ 1,
1004      .922, ... .612} \\
1005      \hline
1006      14 & \cellcolor[rgb]{ 1, .78, 0/.808} & & & &

```

\cellcolor[rgb]{.776,.937,.808}86 & \cellcolor[rgb]{
1,.922,.612}92 & \cellcolor[rgb]{1,.922,.612}98 &



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985         \cellcolor[rgb]{ 1, .922, ... .612} \\
986     \hline
15 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & & &
        \cellcolor[rgb]{ .776, .937, .808}94 & \cellcolor[rgb]{
1, .922, .612}100 & \cellcolor[rgb]{ 1, .922, .....612}
        \\
987     \hline
988     16 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & & &
        & \cellcolor[rgb]{ .776, .937, .808}102 &
        \cellcolor[rgb]{ 1, .922, ... .612} \\
989     \hline...
990     & \cellcolor[rgb]{ 1, .78, ... .808} & & & & &
        & & \cellcolor[rgb]{ .776, .937, ... .808} \\
991     \hline
992     \end{tabular}%
993
994
995     \label{tab:10 spaces}
996 \end{table}
997
998 \begin{table}[h!]
999     \centering
1000     \caption{Data of the Minimum number of moves for Square and
        Rectangular Klotski with 11 Spaces}
1001     % Table generated by Excel2LaTeX from sheet 'Sheet1'
1002     \begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
1003         \hline
1004         x & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & ... \\
1005         \hline
1006         10 & \cellcolor[rgb]{ 1, .78, 0 .808} & \cellcolor[rgb]{ 1,
        .78, 0/ .808} & \cellcolor[rgb]{ 1, .78, 0 .808} &
        \cellcolor[rgb]{ 1, .78, 0 .808} & \cellcolor[rgb]{ 1,
        .78, 0/ .808} & \cellcolor[rgb]{ 1, .78, 0 .808} &
        \cellcolor[rgb]{ 1, .78, 0 .808} & \cellcolor[rgb]{ 1,
        .78, 0/ .808} & \cellcolor[rgb]{ 1, .78, ... .808} \\
1007     \hline
1008     11 & \cellcolor[rgb]{ 1, .78, 0 .808} & \cellcolor[rgb]{ .776,
        .937, .808}61 & \cellcolor[rgb]{ 1, .922, .612}67 &
        \cellcolor[rgb]{ 1, .922, .612}73 & \cellcolor[rgb]{ 1,
        .922, .612}79 & \cellcolor[rgb]{ 1, .922, .612}85 &
        \cellcolor[rgb]{ 1, .922, .612}91 & \cellcolor[rgb]{ 1,
        .922, .612}97 & \cellcolor[rgb]{ 1, .922, ... .612} \\
1009     \hline
1010     12 & \cellcolor[rgb]{ 1, .78, 0 .808} & & \cellcolor[rgb]{

```

```

.776, .937, .808}69 & \cellcolor[rgb]{ 1, .922, .612}75
& \cellcolor[rgb]{ 1, .922, .612}81 & \cellcolor[rgb]{ 1,
.922, .612}87 & \cellcolor[rgb]{ 1, .922, .612}93 &
\cellcolor[rgb]{ 1, .922, .612}99 & \cellcolor[rgb]{ 1,
.922, .612}... \\

```

1011

```
\hline
```

1012

```
13 & \cellcolor[rgb]{ 1, .78, 0 .808} & & &
```

```

\cellcolor[rgb]{ .776, .937, .808}77 & \cellcolor[rgb]{
1, .922, .612}83 & \cellcolor[rgb]{ 1, .922, .612}89 &
\cellcolor[rgb]{ 1, .922, .612}95 & \cellcolor[rgb]{ 1,

```

```

1013         .922, .612}101 & \cellcolor[rgb]{ 1, .922, ... .612} \\
\hline
1014     14 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & & &
        \cellcolor[rgb]{ .776, .937, .808}85 & \cellcolor[rgb]{
1, .922, .612}91 & \cellcolor[rgb]{ 1, .922, .612}97 &
\cellcolor[rgb]{ 1, .922, .612}103 & \cellcolor[rgb]{ 1,
.922, ... .612} \\
1015 \hline
1016     15 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & & &
        \cellcolor[rgb]{ .776, .937, .808}93 & \cellcolor[rgb]{
1, .922, .612}99 & \cellcolor[rgb]{ 1, .922, .612}105 &
\cellcolor[rgb]{ 1, .922, ... .612} \\
1017 \hline
1018     16 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & & &
        \cellcolor[rgb]{ .776, .937, .808}101 & \cellcolor[rgb]{
1, .922, .612}107 & \cellcolor[rgb]{ 1, .922, .....612}
\\
1019 \hline
1020     17 & \cellcolor[rgb]{ 1, .78, 0 .808} & & & & &
        & \cellcolor[rgb]{ .776, .937, .808}109 &
\cellcolor[rgb]{ 1, .922, ... .612} \\
1021 \hline...
1022     & \cellcolor[rgb]{ 1, .78, ... .808} & & & & &
        & & \cellcolor[rgb]{ .776, .937, ... .808} \\
1023 \hline
1024 \end{tabular}%
1025
1026
1027 \label{tab:11 spaces}
1028 \end{table}
1029 \newpage
1030 \begin{center}
1031 \begin{tabular}{cc}
1032     No. of Spaces ($$$) & Minimum no. of moves ($M$) \\
1033 \hline
1034     5 & $8s-21$ \\
1035     6 & $8s-22$ \\
1036     7 & $8s-23$ \\
1037     8 & $8s-24$ \\
1038     9 & $8s-25$ \\
1039     10 & $8s-26$ \\
1040     11 & $8s-27$ \\
1041     ... & ... \\
1042 \hline
1043     & $M=8s-5-16$

```

```

1044 \end{tabular}
1045 \end{center}
1046
1047 Formulation of  $M=8s-S-16$ 
1048 \begin{align}
1049 \quad \text{\label{eqn: gen formula square 8s-S-16}}
1050 \quad M=&(8s-21)+(S-5)(-1) \backslash \text{nonumber} \quad \backslash\backslash
1051 \quad M=&8s-21-S+5 \backslash \text{nonumber} \quad \backslash\backslash
1052 \quad \text{\alignedbox{M= }}&8s-S-16 \quad \backslash\backslash
\end{align}

1053
1054
1055 The above-mention formula (Eq. \ref{eqn: gen formula square
8s-S-16}) calculates the minimum number of moves for a square
Klotski given the number of rows or columns ( $s$ ) and the
number of spaces ( $S$ ) and  $4 \leq S \leq \infty$ 

1056 \backslash\backslash
1057 \backslash\backslash
1058 \textbf{Formula for the Minimum Number of Moves of a Rectangular
Klotski given the number of spaces;}

1059 \backslash\backslash
1060 \backslash\backslash
1061 \textbf{2 Spaces}
1062
1063 Using the data in table \ref{tab:2spaces} in formulating the
formula for the minimum number of moves for rectangular Klotski
containing 2 spaces, the investigators was able to derived the
formula through the same process as equation \ref{eqn:1 Spaces
Rec Explicit}, \ref{eqn:1 spaces Rec N}, \ref{eqn:1 spaces rec
quad }, \ref{eqn:1 spaces rec MP}, \ref{eqn:1 spaces rec k},
\ref{eqn:1 spaces rec MPD}, and \ref{eqn:1 spaces rec finalf}.

1064
1065 \begin{center}
1066 \begin{tabular}{cc}
1067 \multicolumn{2}{c}{ $\{n|n \text{ in } \mathbb{Z}^+\} \backslash$ 
 $\{k|k \text{ in } \mathbb{Z}^+, k > 1\}$  } \backslash\backslash
1068 & \backslash\backslash
1069 Product ( $P$ ) & Minimum no. of moves ( $M$ ) \backslash\backslash
1070 \hline
1071 \cellcolor{RED}  $(n+1)(n+2)$  & \cellcolor{RED}  $8n-2$  \backslash\backslash
1072  $(n+1)(n+3)$  &  $8n+2$  \backslash\backslash
1073  $(n+1)(n+4)$  &  $8n+8$  \backslash\backslash
1074  $(n+1)(n+5)$  &  $8n+14$  \backslash\backslash
1075  $(n+1)(n+6)$  &  $8n+20$  \backslash\backslash
1076  $(n+1)(n+7)$  &  $8n+26$  \backslash\backslash
1077 ... & ... \backslash\backslash
1078 \hline
1079  $P=(n+1)(n+k)$  &
1080 \end{tabular}
1081 \end{center}
1082
1083 In the above-shown table, the row that is highlighted in red has a
separated formula. Hence, only calculates the minimum number of
moves in a rectangular Klotski that has the dimension of
 $(n+1)(n+2)$ . This is due to the constant of the explicit
formula  $8n-2$  since it does not coincide with the common
difference of  $6$  if it is included in the sequence.

```

```

1084      \\
1085      \\
1086      \textbf{Formulation of the Formula of $n-2$}
1087      \begin{align}
1088          \label{eqn:2spaces Exception Derivation of n}
1089          P=&(n+1)(n+k) \nonumber \\
1090          P=&n^2+kn+n+k \nonumber

```

```

1091      P=&n^2+(k+1)n+k\ nonumber \\\
1092      0=&n^2+(k+1)n+k-P
1093 \end{align}
1094
1095 using the quadratic equation to find $n$
1096 \begin{align}
1097   \label{eqn:2spaces Exception quadratic formula to find n}
1098   n=&\frac{-(k+1)\pm \sqrt{(k+1)^2+4(1)(k-P)}}{2(1)}\ nonumber \\\
1099   n=&\frac{-k-1\pm \sqrt{k^2+2k+1-4k+4P}}{2}\ nonumber \\\
1100   n=&\frac{-k-1\pm \sqrt{k^2-2k+1+4P}}{2}\ nonumber \\\
1101   n=&\frac{-k-1\pm \sqrt{(k-1)^2+4P}}{2}
1102 \end{align}
1103
1104
1105 Minimum number of moves in terms of the product ($M_P$)
1106 \begin{align}
1107   \label{eqn:2spaces Exception Minimum number of moves in terms of
1108     the product}
1109   M_P=&8(\frac{-k-1\pm \sqrt{(k-1)^2+4P}}{2})-2\ nonumber \\\
1110   M_P=&4(-k-1\pm \sqrt{(k-1)^2+4P})-2\ nonumber \\\
1111   M_P=&-4k-4\pm 4\sqrt{(k-1)^2+4P}-2\ nonumber \\\
1112   M_P=&4\sqrt{(k-1)^2+4P}-4k-6
1113 \end{align}
1114
1115 since, $k=2$. then,
1116 \begin{align}
1117   \label{eqn:2spaces Exception k=2}
1118   M_P=&4\sqrt{(k-1)^2+4P}-4k-6\ nonumber \\\
1119   M_P=&4\sqrt{(2-1)^2+4P}-4(2)-6\ nonumber \\\
1120   M_P=&4\sqrt{(1)^2+4P}-8-6\ nonumber \\\
1121   \alignedbox{M_P=}{4\sqrt{1+4P}-14}{ }
1122 \end{align}
1123
1124 Furthermore, if we let $l$ and $w$ the number of columns and rows
1125 and $l>w$, then
1126 \begin{align}
1127   M=&4\sqrt{1+4P}-14\ nonumber \\\
1128   M=&4\sqrt{1+4(l)(w)}-14\ nonumber \\\
1129   \alignedbox{M=}{4\sqrt{1+4lw}}{-14}
1130 \end{align}
1131
1132 \textbf{Derivation of the formula for the sequence starting from}
1133   $8n+2$;
1134   $[(n+1)(n+k)]$ in $\mathbb{Z}^+$, $\{k|k$ in $\mathbb{Z}^+, $
1135   $k>2\}$
1136
1137 The following table summarizes the product and the minimum number
1138 of moves for a $(n+1)(n+k)$ Klotski, where $k>2$:
1139 \begin{center}
1140   \begin{tabular}{cc}
1141     Product ($P$) & Minimum no. of moves ($M$) \\\
1142     \hline
1143     $(n+1)(n+3)$ & $8n+2$ \\\
1144     $(n+1)(n+4)$ & $8n+8$ \\\

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1140      $(n+1)(n+5)$ & $8n+14$ \\
1141      $(n+1)(n+6)$ & $8n+20$ \\
1142      $(n+1)(n+7)$ & $8n+26$ \\
1143      ... & ... \\
1144      \hline
1145      $P=(n+1)(n+k)$ & $M=8n+6k-16$
1146    \end{tabular}
1147  \end{center}
1148
1149  Formulation of $M=8n+6k-16$
1150  \begin{align}
1151    \label{eqn:2 spaces 8n+6k-16}
1152    M=&(8n+2)+(k-3)6 \nonumber \\
1153    M=&8n+2+6k-18 \nonumber \\
1154    M=&8n+6k-16
1155  \end{align}
1156
1157  using the quadratic equation to find $n$ (See Eq. \ref{eqn:2 spaces
    Exception quadratic formula to find $n$ for the solution)
1158  \begin{align}
1159    n=&\frac{-k-1 \pm \sqrt{(k-1)^2+4P}}{2} \nonumber
1160  \end{align}
1161
1162  Minimum number of moves in terms of the product ($M_P$)
1163  \begin{align}
1164    \label{eqn:2 spaces Minimum number of moves in terms of the
      product 8n+6k-16}
1165    M_P=&8\left(\frac{-k-1 \pm \sqrt{(k-1)^2+4P}}{2}\right)+6k-16 \nonumber \\
1166    M_P=&4(-k-1 \pm \sqrt{(k-1)^2+4P})+6k-16 \nonumber \\
1167    M_P=&-4k-4 \pm 4\sqrt{(k-1)^2+4P}+6k-16 \nonumber \\
1168    M_P=&4\sqrt{(k-1)^2+4P}+2k-20
1169  \end{align}
1170
1171  the difference of the two sides $+1$ (See Eq. \ref{eqn:1 spaces rec
    k} for the solution),
1172  \begin{align}
1173    k=&d+1 \nonumber
1174  \end{align}
1175
1176  hence the minimum number of moves given the product and difference
    ($M_{Pd}$) is
1177  \begin{align}
1178    \label{eqn:2 spaces hence the minimum number of moves given the
      product and difference ($M_{Pd}$) is rec 8n+6k-16}
1179    M_{Pd}=&4\sqrt{((d+1)-1)^2+4P}+2(d+1)-20 \nonumber \\
1180    M_{Pd}=&4\sqrt{(d+1-1)^2+4P}+2d+2-20 \nonumber \\
1181    M_{Pd}=&4\sqrt{(d)^2+4P}+2d-18 \nonumber \\
1182    \alignedbox{M_{Pd}}{=}{4\sqrt{d^2+4P}+2d-18}
1183  \end{align}
1184
1185  Furthermore, if we let $l$ and $w$ be the number of columns and
    rows and $l>w$, then the minimum number of moves of a
    rectangular Klotski ($M$) is
1186  \begin{align}

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1187 \label{eqn:2spaces 8n+6k-16 rec finalf lw}
1188 M_{Pd}=&4 \sqrt{d^2+4P}+2d-18 \nonumber \\
1189 M=&4 \sqrt{(l-w)^2+4(l)(w)}+2(l-w)-18 \nonumber \\
1190 M=&4 \sqrt{l^2-2lw+w^2+4lw}+2l-2w-18 \nonumber \\
1191 M=&4 \sqrt{l^2+2lw+w^2}+2l-2w-18 \nonumber \\
1192 M=&4 \sqrt{(l+w)^2}+2l-2w-18 \nonumber \\
1193 M=&4(l+w)+2l-2w-18 \nonumber \\
1194 M=&4l+4w+2l-2w-18 \nonumber \\
1195 \alignedbox {M =}{6l+2w-18}
1196 \end{align}
1197 \\
1198 \textbf{3 Spaces}
1199
1200 Using the data in table \ref{tab:3spaces} and following the same
1201 solutions , here are the derived formulae;
1202
1203 \[(n+1)(n+k) \ \{n|n \text{ in } \mathbb{Z}^+\} \ \{k|k \text{ in } \mathbb{Z}^+, \\\
1204 k>2\} \]
1205
1206 The following table summarizes the product and the minimum number
1207 of moves in a  $(n+2)(n+k)$  Klotski:
1208 \begin{center}
1209 \begin{tabular}{cc}
1210 Product ( $P$ ) & Minimum no. of moves ( $M$ ) \\
1211 \hline
1212  $(n+2)(n+3)$  &  $8n+3$  \\
1213  $(n+2)(n+4)$  &  $8n+9$  \\
1214  $(n+2)(n+5)$  &  $8n+15$  \\
1215  $(n+2)(n+6)$  &  $8n+21$  \\
1216  $(n+2)(n+7)$  &  $8n+27$  \\
1217  $(n+2)(n+8)$  &  $8n+33$  \\
1218 ... & ... \\
1219 \hline
1220  $P=(n+2)(n+k)$  &  $M=8n+6k-15$ 
1221 \end{tabular}
1222 \end{center}
1223 Formulation of  $M=8n+6k-15$ 
1224 \begin{align}
1225 \label{eqn:3spaces rec Formulation of $M=8n+6k-15$}
1226 M =& 8n+(3+[k-3]6) \nonumber \\
1227 M =& 8n+(3+6k-18) \nonumber \\
1228 M =& 8n+3+6k-18 \nonumber \\
1229 M =& 8n+6k-15
1230 \end{align}
1231 \\
1232 Derivation of  $n$ 
1233 \begin{align}
1234 \label{eqn:3spaces rec Derivation of $n$}
1235 P=&(n+2)(n+k) \nonumber \\
1236 P=&n^2+kn+2n+2k \nonumber \\
1237 P=&n^2+(k+2)n+2k \nonumber \\
1238 0=&n^2+(k+2)n+2k-P
1239 \end{align}
1240 \\

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1238 using the quadratic equation to find $n$
1239 \begin{align}
1240   \label{eqn:3spaces rec using the quadratic equation to find $n$}
1241   n=&\frac{-(k+2)\pm \sqrt{(k+2)^2+4(1)(2k-P)}}{2}\nonumber \\
1242   n=&\frac{-k-2\pm \sqrt{k^2+4k+4-8k+4P}}{2}\nonumber \\
1243   n=&\frac{-k-2\pm \sqrt{k^2-4k+4+4P}}{2}\nonumber \\
1244   n=&\frac{-k-2\pm \sqrt{(k-2)^2+4P}}{2}
1245 \end{align}
1246 \\
1247 Minimum number of moves in terms of the product ($M_P$)
1248 \begin{align}
1249   \label{eqn:3spaces rec Minimum number of moves in terms of the
1250     product ($M_P$)}
1251   M_P=&8\frac{-k-2\pm \sqrt{(k-2)^2+4P}}{2}+6k-15\nonumber \\
1252   M_P=&4(-k-2\pm \sqrt{(k-2)^2+4P})+6k-15\nonumber \\
1253   M_P=&-4k-8\pm 4\sqrt{(k-2)^2+4P}+6k-15\nonumber \\
1254   M_P=&4\sqrt{(k-2)^2+4P}+2k-23
1255 \end{align}
1256 \\
1257 the difference of the two sides $+2$
1258 \begin{align}
1259   \label{eqn:3spaces rec the difference of the two sides $+2$}
1260   d=&(n+k)-(n+2)\nonumber \\
1261   d=&n+k-n-2\nonumber \\
1262   d=&k-2\nonumber \\
1263   k=&d+2
1264 \end{align}
1265 \\
1266 hence the minimum number of moves given the product and difference
1267 ($M_{Pd}$) is
1268 \begin{align}
1269   \label{eqn:3spaces rec hence the minimum number of moves given
1270     the product and difference ($M_{Pd}$) is}
1271   M_{Pd}=&4\sqrt{((d+2)-2)^2+4P}+2(d+2)-23\nonumber \\
1272   M_{Pd}=&4\sqrt{d^2+4P}+2d+4-23\nonumber \\
1273   \alignedbox{M_{Pd}}{=}{4\sqrt{d^2+4P}+2d-19}
1274 \end{align}
1275 \\
1276 Furthermore, if we let $l$ and $w$ be the number of columns and
1277 rows and $l>w$,
1278 then the minimum number of moves of a rectangular Klotski ($M$) is
1279 \begin{align}
1280   \label{3spaces rec if we let $l$ and $w$}
1281   M_{Pd}=&4\sqrt{d^2+4P}+2d+4-23\nonumber \\
1282   M=&4\sqrt{(l-w)^2+4(l)(w)}+2(l-w)-19\nonumber \\
1283   M=&4\sqrt{l^2+2lw+w^2+4lw}+2l-2w-19\nonumber \\
1284   M=&4\sqrt{l^2+2lw+w^2}+2l-2w-19\nonumber \\
1285   M=&4(l+w)+2l-2w-19\nonumber \\
1286   M=&4l+4w+2l-2w-19\nonumber \\
1287   \alignedbox{M}{=}{6l+2w-19}
1288 \end{align}
1289 \\
1290 \textbf{4 Spaces}

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Using the data from table \ref{tab:4spaces}, the derived formulae are;

$$\{[(n+1)(n+k) \mid n \in \mathbb{Z}^+, k \in \mathbb{Z}^+, k > 3]\}$$

The following tables summarizes the product and the minimum number of moves for a $(n+3)(n+k)$ Klotski:

Product (\$P\$) & Minimum no. of moves (\$M\$)	
$(n+3)(n+4)$	$8n+10$
$(n+3)(n+5)$	$8n+16$
$(n+3)(n+6)$	$8n+22$
$(n+3)(n+7)$	$8n+28$
$(n+3)(n+8)$	$8n+34$
$(n+3)(n+9)$	$8n+40$
...	...
$P=(n+3)(n+k)$	$M=8n+6k-14$

Formulation of $M=8n+6k-14$

$$\begin{aligned} M &= 8n+(10+[k-4]6) \\ M &= 8n+(10+6k-24) \\ M &= 8n+10+6k-24 \\ M &= 8n+6k-14 \end{aligned}$$

Derivation of n

$$\begin{aligned} P &= (n+3)(n+k) \\ P &= n^2 + kn + 3n + 3k \\ P &= n^2 + (k+3)n + 3k \\ 0 &= n^2 + (k+3)n + 3k - P \end{aligned}$$

using the quadratic equation to find n

$$\begin{aligned} n &= \frac{-(k+3) \pm \sqrt{(k+3)^2 + 4(1)(3k-P)}}{2(1)} \\ n &= \frac{-k-3 \pm \sqrt{k^2 + 6k + 9 - 12k + 4P}}{2} \\ n &= \frac{-k-3 \pm \sqrt{k^2 - 6k + 9 + 4P}}{2} \\ n &= \frac{-k-3 \pm \sqrt{(k-3)^2 + 4P}}{2} \end{aligned}$$

Minimum number of moves in terms of the product (M_P)

$$\begin{aligned} M_P &= 8\left(\frac{-k-3 \pm \sqrt{(k-3)^2 + 4P}}{2}\right) + 6k - 14 \\ M_P &= 4(-k-3 \pm \sqrt{(k-3)^2 + 4P}) + 6k - 14 \\ M_P &= -4k - 12 \pm 4\sqrt{(k-3)^2 + 4P} + 6k - 14 \\ M_P &= 4\sqrt{(k-3)^2 + 4P} + 2k - 26 \end{aligned}$$

```

1339 \end{align}
1340
1341 the difference of the two sides  $+3$ 
1342 \begin{align}
1343 d=&(n+k)-(n+3)\nonumber \\
1344 d=&n+k-n-3\nonumber \\
1345 d=&k-3\nonumber \\
1346 k=&d+3 \\
1347 \end{align}
1348
1349 hence the minimum number of moves given the product and difference
1350 ( $M_{Pd}$ ) is
1351 \begin{align}
1352 M_{Pd}=&4\sqrt{(d+3)^2+4P}+2(d+3)-26\nonumber \\
1353 M_{Pd}=&4\sqrt{d^2+4P}+2d+6-26\nonumber \\
1354 \alignedbox{M_{Pd}}{4\sqrt{d^2+4P}+2d-20} \\
1355 \end{align}
1356
1357 Furthermore, if we let  $l$  and  $w$  be the number of columns and
1358 rows and  $l > w$ ,
1359 then the minimum number of moves of a rectangular Klotski ( $M$ ) is
1360 \begin{align}
1361 M_{Pd}=&4\sqrt{d^2+4P}+2d-20\nonumber \\
1362 M=&4\sqrt{(l-w)^2+4(l)(w)}+2(l-w)-20\nonumber \\
1363 M=&4\sqrt{l^2+2lw+w^2+4lw}+2l-2w-20\nonumber \\
1364 M=&4\sqrt{l^2+2lw+w^2}+2l-2w-20\nonumber \\
1365 M=&4\sqrt{(l+w)^2}+2l-2w-20\nonumber \\
1366 M=&4(l+w)+2l-2w-20\nonumber \\
1367 \alignedbox{M}{6l+2w-20} \\
1368 \end{align}
1369 \textbf{5 Spaces}
1370
1371  $\forall (n+1)(n+k) \ \forall n \in \mathbb{Z}^+ \ \forall k \in \mathbb{Z}^+, k > 4$ 
1372
1373 The following tables summarizes the product and the minimum number
1374 of moves in a  $(n+4)(n+k)$  Klotski:
1375 \begin{center}
1376 \begin{tabular}{cc}
1377 Product ( $P$ ) & Minimum no. of moves ( $M$ ) \\
1378 \hline
1379  $(n+4)(n+5)$  &  $8n+17$  \\
1380  $(n+4)(n+6)$  &  $8n+23$  \\
1381  $(n+4)(n+7)$  &  $8n+29$  \\
1382  $(n+4)(n+8)$  &  $8n+35$  \\
1383  $(n+4)(n+9)$  &  $8n+41$  \\
1384  $(n+4)(n+10)$  &  $8n+47$  \\
1385 ... & ... \\
1386 \hline
1387  $P=(n+4)(n+k)$  &  $M=8n+6k-13$  \\
1388 \end{tabular}
1389 \end{center}

```

```

1389
1390 Formulation of  $M=8n+6k-13$  \\
1391 \begin{align}
1392 M =& 8n+(17+[k-5]6)\nonumber \\
1393 M =& 8n+(17+6k-30)\nonumber \\
1394 M =& 8n+17+6k-30\nonumber \\
1395 M =& 8n+6k-13 \\
1396 \end{align}
1397
1398 Derivation of  $n$ 
1399 \begin{align}
1400 P=&(n+4)(n+k)\nonumber \\
1401 P=&n^2+kn+4n+4k\nonumber \\
1402 P=&n^2+(k+4)n+4k\nonumber \\
1403 0=&n^2+(k+4)n+4k-P \\
1404 \end{align}
1405
1406 using the quadratic equation to find  $n$ 
1407 \begin{align}
1408 n =& \frac{-(k+4) \pm \sqrt{(k+4)^2+4(1)(4k-P)}}{2}\nonumber \\
1409 n =& \frac{-k-4 \pm \sqrt{k^2+8k+16-16k+4P}}{2}\nonumber \\
1410 n =& \frac{-k-4 \pm \sqrt{k^2-8k+16+4P}}{2}\nonumber \\
1411 n =& \frac{-k-4 \pm \sqrt{(k-4)^2+4P}}{2} \\
1412 \end{align}
1413
1414 Minimum number of moves in terms of the product ( $M_P$ )
1415 \begin{align}
1416 M_P =& 8\left(\frac{-k-4 \pm \sqrt{(k-4)^2+4P}}{2}\right)+6k-13\nonumber \\
1417 M_P =& 4(-k-4 \pm \sqrt{(k-4)^2+4P})+6k-13\nonumber \\
1418 M_P =& -4k-16 \pm 4\sqrt{(k-4)^2+4P}+6k-13\nonumber \\
1419 M_P =& 4\sqrt{(k-4)^2+4P}+2k-29 \\
1420 \end{align}
1421
1422 the difference of the two sides  $d$ 
1423 \begin{align}
1424 d =& (n+k)-(n+4)\nonumber \\
1425 d =& n+k-n-4\nonumber \\
1426 d =& k-4\nonumber \\
1427 k =& d+4 \\
1428 \end{align}
1429
1430 hence the minimum number of moves given the product and difference
1431 ( $M_{Pd}$ ) is
1432 \begin{align}
1433 M_{Pd} =& 4\sqrt{((d+4)-4)^2+4P}+2(d+4)-29\nonumber \\
1434 M_{Pd} =& 4\sqrt{d^2+4P}+2d+8-29\nonumber \\
1435 \alignedbox{M_{Pd}}{4\sqrt{d^2+4P}+2d-21}{} \\
1436 \end{align}
1437
1438 Furthermore, if we let  $l$  and  $w$  be the number of columns and
1439 rows and  $l>w$ ,
1440 then the minimum number of moves of a rectangular Klotski ( $M$ ) is
1441 \begin{align}
1442 M_{Pd} =& 4\sqrt{d^2+4P}+2d-21\nonumber

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1441 M=&4\sqrt{(l-w)^2+4(l)(w)}+2(l-w)-21\ nonumber \\\
1442 M=&4\sqrt{l^2+2lw+w^2+4lw}+2l-2w-21\ nonumber \\\
1443 M=&4\sqrt{l^2+2lw+w^2}+2l-2w-21\ nonumber \\\
1444 M=&4\sqrt{(l+w)^2}+2l-2w-21\ nonumber \\\
1445 M=&4(l+w)+2l-2w-21\ nonumber \\\
1446 M=&4l+4w+2l-2w-21\ nonumber \\\
1447 \alignedbox{M=}{6l+2w-21}{ }
1448 \end{align}
1449 \\\
1450
1451 The following table summarizes the product and the minimum number
      of moves given the product and the difference for a
      $(n+f)(n+k)$ rectangular Klotski, where $f\geq 1$, $k>2$,
      $k>f$, $f<S$, and $2\leq S\leq \infty$:
1452 \begin{center}
1453 \begin{tabular}{cc}
1454 No. of Spaces ($S$) & Minimum no. of moves ($M_{Pd}$) \\\
1455 \hline
1456 $2$ & $4\sqrt{d^2+4P}+2d-18$ \\\
1457 $3$ & $4\sqrt{d^2+4P}+2d-19$ \\\
1458 $4$ & $4\sqrt{d^2+4P}+2d-20$ \\\
1459 $5$ & $4\sqrt{d^2+4P}+2d-21$ \\\
1460 $6$ & $4\sqrt{d^2+4P}+2d-22$ \\\
1461 ... & ... \\\
1462 \hline
1463 \end{tabular}
1464 \end{center}
1465
1466 Formulation of $M_{Pd}=4\sqrt{d^2+4P}+2d-S-16$
1467 \begin{align}
1468 \label{eqn:M_Pd=4d^2+4P+2d-S-16}
1469 M_{Pd}=& (4\sqrt{d^2+4P}+2d-18)+(S-2)(-1)\ nonumber \\\
1470 M_{Pd}=& 4\sqrt{d^2+4P}+2d-18-S+2\ nonumber \\\
1471 \alignedbox{M_{Pd}={}}{4\sqrt{d^2+4P}+2d-S-16}{ }
1472 \end{align}
1473 \\\
1474 The above-mentioned formula (Eq. \ref{eqn:M_Pd=4d^2+4P+2d-S-16})
      calculates the minimum number of moves required to solve the
      rectangular Klotski puzzle whose product is $(n+f)(n+k)$, where
      $f\geq 1$, $k>2$, $k>f$, $f<S$, and $2\leq S\leq \infty$ and
      if given the product and the difference of the sides.
1475 \\\
1476
1477 The following table summarizes the product and the minimum number
      of moves in a $(n+f)(n+k)$ rectangular Klotski, where $f\geq
      1$, $k>2$, $k>f$, $f<S$, and $2\leq S\leq \infty$:
1478 \begin{center}
1479 \begin{tabular}{cc}
1480 No. of Spaces ($S$) & Minimum no. of moves ($M$) \\\
1481 \hline
1482 $2$ & $6l+2w-18$ \\\
1483 $3$ & $6l+2w-19$ \\\
1484 $4$ & $6l+2w-20$ \\\
1485 $5$ & $6l+2w-21$ \\\

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```

1486      $6$ & $6l+2w-22$ \\
1487      ... & ... \\
1488      \hline
1489    \end{tabular}
1490  \end{center}
1491
1492  Formulation of $M=6l+2w-S-16$
1493  \begin{align}
1494    \label{eqn:6l+2w-S-16}
1495    M=& (6l+2w-18)+(S-2)(-1)\nonumber \\
1496    M=& 6l+2w-18-S+2\nonumber \\
1497    \alignedbox{M=}{6l+2w-S-16}{ }
1498  \end{align}
1499
1500  The above – mentioned formula (Eq. \ref{eqn:6l+2w-S-16}) calculates
    the minimum number of moves required to solve the rectangular
    Klotski puzzle whose product is $(n+f)(n+k)$, where $f\geq 1$,
    $k>2$, $k>f$, $f<S$, and $2\leq S\leq \infty$.
1501  \\
1502  \\
1503  \textbf{General Formulae} \\
1504  \\
1505  General Formula for Square Klotski:
1506  \begin{align}
1507    \label{eqn:Piecewise square}
1508    M=
1509    \begin{dcases}
1510      8s-3S-8 & \text{if } 1\leq S\leq 4, \\
1511      8s-S-16 & \text{if } 4\leq S\leq \infty, \\
1512    \end{dcases}
1513  \end{align}
1514  General Formula for Rectangular Klotski:
1515  \begin{align}
1516    \label{eqn:Piecewise rec}
1517    M=
1518    \begin{dcases}
1519      4\sqrt{1+4lw}-14 & \text{if } (n+1)(n+2) \\
1520      6l+2w-13 & \text{if } S=1 \\
1521      6l+2w-S-16 & \text{if } (n+f)(n+k) \\
1522      & \text{where } f\geq 1, \\
1523      & k>2, k>f, \\
1524      & f<S, \\
1525      & 2\leq S\leq \infty
1526    \end{dcases}
1527  \end{align}
1528  General Formula for Rectangular Klotski given the difference ($d$)
    and the product ($P$):
1529  \begin{align}
1530    \label{eqn:MPD Piecewise}
1531    M_{Pd}=
1532    \begin{dcases}
1533      4\sqrt{1+4P}-14 & \text{if } (n+1)(n+2) \\
1534      4\sqrt{d^2+4P}+2d-13 & \text{if } S=1 \\
1535      4\sqrt{d^2+4P}+2d-S-16 & \text{if } (n+f)(n+k)

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```

1536      & \text{where } f \geq 1, \\
1537      & k > 2, \ k > f, \\
1538      & f < S, \\
1539      & 2 \leq S \leq \infty \\
1540    \end{dcases} \\
1541  \end{align} \\
1542  $M$ - Minimum number of Moves \\
1543  $P$ - Product ($l \times w$) \\
1544  $d$ - Difference ($l - w$) \\
1545  $s$ - Number of rows or columns \\
1546  $l$ - Number of Columns \\
1547  $w$ - Number of Rows \\
1548  $$ - Number of Spaces \\
1549  \\
1550  \textbf{Examples} \\
1551  \\
1552  \\
1553  For Formula $M=8s-3S-8$ (Eq. \ref{eqn:Piecewise square}): \\
1554  \\
1555  In a square Klotski with a $5 \times 5$ dimension with $4$ spaces ,
    by applying the formula $M=8s-3S-8$, the minimum number of
    moves is
1556  \begin{align}
1557    M &= 8(5) - 3(4) - 8 \text{ \nonumber} \\
1558    M &= 40 - 12 - 8 \text{ \nonumber} \\
1559    \alignedboxn{M=}{20} \text{ \nonumber} \\
1560  \end{align} \\
1561  \\
1562  For Formula $M=8s-S-16$ (Eq. \ref{eqn:Piecewise square}): \\
1563  \\
1564  In a square Klotski with a $8 \times 8$ dimension with $6$ spaces ,
    by applying the formula $M=8s-S-16$, the minimum number of
    moves is
1565  \begin{align}
1566    M &= 8(8) - 6 - 16 \text{ \nonumber} \\
1567    M &= 64 - 6 - 16 \text{ \nonumber} \\
1568    \alignedboxn{M=}{42} \text{ \nonumber} \\
1569  \end{align} \\
1570  \\
1571  For Formula $M=4\sqrt{1+4lw}-14$ (Eq. \ref{eqn:Piecewise rec}): \\
1572  \\
1573  In a rectangular Klotski with a $3 \times 4$ dimension , by applying
    the formula $M=4\sqrt{1+4lw}-14$, the minimum number of moves is
1574  \begin{align}
1575    M &= 4\sqrt{1+4(4)(3)} - 14 \text{ \nonumber} \\
1576    M &= 4\sqrt{1+48} - 14 \text{ \nonumber} \\
1577    M &= 4\sqrt{49} - 14 \text{ \nonumber} \\
1578    M &= 4(7) - 14 \text{ \nonumber} \\
1579    M &= 28 - 14 \text{ \nonumber} \\
1580    \alignedboxn{M=}{14} \text{ \nonumber} \\
1581  \end{align} \\
1582  \\
1583  For Formula $M=6l+2w-13$ (Eq. \ref{eqn:Piecewise rec}): \\
1584

```

In a rectangular Klotski with a 3×7 dimension, by applying the formula $M=6l+2w-13$, the minimum number of moves is

$$\backslash \text{begin}\{\text{align}\}$$

`M=&6(7)+2(3)-13 \nonumber \\\`

M=&42+6-13 \nonumber \\\

`\alignedboxn{M=}{35} \nonumber`

$$\backslash \text{end}\{\text{align}\}$$

\\

For Formula $M=6l+2w-S-16$ (Eq. \ref{eqn:Piecewise rec}): \\\

In a rectangular Klotski with a 9×12 dimension with 8 spaces, by applying the formula $M = 6l + 2w - S - 16$, the minimum number of moves is

$$\backslash \text{begin}\{\text{align}\}$$
$$M = 6(12) + 2(9) - 8 - 16 \quad \backslash \text{nonnumber} \quad \backslash \backslash$$

M=&72+18-8-16 \ nonumber \\\

$$\backslash alignedboxn \{M=\}\{66\} \quad \backslash nonumber$$
$$\end{align}$$

//

For Formula $M_{Pd}=4\sqrt{1+4P}-14$ (Eq. \ref{eqn:MPD Piecewise})

//

In a rectangular Klotski that has the product of \$12\$, by applying the formula $M_{\{Pd\}} = 4 \sqrt{1 + 4P} - 14$, the minimum number of moves is

$$\backslash \text{begin}\{\text{align}\}$$
$$M_{\{Pd\}} = \sqrt{1 + 4(12)} - 14 \quad \text{nonnumber}$$
$$M_{\text{Pd}} = 4\sqrt{49} - 14 \quad \text{nonnumber}$$
$$M_{\{Pd\}} = 4(7) - 14 \quad \backslash \text{nonnumber} \quad \backslash \backslash$$
$$M_{\text{Pd}} = 28 - 14 \quad \backslash \text{nonnumber} \quad \backslash \backslash$$

`\alignedboxn{M_{Pd}}={}{14} \nonumber`

$$\end{align}$$

//

For Formula $M_{\{Pd\}}=4\sqrt{d^2+4P}+2d-13$ (Eq. \ref{eqn:MPD Piecewise}) \\

In a rectangular Klotski whose product and difference of the

dimensions are 21×4 , respectively, by applying the formula for the product and difference, the minimum number of moves is

$$\begin{aligned} & \text{\texttt{\textbackslash begin\{align\}}} \\ & \end{aligned}$$
$$M_{\text{Pd}} = 4 \sqrt{4^2 + 4(21)} + 2(4) - 13 \quad \text{nonnumber}$$
$$M_{\text{Pd}} = 4 \sqrt{16 + 84} + 8 - 13 \quad \text{nonnumber}$$

$M_{\text{Pd}} = 4 \sqrt{100} - 5$ \nonumber \\

$$M_{\{Pd\}} = 4(10) - 5 \quad \backslash \text{nonnumber} \quad \backslash \backslash$$

$M_{Pd} = 40 - 5 \setminus \text{nonnumber} \setminus$

$$\backslash alignedboxn{\mathrm{M}_{\mathrm{Pd}}}{35} \backslash nonnumber$$
$$\backslash \text{end}\{\text{align}\}$$

11

For Formula $M_{\text{Pd}}=4\sqrt{d^2+4P}+2d-S-16$ (Eq. \ref{eqn:MPD Piecewise}) \\\

11

In a rectangular Klotski with a product and a difference of \$108\$ and \$3\$ respectively and a spaces of \$8\$, by applying the

```

1627     formula  $M_{Pd}=4\sqrt{d^2+4P}+2d-S-16$ , the minimum number of
1628     moves is
1629     \begin{align}
1630     M_{Pd}&=&4\sqrt{(3)^2+4(108)}+2(3)-(8)-16 \quad \text{\nonumber}\\
1631     M_{Pd}&=&4\sqrt{9+432}+6-8-16 \quad \text{\nonumber}\\
1632     M_{Pd}&=&4\sqrt{441}+6-8-16 \quad \text{\nonumber}\\
1633     M_{Pd}&=&4(21)+6-8-16 \quad \text{\nonumber}\\
1634     \alignedboxn{M_{Pd}}{66} \quad \text{\nonumber}\\
1635     \end{align}
1636     \section{Chapter IV}
1637     \subsection{Summary of the Study}
1638
1639     Therefore, the investigators came up with a technique in the game
1640     Klotski Puzzle where the player must move the tile diagonally
1641     from its starting point to its predetermined spot in order to
1642     obtain the minimum number of moves. Varied formulae were
1643     derived and formulated to determine the minimum number of moves
1644     based on the dimensions and number of spaces in the
1645     predetermined spot of the Klotski puzzle. The investigators
1646     used the piecewise function, the quadratic formula, and the
1647     arithmetic sequence formed by the pattern whose common
1648     difference are 8 and 6 as their basis in developing these
1649     formulae.
1650
1651     Thus, this study shows potential not limiting only in the field of
1652     games but also in some real life situation that involves
1653     algorithm that calculate the minimum number of moves needed to
1654     reach a specific goal such as parking assistance system,
1655     traffic management, cargo handling, space optimization, and
1656     other situations that deals with constraints and involves the
1657     process of optimizing movement.
1658     \subsection{Recommendation}
1659
1660     The investigators recommend to study further not just on square and
1661     rectangles but also on triangle and other shaped Klotskis. The
1662     movement of more than 1 block can also be taken into
1663     consideration in the future conduct of related investigation.
1664     Also, blocks with diferent shapes and sizes will be a good
1665     topic for investigation. Lastly, the investigators of this
1666     study recommends future researchers to investigate the
1667     relevance of the results of this study and to assess the
1668     feasibility of the suggested potential real life application.
1669
1670     \bibliographystyle{plain}
1671     \bibliography{MIP}
1672
1673     \end{document}

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