International Journal of Academic Multidisciplinary Research (IJAMR)
ISSN: 2643-9670
Vol. 9 Issue 2 February - 2025, Pages: 194-194

Exploring Klotski: An Investigation of the Minimum Number of Moves in a Special Case of Slide Puzzles

Axelcris G. Suladay[1], Mergel Ann G. Millendez[1], Trisha B. Vista[1] [1] Francisco Ramos National High School

Neil Van M. Salinas - Coach

December 2024

Abstract: *Klotski is a special type of sliding puzzle that first emerged in the early 20th century. It refers to a variety of sliding block or tile games with the goal of moving a specific block to a predetermined spot. By changing the game's rule that requires a player to move a certain tile or block from one corner of the puzzle to its opposite corner, providing that it contains spaces more than or equal to one, this inquiry sought to ascertain the minimum number of moves. It was also recognized that the Klotski puzzle's dimension was directly influenced by the number of spaces. The results of the investigation proposed the use of these formulae in determining the minimum number of moves (M) given the number of columns/rows of a square-shaped Klotski: a. M = 8s˘11 if the number of spaces (S) is equal to 1, b. M = 8s˘3S˘8 if the number of spaces (S) is greater than or equal to 1 but less than or equal to 4, c. M = 8s˘S˘16 if the number of spaces (S) is greater than or equal to 4 but less than or equal to infinity and the number of rows/columns (s) is greater than or equal to the number of spaces (S). Alternatively, 3 formulae were proposed to determine the minimum number of moves (M) of a rectangular-shaped Klotski: a. $M = 6l + 2w - 13$ if the number of spaces is equal to 1, b. M = 4 1 + 4lw − 14 if the product of the dimension is the quantity of n plus 1 multiplied by the quantity of n plus 2, and c. M = 6l˘2w˘S˘16 if the product of the dimension is the quantity of n plus f multiplied by the quantity of n plus k, where f is greater than or equal to 1, k is greater than 2, k is greater than f, f is less than the number of spaces (S), and lastly if the number of spaces is greater than or equal to 2 but less than or equal to infinity. In the case where only the product (P) and difference (d) of the dimensions are given, the formula $M_{Pd} = 4\ d^2 + 4P + 2d - 13$ may be used to determine the minimum number of moves if the number of spaces is equal to 1, while $M_{Pd} = 4\ 1 + 4P - 14$ is used only if the product of the dimension is the quantity of n plus 1 multiplied by the quantity of n plus 2, and $M_{Pd} = 4\ d^2 + 4P + 2d - S - 16$ will be utilized if the product of the dimension is the quantity of n plus f multiplied by the quantity of n plus k, where f is greater than or equal to 1, k is greater than 2, k is greater than f, f is less than the number of spaces (S), and the number of spaces is greater than or equal to 2 but less than or equal to infinity. It is recommended that additional investigations must be done for other cases of the puzzle, such as with triangles and other shaped Klotski puzzles, with consideration of the movement of more than 1 block and to determine the other possible real-life application of the concepts and result in this study.*

Arithmetic, Klotski, Minimum Number of Moves, Pattern, Sliding Puzzle, Quadratic Formula, Piecewise function

Exploring Klotski: An Investigation of the Minimum Number of Moves in a Special Case of Slide Puzzles

Axelcris G. Suladay[1], Mergel Ann G. Millendez[1], Trisha B. Vista[1]
[1] Francisco Ramos National High School

Neil Van M. Salinas - Coach

Exploring Klotski: An Investigation of the Minimum Number of Moves in a Special Case of Slide Puzzles

Axelcris G. Suladay
Mergel Ann G. Millendez
Trisha B. Vista

December 2024

Abstract

Klotski is a special type of sliding puzzle that first emerged in the early 20th century. It refers to a variety of sliding block or tile games with the goal of moving a specific block to a predetermined spot. By changing the game's rule that requires a player to move a certain tile or block from one corner of the puzzle to its opposite corner, providing that it contains spaces more than or equal to one, this inquiry sought to ascertain the minimum number of moves. It was also recognized that the Klotski puzzle's dimension was directly influenced by the number of spaces. The results of the investigation proposed the use of these formulae in determining the minimum number of moves (M) given the number of columns/rows of a square-shaped Klotski: a. $M = 8s - 11$ if the number of spaces (S) is equal to 1, b. $M = 8s - 3S - 8$ if the number of spaces (S) is greater than or equal to 1 but less than or equal to 4, c. $M = 8s - S - 16$ if the number of spaces (S) is greater than or equal to 4 but less than or equal to infinity and the number of rows/columns (s) is greater than or equal to the number of spaces (S). Alternatively, 3 formulae were proposed to determine the minimum number of moves (M) of a rectangular-shaped Klotski: a. $M = 6l + 2w - 13$ if the number of spaces is equal to 1, b. $M = 4\sqrt{1 + 4lw} - 14$ if the product of the dimension is the quantity of n plus 1 multiplied by the quantity of n plus 2, and c. $M = 6l - 2w - S - 16$ if the product of the dimension is the quantity of n plus f multiplied by the quantity of n plus k, where f is greater than or equal to 1, k is greater than 2, k is greater than f, f is less than the number of spaces (S), and lastly if the number of spaces is greater than or equal to 2 but less than or equal to infinity. In the case where only the product (P) and difference (d) of the dimensions are given, the formula $M_{Pd} = 4\sqrt{d^2 + 4P} + 2d - 13$ may be used to determine the minimum number of moves if the number of spaces is equal to 1, while $M_{Pd} = 4\sqrt{1 + 4P} - 14$ is used only if the product of the dimension is the quantity of n plus 1 multiplied by the quantity of n plus 2, and $M_{Pd} = 4\sqrt{d^2 + 4P} + 2d - S - 16$ will be utilized if the product of the dimension is the quantity of n plus f multiplied by the quantity of n plus k, where f is greater than or equal to 1, k is greater than 2, k is greater than f, f is less than the number of spaces (S), and the number of spaces is greater than or equal to 2 but less than or equal to infinity. It is recommended that additional investigations must be done for other cases of the puzzle, such as with triangles and other shaped Klotski puzzles, with consideration of the movement of more than 1 block and to determine the other possible real-life application of the concepts and result in this study.

Arithmetic, Klotski, Minimum Number of Moves, Pattern, Sliding Puzzle, Quadratic Formula, Piecewise function

1 Chapter I

Introduction

Klotski is a type of a sliding puzzle that have started and gained popularity in the early 20th century. It refers to a whole group of similar sliding-block puzzles with a same objective of moving a certain block to a predetermined location [3]. Within the frame, the blocks can be moved anywhere by sliding—not turning, lifting or jumping. To complete the sliding puzzle, the starting and finishing positions are typically provided [2].

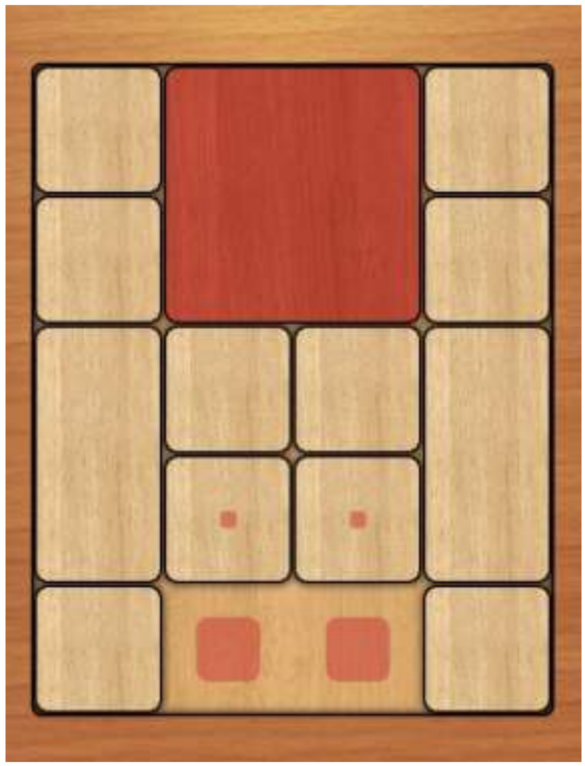

Figure 1: Sample of a Mobile Game Klotski

Upon watching a player do the Klotski puzzle by moving a tile from one place to another, the investigators observed some patterns that are related and was tackled in their Mathematics class in the previous school year. This urges them to perform a short experimentation of moving a tile in a diagonal direction in order to find the minimum number of moves and to formulate a mathematical investigation about the Klotski puzzle in hopes of discovering concepts that may contribute in the study of mathematics.

Thus, this study focuses on exploring the Klotski puzzle and to generate formulas out of the number of moves it needed to complete the puzzle. This investigation also aims to determine the minimum possible number of moves in solving a square or rectangular-shaped Klotski with the aid of the different mathematical concepts, particularly the arithmetic sequence.

Like other varieties of sliding puzzle, Klotski is also known as a challenging game that brings entertainment to people in all ages. This game also enhances an individual's problem solving skills, critical thinking skills and other mathematical related skills. Moreover, this game might not just be a "simple game" for it has the potential to unfold mathematical concepts that have a significant contribution to mathematical research.

Significance of the Study

This study adds to the corpus of information for everyone, especially for gamers, and it is noteworthy since the idea might be used in other games. This investigation makes fresh contribution to the field of games and geometry, due to its usage and relation to the concept of sequence and patterns. With the success of this study, not only students but everyone else may be able to benefit from this study by using the concepts in this investigation in the future and pass this knowledge to the next generation. Lastly, the investigators of this study are hoping that other investigators would find the relevance of this study into its real life application.

Statement of the Problem

The aim of the game under investigation is to slide a corner tile/block from its starting position (upper left corner) to its opposite corner (bottom right) in the least possible number of moves. In addition, each block/tile can only be moved up, down, left, right and one at a time. Diagonal movement and lifting the blocks are not permitted. There can be 1 or more spaces and the starting free space of the puzzle is located in the bottom right corner. If there are 2 or more spaces, the spaces should be vertically aligned above the predetermined spot.

Figure 2: Sample Predetermined State of a Square Klotski

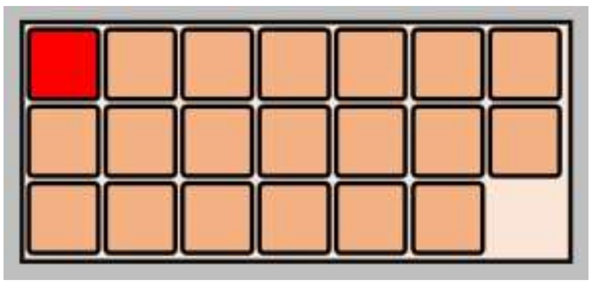

Figure 3: Sample Predetermined State of a Rectangular Klotski

Based on the aforementioned rule of the game, the following objectives were identified to facilitate the conduct of this investigation:

- To determine the minimum number of moves in the Klotski game
- To determine whether a pattern is formed by the minimum number of moves
- To create a formula in calculating the minimum number of moves for all square Klotski

· To create a formula in calculating the minimum number of moves for all rectangular Klotski

· To determine the relationship between the number of spaces and the number of rows and columns

. To create a formula in calculating the minimum number of moves given the number of spaces for all square Klotski

. To create a formula in calculating the minimum number of moves given the number of spaces for all rectangular Klotski

Review of Related Literature

Klotski and other Sliding Puzzles

A frame enclosed collection of forms makes up a sliding puzzle. The only way to move the shapes inside that frame is to slide; turning, lifting, or jumping are not permitted. The 15 puzzle, created in 1878 in the USA (by an unidentified person), was the one that actually kicked off the genre of puzzles.

Figure 4: Sample of a 3×3 Mobile Game Sliding Puzzle

Figure 5: Sample of a 5×5 Mobile Game Sliding Puzzle

The traditional Klotski puzzle is one of the notable sliding puzzle that was known as "Hua Rong Dao" in China and "Dad's Puzzler" of USA. In China, this traditional Klotski gained popularity in the 1930's based on the warrior from the Eastern Han Dynasty and well-known Chinese strategist Cao Cao. The Cao Cao piece serves as the main header in sliding block puzzles, and the other nine pieces represent armed guards and other barriers. The player's task is to overcome these challenges and deliver Cao Cao to freedom at the bottom of the board [8]. On the other hand, Dad's puzzler was invented by one of America's leading mathematician, Olive C. Hazlett during the 1920s. There is no pictures to put together or number to get in order in this puzzle, it's just simply uses pieces of wood. To play this puzzle, the large square must be moved from position A to position C in order to complete the puzzle. Also, no pieces may be turned, raised off the bottom of the frame, or jumped [7].

The frame and base of the Klotski puzzle and Dad's Puzzler serves as the foundation upon which the puzzle is built, it also have the crucial role to constraints within which blocks can be shifted. It is commonly made out of wood or durable plastic. China's Klotski puzzle have one large block(2×2), representing the protagonist. It also have 4 medium rectangles(2×1) and 4 small squares(1×1) that can be slid in a specific direction to help the protagonist reach the exit. While Dad's puzzler's blocks have one 2×2 square, four 2×1 rectangles with the long side horizontal, two 2×1 rectangles with the long side vertical, and two 1×1 squares [6].

The Klotski puzzle has changed over time, giving rise to a variety of versions and adaptations that continue to intrigue and test puzzle enthusiasts. Some example of these are Klotski puzzle with Animated images, Klotski puzzle with a hexagonal frame, and a Klotski puzzle that holds more than 10 blocks that the traditional one have. The well-known Klotski puzzle game was also included in the improvement of technology after becoming part of Windows 3.1's Microsoft entertainment pack and has since become a classic [6].

A puzzle is a type of educational game that brings entertainment and challenges everyone to solve it. To find the right and enjoyable solution to a puzzle it requires the development of certain skill that may not only help solve the puzzle but also enhances one's congnitive abilities. This will put the observation skill, planning skill, critical thinking skill and patience up to a test. Playing the puzzle will help us develop our skills that we can also apply in our daily lives without taking away the fun [6].

According to the recent article of Zhong (2023), studying new algorithms to solve sliding puzzles has become increasingly important as sliding puzzle sports have developed. Zhong along with other investigators hope to develop an additive approximation method because it is difficult to solve the puzzle optimally. The length of the answer produced by this algorithm should only be a low-order term longer than the ideal solution [15].

Arithmetic Sequence

Carl Friedrich Gauss is a German Mathematician who contributed in many fields of Math and Science. When he was a young boy he was tasked to add the integers 1 to 100 by his teacher. As a Math prodigy he is, he was able to create his own formula which leads to the discovery of arithmetic sequence. It is an ordered set of numbers that have a common difference between each consecutive term [1].

Mathematicians and others who work with numbers can solve difficult mathematical problems by using arithmetic sequences, which are employed in algebra and geometry. Arithmetic sequences can be used to solve simple or complex problems but it's necessary to have the basic understanding about it to ensure that it is applied correctly. Finding patterns, such as a specific number of extra seats each row, is how this is done [4].

In the study of Oded Goldreich in 2011, They take into account a game in which players move various stones along an undirected graph's edges. Only one pebble may be present in each vertex at any one moment, and only one pebble may be transferred at a time (i.e., the pebble must be moved to an empty vertex). They demonstrate that it is NP-Hard to determine the shortest sequence of moves between two given "pebble configurations" [10].

Piecewise Function

A piecewise function is a function in which the outcome is defined by more than one formulas. It describes situations where a rule or relationship changes as the input value passes certain “boundaries"[11]. The earliest known use of the word piecewise is in the late 1600’s in the writing of Nathaniel Fairfax, a physician and antiquary. Since then, it has been significant and useful in different fields as long as there is a change based on its domain value; specifically, in mathematics, this function is employed when different expressions or formulas apply to various parts of the domain [5].

In the study of Nagayama, Sasao, and Butler (2014), to represent a numeric function compactly, they partition the domain of the function into uniform segments and transform the sub-function in each segment into an arithmetic spectrum. An arithmetic expression is derived from this, and a piecewise arithmetic expression for the function is obtained. This is later on widely used in the different parts of their study [13].

Quadratic Formula

The development or derivation of mathematical ideas is similarly as continuous as the historical evolution, as one mathematician picks up an idea where another mathematician leaves it. One of these is the quadratic formula, which is used to find the roots of a quadratic equation [12]. The Quadratic Formula uses the "a", "b", and "c" from "$ax^2 + bx + c$", where "a", "b", and "c" are just numbers; they are the "numerical coefficients" of the quadratic equation that is given to solve [14].

The quadratic formula is the easiest method to determine the roots of a quadratic equation. It provides a straightforward way to solve the equation, offering an alternative to factorization. In cases where a quadratic equation is difficult to factor, the quadratic formula is a convenient and efficient tool for quickly finding the roots [9].

Scope and Delimitations

This investigation aims to determine the minimum number of moves required to move a block in a Klotski puzzle from its starting location to its designated spot. The predetermined spot occupies one or more than spaces that is vertically aligned. A block or tile must be moved diagonally from the upper-left corner to the opposite bottom-right corner. The legal moves are limited to: (a) sliding the block/tile up, down, left, or right, and (b) only one block can be moved at a time. All blocks are of equal size, but the puzzle’s dimensions differ as the number of columns should be greater than the number of rows, it also vary depending on the number of spaces in the predetermined spot, where the puzzle’s dimension must be equal to or greater than the number of spaces in the predetermined spot disregarding the rows and column that is less than the chosen dimension. Additionally, this study aims to derive or create formulae based on observed patterns or sequences. Finally, the scope of this study is not limited to square Klotski puzzles; it also considers rectangular Klotski puzzles with varying dimensions and number of spaces in the predetermined spot.

2 Chapter II

2.1 Definition

Arithmetic Sequence- a sequence where the next terms after the first term is obtained by adding a constant number. The difference between each consecutive terms is called the common difference.

Klotski- comes from the polish word "klocki" which means wooden blocks. It is a sliding puzzle that aims to move the red block to exit at the bottom of the board or to move the red block from and into a specified location.

Minimum moves- the least quantity, amount possible, assignable, allowable, or attainable in moving a block from a Klotski puzzle.

Pattern- is a repeated arrangement of number or the set of numbers that are related to each other by following a specific rule.

Sliding puzzle- is a type of simple combination puzzle. It challenges the player to slide pieces along a certain routes on a board to reach a certain end arrangement.

Piecewise Function- is a function defined by multiple sub-functions each applying to a specific intervals of its domain.

Quadratic Formula- is a mathematical rule that helps find the solutions to equations of the form $ax^2 + bx + c = 0$, to figure out the value "x" that makes the equation true.

2.2 Known Results or Conjecture

The following are the conjectures generated by the investigators in the conduct of the investigation:

- The minimum number of moves for square and rectangular Klotski with varying dimensions forms an arithmetic sequence

- The dimension of a Klotski puzzle are directly influenced by the number of spaces it contains

- The formula to find the minimum number of moves (M) of a square Klotski is $M = 8s - 11$, where s is the number of columns or rows given that the number of space is equal to 1

- The formula to find the minimum number of moves of a rectangular Klotski are:

a. Given the difference and the product:

1) $M_{Pd} = 4\sqrt{d^2 + 4P} + 2d - 13$, where d is the positive difference and P is the product between the number of rows and columns

b. Given the number of rows or columns:

2) $M = 6l + 2w - 13$, where l and w are the number of rows and columns, and $l > w$

- The formula to find the minimum number of moves (M) of a square Klotski given the spaces are greater than or equal to 2 are:

1. $M = 8s - 3S - 8$, this formula can be use if $1 \leq S \leq 4$ where s is the number of rows or columns and S is the number of spaces, and $s \geq S$

2. $M = 8s - S - 16$, this formula can be use if $4 \leq S \leq \infty$ where s is the number of rows or columns and S is the number of spaces, and $s \geq S$

. The formula to find the minimum number of moves (M) of a rectangular Klotski given the spaces are greater than or equal to 2 are:

a. Given the difference and the product

1) $M_{Pd} = 4\sqrt{1+4P} - 14$, this formula can be used if the product of the dimension is $(n+1)(n+2)$
2) $M_{Pd} = 4\sqrt{d^2+4P+2d-S} - 16$, this formula can be used if the product of the dimension is $(n+f)(n+k)$, where $f \geq 1$, $k > 2$, $k > f$, $f < S$ and $2 \leq S \leq \infty$

b. Given the number of rows and columns

3) $M = 4\sqrt{1+4lw} - 14$, this formula can be use if $(n+1)(n+2)$ where l is the number of columns and w is the number of rows, $l > w$
4) $M = 6l + 2w - S - 16$, this formula can be use if $(n+f)(n+k)$ where $f \geq 1$, $k > 2$, $k > f$, $f < S$, $2 \leq S \leq \infty$, where l is the number of columns, w is the number of rows, S is the number of spaces, and $l > w$.

3 Chapter III

3.1 Researchers' Generated Data and Results

This section presents the data that the investigators gathered, the pattern uncovered and the formulation and derivation of the formula to find the minimum number of moves on the Klotski Puzzle. The following data and results are organized and presented based on the order of the specific objectives identified in section 1.3

Data for the Minimum Number of Moves

Table 1: Minimum Number of Moves based on the Number of Rows and Columns

x	1	2	3	4	5	6	7	...
1	0/	1	0/	0/	0/	0/	0/	...
2	1	5	9	15	21	27	33	...
3	0/	9	13	17	23	29	35	...
4	0/	15	17	21	25	31	37	...
5	0/	21	23	25	29	33	39	...
6	0/	27	29	31	33	37	41	...
7	0/	33	35	37	39	41	45	...
...	...	...	...	...	...	...	...	...

The minimum number of moves in a square Klotski is always obtained by traveling the main block/tile around the diagonal path of the puzzle or it follows a stair-like path. For the rectangular Klotski, a similar strategy can be used were the main block/tile should travel also around the diagonal path of the puzzle.

Pattern for the Minimum Number of Moves

As shown in Table 2, the minimum number of moves of a square Klotski (in an increasing number of sides) forms an Arithmetic Sequence whose common difference is 8.

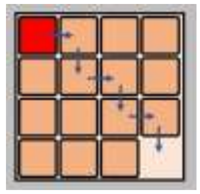

Figure 6: Path to obtain the Minimum Number of Moves of a Square Klotski

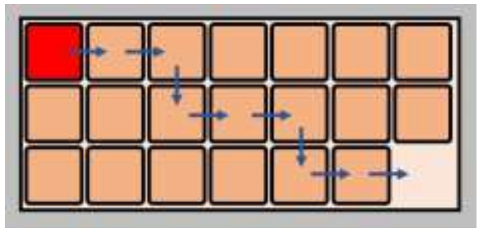

Figure 7: Path to obtain the Minimum Number of Moves of a Rectangular Klotski

Table 2: Pattern for the Minimum Number of Moves for a Square Klotski

x	1	2	3	4	5	6	7	...
1	0/	1	0/	0/	0/	0/	0/	...
2	1	5	9	15	21	27	33	...
3	0/	9	13	17	23	29	35	...
4	0/	15	17	21	25	31	37	...
5	0/	21	23	25	29	33	39	...
6	0/	27	29	31	33	37	41	...
7	0/	33	35	37	39	41	45	...
...	...	...	...	...	...	...	...	...

Table 3: Pattern for the Minimum Number of Moves for a Rectangular Klotski

x	1	2	3	4	5	6	7	...
1	0/	1	0/	0/	0/	0/	0/	...
2	1	5	9	15	21	27	33	...
3	0/	9	13	17	23	29	35	...
4	0/	15	17	21	25	31	37	...
5	0/	21	23	25	29	33	39	...
6	0/	27	29	31	33	37	41	...
7	0/	33	35	37	39	41	45	...
...	...	...	...	...	...	...	...	...

As shown in Table 3, the minimum number of moves of a rectangular Klotski (in an increasing number of sides) forms also an Arithmetic Sequence whose common difference is 8 just like the pattern of the square Klotski. Furthermore, the first term of each sequence in the rectangular Klotski also forms an arithmetic sequence whose common difference is 6.

Formula for the Minimum Number of Moves of a Square Klotski;

$$(n+1)(n+1) \text{ or } (n+1)^2 \quad \{n | n \in \mathbb{Z}^+\}$$

Product(P)	Minimum No. of Moves (M)
4	5
9	13
16	21
25	29
36	37
49	45
64	53
81	61
$P = (n+1)^2$	$M = 8n - 3$

Formulation of $M = 8n - 3$

$$M = 5 + (n-1)8$$
$$M = 5 + 8n - 8$$
$$M = 8n - 3 \qquad (1)$$

Derivation of n

$$P = (n+1)^2$$
$$\sqrt{P} = \sqrt{(n+1)^2}$$
$$\sqrt{P} = n + 1$$
$$\sqrt{P} - 1 = n$$
$$n = \sqrt{P} - 1 \qquad (2)$$

Minimum number of moves in terms of the product (M_P)

$$M = 8n - 3$$
$$M_P = 8(\sqrt{P} - 1) - 3$$
$$M_P = 8\sqrt{P} - 8 - 3$$
$$M_P = 8\sqrt{P} - 11 \qquad (3)$$

Since $\sqrt{P}$ is just equal to the number of rows/columns (s) of the square Klotski, then the minimum number of moves (M) is

$$M = 8s - 11 \qquad (4)$$

Formula for the Minimum Number of Moves of a Rectangular Klotski;

$$(n+1)(n+k) \quad \{n | n \in \mathbb{Z}^+\} \ \{k | k \in \mathbb{Z}^+, k > 1\}$$

The arithmetic sequence formula of the different minimum number of moves (M) for each $(n+1)(n+k)$

Minimum no. of moves for $(n+1)(n+2)$	
$P_{(n+1)(n+2)}$	$M_{(n+1)(n+2)}$
6	9
12	17
20	25
30	33
...	...
$(n+1)(n+2)$	$8n+1$

Minimum no. of moves for $(n+1)(n+3)$	
$P_{(n+1)(n+3)}$	$M_{(n+1)(n+3)}$
8	15
15	23
24	31
35	39
...	...
$(n+1)(n+3)$	$8n+7$

Minimum no. of moves for $(n+1)(n+4)$	
$P_{(n+1)(n+4)}$	$M_{(n+1)(n+4)}$
10	21
18	29
28	37
40	45
...	...
$(n+1)(n+4)$	$8n+13$

Minimum no. of moves for $(n+1)(n+5)$	
$P_{(n+1)(n+5)}$	$M_{(n+1)(n+5)}$
12	27
21	35
32	43
45	51
...	...
$(n+1)(n+5)$	$8n+19$

The following table summarizes the product and the minimum number of moves for $(n+1)(n+k)$ Klotski:

Product (P)	Minimum no. of moves (M)
$(n+1)(n+2)$	$8n+1$
$(n+1)(n+3)$	$8n+7$
$(n+1)(n+4)$	$8n+13$
$(n+1)(n+5)$	$8n+19$
$(n+1)(n+6)$	$8n+25$
$(n+1)(n+7)$	$8n+31$
...	...
$P=(n+1)(n+k)$	$M=8n+6k-11$

Formulation of $M=8n+6k-11$

$$M = 8n + (1 + [k - 2]6)$$
$$M = 8n + (1 + 6k - 12)$$
$$M = 8n + 1 + 6k - 12$$
$$M = 8n + 6k - 11 \qquad (5)$$

Derivation of n

$$P = (n+1)(n+k)$$
$$P = n^2 + kn + n + k$$
$$P = n^2 + (k+1)n + k$$
$$0 = n^2 + (k+1)n + k - P \qquad (6)$$

using quadratic formula to find n

$$n = \frac{-(k+1) \pm \sqrt{(k+1)^2 - 4(1)(k-P)}}{2(1)}$$

$$n = \frac{-k-1 \pm \sqrt{k^2+2k+1-4k+4P}}{2}$$

$$n = \frac{-k-1 \pm \sqrt{k^2-2k+1+4P}}{2}$$

$$n = \frac{-k-1 \pm \sqrt{(k-1)^2+4P}}{2} \tag{7}$$

Minimum number of moves in terms of the product (M_P)

$$M = 8n + 6k - 11$$

$$M_P = 8\left(\frac{-k-1 \pm \sqrt{(k-1)^2+4P}}{2}\right) + 6k - 11$$

$$M_P = 4\left(-k-1 \pm \sqrt{(k-1)^2+4P}\right) + 6k - 11$$

$$M_P = -4k - 4 \pm 4\sqrt{(k-1)^2+4P} + 6k - 11$$

$$M_P = 4\sqrt{(k-1)^2+4P} + 2k - 15 \tag{8}$$

Since k is the difference (d) of the two sides $+1$,

$$d = (n+k) - (n+1)$$

$$d = n + k - n - 1$$

$$d = k - 1$$

$$d + 1 = k$$

$$k = d + 1 \tag{9}$$

hence the minimum number of moves given the product and difference (M_{Pd}) is

$$M_{Pd} = 4\sqrt{((d+1)-1)^2+4P} + 2(d+1) - 15$$

$$M_{Pd} = 4\sqrt{(d+1-1)^2+4P} + 2d + 2 - 15$$

$$M_{Pd} = 4\sqrt{(d)^2+4P} + 2d - 13$$

$$M_{Pd} = 4\sqrt{d^2+4P} + 2d - 13 \tag{10}$$

Furthermore, if we let l and w be the number of columns and rows and $l > w$, then the minimum number of moves of a rectangular Klotski (M) is

$$M_{Pd} = 4\sqrt{d^2 + 4P} + 2d - 13$$

$$M = 4\sqrt{(l-w)^2 + 4(l)(w)} + 2(l-w) - 13$$

$$M = 4\sqrt{l^2 - 2lw + w^2 + 4lw} + 2l - 2w - 13$$

$$M = 4\sqrt{l^2 + 2lw + w^2} + 2l - 2w - 13$$

$$M = 4\sqrt{(l+w)^2} + 2l - 2w - 13$$

$$M = 4(l+w) + 2l - 2w - 13$$

$$M = 4l + 4w + 2l - 2w - 13$$

$$M = 6l + 2w - 13 \tag{11}$$

Formula for the Minimum Number of Moves of a Square Klotski given the number of spaces;

Table 4: Data of the Minimum number of moves for Square and Rectangular Klotski with 2 Spaces

x	1	2	3	4	5	6	7	...
1	0/	0/	0/	0/	0/	0/	0/	...
2	0/	2	6	10	16	22	28	...
3	0/		10	14	18	24	30	...
4	0/			18	22	26	32	...
5	0/				26	30	34	...
6	0/					34	38	...
7	0/						42	...
...	...	...	...	...	...	...	...	...

$$(n+1)(n+1) \text{ or } (n+1)^2 \; \{n \mid n \in \mathbb{Z}^+\}$$

Product(P)	Minimum No. of Moves (M)
4	2
9	10
16	18
25	26
36	34
49	42
$P = (n+1)^2$	$M = 8n - 6$

Formulation of $M = 8n - 6$

$$M = 2 + (n-1)8$$

$$M = 2 + 8n - 8$$

$$M = 8n - 6 \tag{12}$$

Derivation of n

$$P = (n+1)^2$$
$$\sqrt{P} = \sqrt{(n+1)^2}$$
$$\sqrt{P} = n+1$$
$$\sqrt{P} - 1 = n$$
$$n = \sqrt{P} - 1 \tag{13}$$

Minimum number of moves in terms of the product (M_P)

$$M_P = 8(\sqrt{P} - 1) - 6$$
$$M_P = 8\sqrt{P} - 8 - 6$$
$$M_P = 8\sqrt{P} - 14 \tag{14}$$

Since $\sqrt{P}$ is just equal to the number of rows/columns (s) of the square Klotski, then the minimum number of moves (M) given the 2 spaces is

$$M = 8s - 14 \tag{15}$$

Table 5: Data of the Minimum number of moves for Square and Rectangular Klotski with 3 Spaces

x	1	2	3	4	5	6	7	...
1	0/	0/	0/	0/	0/	0/	0/	...
2	0/	2	3	7	13	19	25	...
3	0/		7	11	17	23	29	...
4	0/			15	19	25	31	...
5	0/				23	27	33	...
6	0/					31	35	...
7	0/						39	...
...	...							...

$$(n+2)(n+2) \ or \ (n+2)^2 \ \{n | n \in \mathbb{Z}^+\}$$

Formulation of $M = 8n - 1$

$$M = 7 + (n-1)8$$
$$M = 7 + 8n - 8$$
$$M = 8n - 1 \tag{16}$$

Derivation of n

$$P = (n+2)^2$$
$$\sqrt{P} = \sqrt{(n+2)^2}$$
$$\sqrt{P} = n+2$$
$$\sqrt{P} - 2 = n$$
$$n = \sqrt{P} - 2 \tag{17}$$

Minimum number of moves in terms of the product (M_P)

$$M_P = 8(\sqrt{P} - 2) - 1$$
$$M_P = 8\sqrt{P} - 16 - 1$$
$$M_P = 8\sqrt{P} - 17 \quad (18)$$

Since $\sqrt{P}$ is just equal to the number of rows/columns (s) of the square Klotski, then the minimum number of moves (M) given the 3 spaces is

$$M = 8s - 17 \quad (19)$$

Table 6: Data of the Minimum number of moves for Square and Rectangular Klotski with 4 Spaces

x	1	2	3	4	5	6	7	...
1	0/	0/	0/	0/	0/	0/	0/	...
2	0/	0/	3	8	10	17	20	...
3	0/	3	7	8	15	16	22	...
4	0/	8	8	12	18	24	30	...
5	0/	8	15		20	26	32	...
6	0/	14	16			28	34	...
7	0/	20	22				36	...
...	...	...	...					...

$$(n+3)(n+3) \ or \ (n+3)^2 \ \{n | n \in \mathbb{Z}^+\}$$

Formulation of $M = 8n - 1$

$$M = 12 + (n-1)8$$
$$M = 12 + 8n - 8$$
$$M = 8n + 4 \quad (20)$$

Derivation of n

$$P = (n+3)^2$$
$$\sqrt{P} = \sqrt{(n+3)^2}$$
$$\sqrt{P} = n + 3$$
$$\sqrt{P} - 3 = n$$
$$n = \sqrt{P} - 3 \quad (21)$$

Minimum number of moves in terms of the product (M_P)

$$M_P = 8(\sqrt{P} - 3) + 4$$
$$M_P = 8\sqrt{P} - 24 + 4$$
$$M_P = 8\sqrt{P} - 20 \quad (22)$$

Since $\sqrt{P}$ is just equal to the number of rows/columns (s) of the square Klotski, then the minimum number of moves (M) given the 4 spaces is

$$M = 8s - 20 \tag{23}$$

The following table summarizes the number of spaces and the minimum number of moves for square Klotski whose number of sides is greater than or equal to the number of spaces:

No. of Spaces (S)	Minimum no. of moves (M)
1	$8s-11$
2	$8s-14$
3	$8s-17$
4	$8s-20$

$M = 8s - 3S - 8$

Formulation of $M = 8s - 3S - 8$

$M = (8s - 11) + (S - 1)(-3)$

$M = 8s - 11 - 3S + 3$

$$M = 8s - 3S - 8 \tag{24}$$

The above-mentioned formula (Eq. 24) calculates the minimum number of moves in a square Klotski which has spaces ranging from 1 to 4, where s is the number of rows or columns and S is the number of spaces in the puzzle.

Square Klotski with spaces ranging from 5 to ∞

Using the data from the following tables and following the same process in deriving the formula for the square Klotski that contains spaces ranging from 1 to 4, the investigators was able to derived a general formula that calculates the minimum number of moves for square Klotski that contains spaces ranging from 5 to infinity.

Table 7: Data of the Minimum number of moves for Square and Rectangular Klotski with 5 Spaces

x	4	5	6	7	8	9	10	. . .
4	0/	0/	0/	0/	0/	0/	0/	. . .
5	0/	19	25	31	37	43	49	. . .
6	0/		27	33	39	45	51	. . .
7	0/			35	41	47	53	. . .
8	0/				43	49	55	. . .
9	0/					51	57	. . .
10	0/						59	. . .
. . .	. . .							. . .

Table 8: Data of the Minimum number of moves for Square and Rectangular Klotski with 6 Spaces

x	5	6	7	8	9	10	11	12	...
5	0/	0/	0/	0/	0/	0/	0/	0/	...
6	0/	26	32	38	44	50	56	62	...
7	0/		34	40	46	52	58	64	...
8	0/			42	48	54	60	66	...
9	0/				50	56	62	68	...
10	0/					58	64	70	...
11	0/						66	72	...
12	0/							74	...
...	...								...

Table 9: Data of the Minimum number of moves for Square and Rectangular Klotski with 7 Spaces

x	6	7	8	9	10	11	12	13	...
6	0/	0/	0/	0/	0/	0/	0/	0/	...
7	0/	33	39	45	51	57	63	69	...
8	0/		41	47	53	59	65	71	...
9	0/			49	55	61	67	73	...
10	0/				57	63	69	75	...
11	0/					65	71	77	...
12	0/						73	79	...
13	0/							81	...
...	...								...

No. of Spaces (S)	Minimum no. of moves (M)
5	$8s - 21$
6	$8s - 22$
7	$8s - 23$
8	$8s - 24$
9	$8s - 25$
10	$8s - 26$
11	$8s - 27$
...	...
	$M = 8s - S - 16$

Formulation of $M = 8s - S - 16$

$$M = (8s - 21) + (S - 5)(-1)$$

$$M = 8s - 21 - S + 5$$

$$M = 8s - S - 16 \tag{25}$$

The above-mention formula (Eq. 25) calculates the minimum number of moves for a square Klotski given the number of rows or columns (s) and the number of spaces (S) and $4 \leq S \leq \infty$

Formula for the Minimum Number of Moves of a Rectangular Klotski given the number of spaces;

Table 10: Data of the Minimum number of moves for Square and Rectangular Klotski with 8 Spaces

x	7	8	9	10	11	12	13	14	...
7	0/	0/	0/	0/	0/	0/	0/	0/	...
8	0/	40	46	52	58	64	70	76	...
9	0/		48	54	60	66	72	78	...
10	0/			56	62	68	74	80	...
11	0/				64	70	76	82	...
12	0/					72	78	84	...
13	0/						80	86	...
14	0/							88	...
...	...								...

Table 11: Data of the Minimum number of moves for Square and Rectangular Klotski with 9 Spaces

x	8	9	10	11	12	13	14	15	...
8	0/	0/	0/	0/	0/	0/	0/	0/	...
9	0/	47	53	59	65	71	77	83	...
10	0/		55	61	67	73	79	85	...
11	0/			63	69	75	81	87	...
12	0/				71	77	83	89	...
13	0/					79	85	91	...
14	0/						87	93	...
15	0/							95	...
...	...								...

2 Spaces

Using the data in table 4 in formulating the formula for the minimum number of moves for rectangular Klotski containing 2 spaces, the investigators was able to derived the formula through the same process as equation 5, 6, 7, 8, 9, 10, and 11.

$$(n+1)(n+k)\ \{n|n \in \mathbb{Z}^+\}\ \{k|k \in \mathbb{Z}^+,\ k > 1\}$$

Product (P)	Minimum no. of moves (M)
$(n+1)(n+2)$	$8n-2$
$(n+1)(n+3)$	$8n+2$
$(n+1)(n+4)$	$8n+8$
$(n+1)(n+5)$	$8n+14$
$(n+1)(n+6)$	$8n+20$
$(n+1)(n+7)$	$8n+26$
...	...

$P = (n+1)(n+k)$

In the above-shown table, the row that is highlighted in red has a separated formula. Hence, only calculates the minimum number of moves in a rectangular Klotski that has the dimension of $(n+1)(n+2)$. This is due to the constant of the explicit formula $8n-2$ since it does not coincide with the common difference of 6 if it is included in the sequence.

Table 12: Data of the Minimum number of moves for Square and Rectangular Klotski with 10 Spaces

x	9	10	11	12	13	14	15	16	...
9	0/	0/	0/	0/	0/	0/	0/	0/	...
10	0/	54	60	66	72	78	84	90	...
11	0/		62	68	74	80	86	92	...
12	0/			70	76	82	88	94	...
13	0/				78	84	90	96	...
14	0/					86	92	98	...
15	0/						94	100	...
16	0/							102	...
...	...								...

Table 13: Data of the Minimum number of moves for Square and Rectangular Klotski with 11 Spaces

x	10	11	12	13	14	15	16	17	...
10	0/	0/	0/	0/	0/	0/	0/	0/	...
11	0/	61	67	73	79	85	91	97	...
12	0/		69	75	81	87	93	99	...
13	0/			77	83	89	95	101	...
14	0/				85	91	97	103	...
15	0/					93	99	105	...
16	0/						101	107	...
17	0/							109	...
...	...								...

Formulation of the Formula of $8n-2$

$$\begin{aligned} P &= (n+1)(n+k) \\ P &= n^2 + kn + n + k \\ P &= n^2 + (k+1)n + k \\ 0 &= n^2 + (k+1)n + k - P \end{aligned} \tag{26}$$

using the quadratic equation to find n

$$\begin{aligned} n &= \frac{-(k+1) \pm \sqrt{(k+1)^2 + 4(1)(k-P)}}{2(1)} \\ n &= \frac{-k-1 \pm \sqrt{k^2 + 2k + 1 - 4k + 4P}}{2} \\ n &= \frac{-k-1 \pm \sqrt{k^2 - 2k + 1 + 4P}}{2} \\ n &= \frac{-k-1 \pm \sqrt{(k-1)^2 + 4P}}{2} \end{aligned} \tag{27}$$

Minimum number of moves in terms of the product (M_P)

$$M_P = 8\left(\frac{-k-1 \pm \sqrt{(k-1)^2+4P}}{2}\right) - 2$$
$$M_P = 4\left(-k-1 \pm \sqrt{(k-1)^2+4P}\right) - 2$$
$$M_P = -4k - 4 \pm 4\sqrt{(k-1)^2+4P} - 2$$
$$M_P = 4\sqrt{(k-1)^2+4P} - 4k - 6 \tag{28}$$

since, $k = 2$. then,

$$M_P = 4\sqrt{(k-1)^2+4P} - 4k - 6$$
$$M_P = 4\sqrt{(2-1)^2+4P} - 4(2) - 6$$
$$M_P = 4\sqrt{(1)^2+4P} - 8 - 6$$
$$M_P = 4\sqrt{1+4P} - 14 \tag{29}$$

Furthermore, if we let l and w the number of columns and rows and $l > w$, then

$$M = 4\sqrt{1+4P} - 14$$
$$M = 4\sqrt{1+4(l)(w)} - 14$$
$$M = 4\sqrt{1+4lw} - 14 \tag{30}$$

Derivation of the formula for the sequence starting from $8n+2$;

$$(n+1)(n+k)\ \{n|n \in \mathbb{Z}^+\}\ \{k|k \in \mathbb{Z}^+,\ k > 2\}$$

The following table summarizes the product and the minimum number of moves for a $(n+1)(n+k)$ Klotski, where $k > 2$:

Product (P)	Minimum no. of moves (M)
$(n+1)(n+3)$	$8n+2$
$(n+1)(n+4)$	$8n+8$
$(n+1)(n+5)$	$8n+14$
$(n+1)(n+6)$	$8n+20$
$(n+1)(n+7)$	$8n+26$
...	...
$P = (n+1)(n+k)$	$M = 8n+6k-16$

Formulation of $M = 8n+6k-16$

$$M = (8n+2) + (k-3)6$$
$$M = 8n+2+6k-18$$
$$M = 8n+6k-16 \tag{31}$$

using the quadratic equation to find n (See Eq. 27 for the solution)

$$n = \frac{-k-1 \pm \sqrt{(k-1)^2+4P}}{2}$$

Minimum number of moves in terms of the product (M_P)

$$M_P = 8\left(\frac{-k-1 \pm \sqrt{(k-1)^2+4P}}{2}\right) + 6k - 16$$

$$M_P = 4\left(-k-1 \pm \sqrt{(k-1)^2+4P}\right) + 6k - 16$$

$$M_P = -4k - 4 \pm 4\sqrt{(k-1)^2+4P} + 6k - 16$$

$$M_P = 4\sqrt{(k-1)^2+4P} + 2k - 20 \tag{32}$$

the difference of the two sides $+1$ (See Eq. 9 for the solution),

$$k = d + 1$$

hence the minimum number of moves given the product and difference (M_{Pd}) is

$$M_{Pd} = 4\sqrt{((d+1)-1)^2+4P} + 2(d+1) - 20$$

$$M_{Pd} = 4\sqrt{(d+1-1)^2+4P} + 2d + 2 - 20$$

$$M_{Pd} = 4\sqrt{(d)^2+4P} + 2d - 18$$

$$M_{Pd} = 4\sqrt{d^2+4P} + 2d - 18 \tag{33}$$

Furthermore, if we let l and w be the number of columns and rows and $l > w$, then the minimum number of moves of a rectangular Klotski (M) is

$$M_{Pd} = 4\sqrt{d^2+4P} + 2d - 18$$

$$M = 4\sqrt{(l-w)^2+4(l)(w)} + 2(l-w) - 18$$

$$M = 4\sqrt{l^2-2lw+w^2+4lw} + 2l - 2w - 18$$

$$M = 4\sqrt{l^2+2lw+w^2} + 2l - 2w - 18$$

$$M = 4\sqrt{(l+w)^2} + 2l - 2w - 18$$

$$M = 4(l+w) + 2l - 2w - 18$$

$$M = 4l + 4w + 2l - 2w - 18$$

$$M = 6l + 2w - 18 \tag{34}$$

3 Spaces

Using the data in table 5 and following the same solutions, here are the derived formulae;

$$(n+1)(n+k)\ \{n|n \in \mathbb{Z}^+\}\ \{k|k \in \mathbb{Z}^+,\ k > 2\}$$

The following table summarizes the product and the minimum number of moves in a $(n + 2)(n + k)$ Klotski:

Product (P)	Minimum no. of moves (M)
$(n+2)(n+3)$	$8n+3$
$(n+2)(n+4)$	$8n+9$
$(n+2)(n+5)$	$8n+15$
$(n+2)(n+6)$	$8n+21$
$(n+2)(n+7)$	$8n+27$
$(n+2)(n+8)$	$8n+33$
...	...
$P = (n+2)(n+k)$	$M = 8n+6k-15$

Formulation of $M = 8n + 6k - 15$

$$
\begin{aligned}
M &= 8n + (3 + [k-3]6) \\
M &= 8n + (3 + 6k - 18) \\
M &= 8n + 3 + 6k - 18 \\
M &= 8n + 6k - 15 \qquad (35)
\end{aligned}
$$

Derivation of n

$$
\begin{aligned}
P &= (n+2)(n+k) \\
P &= n^2 + kn + 2n + 2k \\
P &= n^2 + (k+2)n + 2k \\
0 &= n^2 + (k+2)n + 2k - P \qquad (36)
\end{aligned}
$$

using the quadratic equation to find n

$$
\begin{aligned}
n &= \frac{-(k+2) \pm \sqrt{(k+2)^2 + 4(1)(2k-P)}}{2(1)} \\
n &= \frac{-k-2 \pm \sqrt{k^2+4k+4-8k+4P}}{2} \\
n &= \frac{-k-2 \pm \sqrt{k^2-4k+4+4P}}{2} \\
n &= \frac{-k-2 \pm \sqrt{(k-2)^2+4P}}{2} \qquad (37)
\end{aligned}
$$

Minimum number of moves in terms of the product (M_P)

$$
\begin{aligned}
M_P &= 8\left(\frac{-k-2 \pm \sqrt{(k-2)^2+4P}}{2}\right) + 6k - 15 \\
M_P &= 4\left(-k-2 \pm \sqrt{(k-2)^2+4P}\right) + 6k - 15 \\
M_P &= -4k - 8 \pm 4\sqrt{(k-2)^2+4P} + 6k - 15
\end{aligned}
$$

$$M_P = 4\sqrt{(k-2)^2 + 4P} + 2k - 23 \tag{38}$$

the difference of the two sides +2

$$\begin{aligned} d &= (n+k) - (n+2) \\ d &= n+k-n-2 \\ d &= k-2 \\ k &= d+2 \end{aligned} \tag{39}$$

hence the minimum number of moves given the product and difference (M_{Pd}) is

$$\begin{aligned} M_{Pd} &= 4\sqrt{((d+2)-2)^2 + 4P} + 2(d+2) - 23 \\ M_{Pd} &= 4\sqrt{d^2 + 4P} + 2d + 4 - 23 \\ M_{Pd} &= 4\sqrt{d^2 + 4P} + 2d - 19 \end{aligned} \tag{40}$$

Furthermore, if we let l and w be the number of columns and rows and $l > w$, then the minimum number of moves of a rectangular Klotski (M) is

$$\begin{aligned} M_{Pd} &= 4\sqrt{d^2 + 4P} + 2d + 4 - 23 \\ M &= 4\sqrt{(l-w)^2 + 4(l)(w)} + 2(l-w) - 19 \\ M &= 4\sqrt{l^2 + 2lw + w^2 + 4lw} + 2l - 2w - 19 \\ M &= 4\sqrt{l^2 + 2lw + w^2} + 2l - 2w - 19 \\ M &= 4\sqrt{(l+w)^2} + 2l - 2w - 19 \\ M &= 4(l+w) + 2l - 2w - 19 \\ M &= 4l + 4w + 2l - 2w - 19 \\ M &= 6l + 2w - 19 \end{aligned} \tag{41}$$

4 Spaces

Using the data from table 6, the derived formulae are;

$$(n+1)(n+k)\ \{n|n \in \mathbb{Z}^+\}\ \{k|k \in \mathbb{Z}^+,\ k > 3\}$$

The following tables summarizes the product and the minimum number of moves for a $(n+3)(n+k)$ Klotski:

Product (P)	Minimum no. of moves (M)
$(n+3)(n+4)$	$8n+10$
$(n+3)(n+5)$	$8n+16$
$(n+3)(n+6)$	$8n+22$
$(n+3)(n+7)$	$8n+28$
$(n+3)(n+8)$	$8n+34$
$(n+3)(n+9)$	$8n+40$

...	...
$P = (n+3)(n+k)$	$M = 8n + 6k - 14$

Formulation of $M = 8n + 6k - 14$

$$\begin{aligned} M &= 8n + (10 + [k-4]6) \\ M &= 8n + (10 + 6k - 24) \\ M &= 8n + 10 + 6k - 24 \\ M &= 8n + 6k - 14 \end{aligned} \tag{42}$$

Derivation of n

$$\begin{aligned} P &= (n+3)(n+k) \\ P &= n^2 + kn + 3n + 3k \\ P &= n^2 + (k+3)n + 3k \\ 0 &= n^2 + (k+3)n + 3k - P \end{aligned} \tag{43}$$

using the quadratic equation to find n

$$\begin{aligned} n &= \frac{-(k+3) \pm \sqrt{(k+3)^2 + 4(1)(3k-P)}}{2(1)} \\ n &= \frac{-k-3 \pm \sqrt{k^2 + 6k + 9 - 12k + 4P}}{2} \\ n &= \frac{-k-3 \pm \sqrt{k^2 - 6k + 9 + 4P}}{2} \\ n &= \frac{-k-3 \pm \sqrt{(k-3)^2 + 4P}}{2} \end{aligned} \tag{44}$$

Minimum number of moves in terms of the product (M_P)

$$\begin{aligned} M_P &= 8\left(\frac{-k-3 \pm \sqrt{(k-3)^2 + 4P}}{2}\right) + 6k - 14 \\ M_P &= 4\left(-k-3 \pm \sqrt{(k-3)^2 + 4P}\right) + 6k - 14 \\ M_P &= -4k - 12 \pm 4\sqrt{(k-3)^2 + 4P} + 6k - 14 \\ M_P &= 4\sqrt{(k-3)^2 + 4P} + 2k - 26 \end{aligned} \tag{45}$$

the difference of the two sides $+3$

$$\begin{aligned} d &= (n+k) - (n+3) \\ d &= n + k - n - 3 \\ d &= k - 3 \\ k &= d + 3 \end{aligned} \tag{46}$$

hence the minimum number of moves given the product and difference (M_{Pd}) is

$$\begin{aligned} M_{Pd} &= 4\sqrt{((d+3)-3)^2 + 4P} + 2(d+3) - 26 \\ M_{Pd} &= 4\sqrt{d^2 + 4P} + 2d + 6 - 26 \\ M_{Pd} &= 4\sqrt{d^2 + 4P} + 2d - 20 \end{aligned} \tag{47}$$

Furthermore, if we let l and w be the number of columns and rows and $l > w$, then the minimum number of moves of a rectangular Klotski (M) is

$$M_{Pd} = 4\sqrt{d^2 + 4P} + 2d - 20$$
$$M = 4\sqrt{(l-w)^2 + 4(l)(w)} + 2(l-w) - 20$$
$$M = 4\sqrt{l^2 + 2lw + w^2 + 4lw} + 2l - 2w - 20$$
$$M = 4\sqrt{l^2 + 2lw + w^2} + 2l - 2w - 20$$
$$M = 4\sqrt{(l+w)^2} + 2l - 2w - 20$$
$$M = 4(l+w) + 2l - 2w - 20$$
$$M = 4l + 4w + 2l - 2w - 20$$
$$M = 6l + 2w - 20 \tag{48}$$

5 Spaces

$$(n+1)(n+k)\ \{n | n \in \mathbb{Z}^+\}\ \{k | k \in \mathbb{Z}^+,\ k > 4\}$$

The following tables summarizes the product and the minimum number of moves in a $(n+4)(n+k)$ Klotski:

Product (P)	Minimum no. of moves (M)
$(n+4)(n+5)$	$8n+17$
$(n+4)(n+6)$	$8n+23$
$(n+4)(n+7)$	$8n+29$
$(n+4)(n+8)$	$8n+35$
$(n+4)(n+9)$	$8n+41$
$(n+4)(n+10)$	$8n+47$
...	...
$P = (n+4)(n+k)$	$M = 8n + 6k - 13$

Formulation of $M = 8n + 6k - 13$

$$M = 8n + (17 + [k-5]6)$$
$$M = 8n + (17 + 6k - 30)$$
$$M = 8n + 17 + 6k - 30$$
$$M = 8n + 6k - 13 \tag{49}$$

Derivation of n

$$P = (n+4)(n+k)$$
$$P = n^2 + kn + 4n + 4k$$
$$P = n^2 + (k+4)n + 4k$$

$$0 = n^2 + (k + 4)n + 4k - P \tag{50}$$

using the quadratic equation to find n

$$n = \frac{-(k+4) \pm \sqrt{(k+4)^2 + 4(1)(4k-P)}}{2(1)}$$
$$n = \frac{-k-4 \pm \sqrt{k^2+8k+16-16k+4P}}{2}$$
$$n = \frac{-k-4 \pm \sqrt{k^2-8k+16+4P}}{2}$$
$$n = \frac{-k-4 \pm \sqrt{(k-4)^2+4P}}{2} \tag{51}$$

Minimum number of moves in terms of the product (M_P)

$$M_P = 8\left(\frac{-k-4 \pm \sqrt{(k-4)^2+4P}}{2}\right) + 6k - 13$$
$$M_P = 4(-k-4 \pm \sqrt{(k-4)^2+4P}) + 6k - 13$$
$$M_P = -4k - 16 \pm 4\sqrt{(k-4)^2+4P} + 6k - 13$$
$$M_P = 4\sqrt{(k-4)^2+4P} + 2k - 29 \tag{52}$$

the difference of the two sides +4

$$d = (n+k) - (n+4)$$
$$d = n+k-n-4$$
$$d = k-4$$
$$k = d+4 \tag{53}$$

hence the minimum number of moves given the product and difference (M_{Pd}) is

$$M_{Pd} = 4\sqrt{((d+4)-4)^2+4P} + 2(d+4) - 29$$
$$M_{Pd} = 4\sqrt{d^2+4P} + 2d + 8 - 29$$
$$M_{Pd} = 4\sqrt{d^2+4P} + 2d - 21 \tag{54}$$

Furthermore, if we let l and w be the number of columns and rows and $l > w$, then the minimum number of moves of a rectangular Klotski (M) is

$$M_{Pd} = 4\sqrt{d^2+4P} + 2d - 21$$

$$M = 4\sqrt{(l-w)^2+4(l)(w)} + 2(l-w) - 21$$
$$M = 4\sqrt{l^2+2lw+w^2+4lw} + 2l - 2w - 21$$
$$M = 4\sqrt{l^2+2lw+w^2} + 2l - 2w - 21$$
$$M = 4\sqrt{(l+w)^2} + 2l - 2w - 21$$
$$M = 4(l+w) + 2l - 2w - 21$$
$$M = 4l + 4w + 2l - 2w - 21$$
$$M = 6l + 2w - 21$$

The following table summarizes the product and the minimum number of moves given the product and the difference for a $(n+f)(n+k)$ rectangular Klotski, where $f \geq 1$, $k > 2$, $k > f$, $f < S$, and $2 \leq S \leq \infty$:

No. of Spaces (S)	Minimum no. of moves (M_{Pd})
2	$4\sqrt{d^2+4P}+2d-18$
3	$4\sqrt{d^2+4P}+2d-19$
4	$4\sqrt{d^2+4P}+2d-20$
5	$4\sqrt{d^2+4P}+2d-21$
6	$4\sqrt{d^2+4P}+2d-22$
...	...

Formulation of $M_{Pd} = 4\sqrt{d^2+4P}+2d-S-16$

$$M_{Pd} = (4\sqrt{d^2+4P}+2d-18)+(S-2)(-1)$$

$$M_{Pd} = 4\sqrt{d^2+4P}+2d-18-S+2$$

$$M_{Pd} = 4\sqrt{d^2+4P}+2d-S-16 \tag{56}$$

The above-mentioned formula (Eq. 56) calculates the minimum number of moves required to solve the rectangular Klotski puzzle whose product is $(n+f)(n+k)$, where $f \geq 1$, $k > 2$, $k > f$, $f < S$, and $2 \leq S \leq \infty$ and if given the product and the difference of the sides.

The following table summarizes the product and the minimum number of moves in a $(n+f)(n+k)$ rectangular Klotski, where $f \geq 1$, $k > 2$, $k > f$, $f < S$, and $2 \leq S \leq \infty$:

No. of Spaces (S)	Minimum no. of moves (M)
2	$6l+2w-18$
3	$6l+2w-19$
4	$6l+2w-20$
5	$6l+2w-21$
6	$6l+2w-22$
...	...

Formulation of $M = 6l+2w-S-16$

$$M = (6l+2w-18)+(S-2)(-1)$$

$$M = 6l+2w-18-S+2$$

$$M = 6l+2w-S-16 \tag{57}$$

The above-mentioned formula (Eq. 57) calculates the minimum number of moves required to solve the rectangular Klotski puzzle whose product is $(n+f)(n+k)$, where $f \geq 1$, $k > 2$, $k > f$, $f < S$, and $2 \leq S \leq \infty$.

General Formulae

General Formula for Square Klotski:

$$M = \begin{cases} 8s - 3S - 8 & \text{if } 1 \le S \le 4,\ s \ge S \\ 8s - S - 16 & \text{if } 4 \le S \le \infty,\ s \ge S \end{cases} \tag{58}$$

General Formula for Rectangular Klotski:

$$M = \begin{cases} 4\sqrt{1+4lw} - 14 & \text{if } (n+1)(n+2) \\ 6l + 2w - 13 & \text{if } S = 1 \\ 6l + 2w - S - 16 & \text{if } (n+f)(n+k) \\ & \text{where } f \ge 1, \\ & k > 2,\ k > f, \\ & f < S, \\ & 2 \le S \le \infty \end{cases} \tag{59}$$

General Formula for Rectangular Klotski given the difference (d) and the product (P):

$$M_{Pd} = \begin{cases} 4\sqrt{1+4P} - 14 & \text{if } (n+1)(n+2) \\ 4\sqrt{d^2+4P} + 2d - 13 & \text{if } S = 1 \\ 4\sqrt{d^2+4P} + 2d - S - 16 & \text{if } (n+f)(n+k) \\ & \text{where } f \ge 1, \\ & k > 2,\ k > f, \\ & f < S, \\ & 2 \le S \le \infty \end{cases} \tag{60}$$

M - Minimum number of Moves
P - Product ($l \times w$)
d - Difference ($l - w$)
s - Number of rows or columns
l - Number of Columns
w - Number of Rows
S - Number of Spaces

Examples

For Formula $M = 8s - 3S - 8$ (Eq. 58):

In a square Klotski with a 5×5 dimension with 4 spaces, by applying the formula $M = 8s - 3S - 8$, the minimum number of moves is

$$M = 8(5) - 3(4) - 8$$
$$M = 40 - 12 - 8$$
$$M = 20$$

For Formula $M = 8s - S - 16$ (Eq. 58):

In a square Klotski with a 8×8 dimension with 6 spaces, by applying the formula $M = 8s - S - 16$, the minimum number of moves is

$$M = 8(8) - 6 - 16$$
$$M = 64 - 6 - 16$$
$$M = 42$$

For Formula $M = 4\sqrt{1 + 4lw} - 14$ (Eq. 59):

In a rectangular Klotski with a 3×4 dimension, by applying the formula $M = 4\sqrt{1 + 4lw} - 14$, the minimum number of moves is

$$M = 4\sqrt{1 + 4(4)(3)} - 14$$
$$M = 4\sqrt{1 + 48} - 14$$
$$M = 4\sqrt{49} - 14$$
$$M = 4(7) - 14$$
$$M = 28 - 14$$
$$M = 14$$

For Formula $M = 6l + 2w - 13$ (Eq. 59):

In a rectangular Klotski with a 3×7 dimension, by applying the formula $M = 6l + 2w - 13$, the minimum number of moves is

$$M = 6(7) + 2(3) - 13$$
$$M = 42 + 6 - 13$$
$$M = 35$$

For Formula $M = 6l + 2w - S - 16$ (Eq. 59):

In a rectangular Klotski with a 9×12 dimension with 8 spaces, by applying the formula $M = 6l + 2w - S - 16$, the minimum number of moves is

$$M = 6(12) + 2(9) - 8 - 16$$
$$M = 72 + 18 - 8 - 16$$
$$M = 66$$

For Formula $M_{Pd} = 4\sqrt{1 + 4P} - 14$ (Eq. 60)

In a rectangular Klotski that has the product of 12, by applying the formula $M_{Pd} = 4\sqrt{1 + 4P} - 14$, the minimum number of moves is

$$M_{Pd} = 4\sqrt{1 + 4(12)} - 14$$
$$M_{Pd} = 4\sqrt{49} - 14$$
$$M_{Pd} = 4(7) - 14$$
$$M_{Pd} = 28 - 14$$
$$M_{Pd} = 14$$

For Formula $M_{Pd} = 4\sqrt{d^2 + 4P} + 2d - 13$ (Eq. 60)

In a rectangular Klotski whose product and difference of the dimensions are 21 and 4, respectively, by applying the formula for the product and difference, the minimum number of moves is

$$M_{Pd} = 4\sqrt{4^2 + 4(21)} + 2(4) - 13$$
$$M_{Pd} = 4\sqrt{16 + 84} + 8 - 13$$
$$M_{Pd} = 4\sqrt{100} - 5$$
$$M_{Pd} = 4(10) - 5$$
$$M_{Pd} = 40 - 5$$
$$M_{Pd} = 35$$

For Formula $M_{Pd} = 4\sqrt{d^2 + 4P} + 2d - S - 16$ (Eq. 60)

In a rectangular Klotski with a product and a difference of 108 and 3 respectively and a spaces of 8, by applying the formula $M_{Pd} = 4\sqrt{d^2 + 4P} + 2d - S - 16$, the minimum number of moves is

$$M_{Pd} = 4\sqrt{(3)^2 + 4(108)} + 2(3) - (8) - 16$$
$$M_{Pd} = 4\sqrt{9 + 432} + 6 - 8 - 16$$
$$M_{Pd} = 4\sqrt{441} + 6 - 8 - 16$$
$$M_{Pd} = 4(21) + 6 - 8 - 16$$
$$M_{Pd} = 66$$

4 Chapter IV

4.1 Summary of the Study

Therefore, the investigators came up with a technique in the game Klotski Puzzle where the player must move the tile diagonally from its starting point to its predetermined spot in order to obtain the minimum number of moves. Varied formulae were derived and formulated to determine the minimum number of moves based on the dimensions and number of spaces in the predetermined spot of the Klotski puzzle. The investigators used the piecewise function, the quadratic formula, and the arithmetic sequence formed by the pattern whose common difference are 8 and 6 as their basis in developing these formulae.

Thus, this study shows potential not limiting only in the field of games but also in some real life situation that involves algorithm that calculate the minimum number of moves needed to reach a specific goal such as parking assistance system, traffic management, cargo handling, space optimization, and other situations that deals with constraints and involves the process of optimizing movement.

4.2 Recommendation

The investigators recommend to study further not just on square and rectangles but also on triangle and other shaped Klotskis. The movement of more than 1 block can also be taken into consideration in the future conduct of related investigation. Also, blocks with diferent shapes and sizes will be a good topic for investigation. Lastly, the investigators of this study recommends future researchers to investigate the relevance of the results of this study and to assess the feasibility of the suggested potential real life application.

5 Appendices

5.1 Source Code

```
\documentclass[11pt]{article}
\usepackage[margin=1in]{geometry}
\usepackage{amsmath, amssymb, bbm}
\usepackage{graphicx}
\usepackage{xcolor, soul}
\usepackage{multicol}
\usepackage{colortbl, booktabs}
\sethlcolor{yellow}
\usepackage{calc}
\usepackage{mathtools}
\usepackage{mathptmx}
\usepackage{color}
\usepackage{amsfonts}
\title{\textbf{Exploring Klotski: An Investigation of the Minimum Number of Moves in a Special Case of Slide Puzzles}}
\author{Axelcris G. Suladay \\ Mergel Ann G. Millendez \\ Trisha B. Vista}
\date{December 2024}
\usepackage[firstpage=true]{background}
\backgroundsetup{
  scale=3,
  angle=0,
  firstpage=false,
  opacity=0.20,
  contents={\begin{tikzpicture}[remember picture,overlay]
      \node at ([yshift=0pt,xshift=0pt]current page.center)
      {\includegraphics[width=1.5in]{"FRNHS LOGO"}};
\end{tikzpicture} }}

\usepackage{xcolor}
\usepackage{listings}

\lstset
{
  language={[LaTeX]TeX},
  %alsolanguage={PGF/TikZ},
  frame=single,
  framesep=\fboxsep,
  framerule=\fboxrule,
  rulecolor=\color{red},
  xleftmargin=\dimexpr\fboxsep+\fboxrule,
  xrightmargin=\dimexpr\fboxsep+\fboxrule,
  breaklines=true,
  basicstyle=\small\tt,
  keywordstyle=\color{red}\sf,
  identifierstyle=\color{magenta},
  commentstyle=\color{gray},
  %backgroundcolor=\color{white},
```

```
    tabsize=2,
    columns=flexible,
}

\DeclareUnicodeCharacter{2205}{$\emptyset$}

\newlength\dlf
\newcommand\alignedbox[2]{
  % #1 = before alignment
  % #2 = after alignment
  &
  \begingroup
  \settowidth\dlf{$\displaystyle #1$}
  \addtolength\dlf{\fboxsep+\fboxrule}
  \hspace{-\dlf}
  \fcolorbox{white}{yellow}{\Large$\displaystyle #1 #2$}
  \endgroup
}

\newcommand\alignedboxn[2]{
  % #1 = before alignment
  % #2 = after alignment
  &
  \begingroup
  \settowidth\dlf{$\displaystyle #1$}
  \addtolength\dlf{\fboxsep+\fboxrule}
  \hspace{-\dlf}
  \fcolorbox{white}{yellow}{$\displaystyle #1 #2$}
  \endgroup
}

\definecolor{RED}{HTML}{FFC7CE}

\begin{document}
  \onecolumn
  \thispagestyle{empty}\setcounter{page}{0}%
  \textcolor{white}{hgghjg}
  \vfill
  \begin{center}
    \Huge \textbf{Exploring Klotski: An Investigation of the Minimum Number of Moves in a Special Case of Slide Puzzles}
    \vspace*{0.25in} \\
    \normalsize Axelcris G. Suladay\textsuperscript{1}, Mergel Ann G. Millendez\textsuperscript{1}, Trisha B. Vista\textsuperscript{1} \\
    \textsuperscript{1} Francisco Ramos National High School
  \end{center}
  \vfill
  \newpage
  \maketitle
  \begin{abstract}
    Klotski is a special type of sliding puzzle that first emerged in the early 20th century. It refers to a variety of sliding block or tile games with the goal of moving a specific block
```

```
        to a predetermined spot. By changing the game's␣rule␣that␣
        requires␣a␣player␣to␣move␣a␣certain␣tile␣or␣block␣from␣one␣
        corner␣of␣the␣puzzle␣to␣its␣opposite␣corner,␣providing␣that␣
        it␣contains␣spaces␣more␣than␣or␣equal␣to␣one,␣this␣inquiry␣
        sought␣to␣ascertain␣the␣minimum␣number␣of␣moves.␣It␣was␣also␣
        recognized␣that␣the␣Klotski␣puzzle's dimension was directly
        influenced by the number of spaces.
    The results of the investigation proposed the use of these
        formulae in determining the minimum number of moves ($M$)
        given the number of columns/rows of a square-shaped Klotski:
        a.  $M = 8-s11$ if the number of spaces ($S$) is equal to $1$,
        b.  $M = 8--s3S8$ if the number of spaces ($S$) is greater than
        or equal to $1$ but less than or equal to $4$, c. $M = 8--
        sS16$ if the number of spaces ($S$) is greater than or equal
        to $4$ but less than or equal to infinity and the
        number of rows/columns ($s$) is greater than or equal to the
        number of spaces ($S$). Alternatively, $3$ formulae were
        proposed to determine the minimum number of moves ($M$) of a
        rectangular-shaped Klotski: a. $M= 6l+2w-13$ if the number of
        spaces is equal to $1$, b. $M=4\sqrt{1+4lw}-14$ if the
        product of the dimension is the quantity of $n$ plus $1$
        multiplied by the quantity of $n$ plus $2$, and c.
        $M=6---l2wS16$ if the product of the dimension is the quantity
        of $n$ plus $f$ multiplied by the quantity of $n$ plus $k$,
        where $f$ is greater than or equal to $1$, $k$ is greater
        than $2$, $k$ is greater than $f$, $f$ is less than the
        number of spaces ($S$), and lastly if the number of spaces is
        greater than or equal to $2$ but less than or equal to
        infinity. In the case where only the product ($P$) and
        difference ($d$) of the dimensions are given, the formula
        $M_{Pd} = 4\sqrt{d^2+4P}+2d-13$ may be used to determine the
        minimum number of moves if the number of spaces is equal to
        $1$, while $M_{Pd} = 4\sqrt{1+4P}-14$ is used only if the
        product of the dimension is the quantity of $n$ plus $1$
        multiplied by the quantity of $n$ plus $2$, and $M_{Pd} =
        4\sqrt{d^2+4P}+2d-S-16$ will be utilized if the product of
        the dimension is the quantity of $n$ plus $f$ multiplied by
        the quantity of $n$ plus $k$, where $f$ is greater than or
        equal to $1$, $k$ is greater than $2$, $k$ is greater than
        $f$, $f$ is less than the number of spaces ($S$), and the
        number of spaces is greater than or equal to $2$ but less
        than or equal to infinity. It is recommended that additional
        investigations must be done for other cases of the puzzle,
        such as with triangles and other shaped Klotski puzzles, with
        consideration of the movement of more than $1$ block and to
        determine the other possible real-life application of the
        concepts and result in this study.
\end{abstract}
\begin{center}
    \textit{Arithmetic, Klotski, Minimum Number of Moves, Pattern,
        Sliding Puzzle, Quadratic Formula, Piecewise function}
\end{center}

\newpage
```

```
\section{Chapter I}
\subsection{Introduction}
\paragraph{ }
Klotski is a type of a sliding puzzle that have started and gained
    popularity in the early 20th century. It refers to a whole
    group of similar sliding-block puzzles with a same objective of
    moving a certain block to a predetermined location
    \cite{sliding}. Within the frame, the blocks can be moved
    anywhere by —slidingnot turning, lifting or jumping. To complete
    the sliding puzzle, the starting and finishing positions are
    typically  provided  \cite{kiddle}.
\begin{figure}[h!]
  \centering
  \includegraphics[width=3in]{Figures/Klotski_Puzzle}
  \caption{Sample of a Mobile Game Klotski}
  \label{fig:klotskipuzzle}
\end{figure}

Upon watching a player do the Klotski puzzle by moving a tile from
     one place to another, the investigators observed some patterns
    that are related and was tackled in their Mathematics class in
    the previous  school year. This urges them to perform a short
    experimentation of moving a tile in a diagonal direction in
    order to find the minimum number of moves and to formulate a
    mathematical investigation about the Klotski puzzle in hopes of
    discovering concepts that may contribute in the study of
    mathematics.

Thus, this study focuses on exploring the Klotski puzzle and to
    generate formulas out of the number of moves it needed to
    complete the puzzle. This investigation also aims to determine
    the minimum possible number of moves in solving a square or
    rectangular-shaped Klotski with the aid of the different
    mathematical concepts, particularly the arithmetic sequence.

Like other varieties of sliding puzzle, Klotski is also known as a
    challenging game that brings entertainment to people in all
    ages. This game also enhances an individual's␣problem␣solving␣
    skills,␣critical␣thinking␣skills␣and␣other␣mathematical␣related␣
    skills.␣Moreover,␣this␣game␣might␣not␣just␣be␣a␣"simple␣game"␣
    for␣it␣has␣the␣potential␣to␣unfold␣mathematical␣concepts␣that␣
    have␣a␣significant␣contribution␣to␣mathematical␣research.
␣␣␣␣␣␣␣␣␣␣\subsection{Significance␣of␣the␣Study}
␣␣␣␣␣␣␣␣␣␣\paragraph{␣}
␣␣␣␣␣␣␣␣␣␣This␣study␣adds␣to␣the␣corpus␣of␣information␣for␣everyone,␣
    especially␣for␣gamers,␣and␣it␣is␣noteworthy␣since␣the␣idea␣might␣be␣used␣
    in␣other␣games.␣This␣investigation␣makes␣fresh␣contribution␣to␣the␣field␣
    of␣games␣and␣geometry,␣due␣to␣its␣usage␣and␣relation␣to␣the␣concept␣of␣
    sequence␣and␣patterns.␣With␣the␣success␣of␣this␣study,␣not␣only␣students␣␣
    but␣everyone␣else␣may␣be␣able␣to␣benefit␣from␣this␣study␣by␣using␣the␣
    concepts␣in␣this␣investigation␣in␣the␣future␣and␣pass␣this␣knowledge␣to␣
    the␣next␣generation.␣Lastly,␣the␣investigators␣of␣this␣study␣are␣hoping␣
    that␣other␣investigators␣would␣find␣the␣relevance␣of␣this␣study␣into␣its␣
    real␣life␣application.
```

```
          \subsection{Statement of the Problem}
          \paragraph{ }
          The aim of the game under investigation is to slide  a corner tile/block from its starting position (upper left corner) to its opposite corner (bottom right) in the least possible number of moves. In addition, each block/tile can only be moved up, down, left, right and one at a time. Diagonal movement and lifting the blocks are not permitted. There can be 1 or more spaces and the starting free space of the puzzle is located in the bottom right corner. If there are 2 or more spaces, the spaces should be vertically aligned above the predetermined spot.
          \begin{figure}[h]
              \centering
            \includegraphics[width=1.5in]{Figures/Slide1}
            \caption{Sample Predetermined State of a Square Klotski}
             \label{fig:slide1}
           \end{figure}
          \begin{figure}[h]
              \centering
            \includegraphics[width=3in]{Figures/Slide2}
            \caption{Sample Predetermined State of a Rectangular Klotski}
             \label{fig:slide2}
           \end{figure}
          
          Based on the aforementioned rule of the game, the following objectives were identified to facilitate the conduct of  this investigation: \\
          
         $\bullet$ To determine the minimum number of moves in the Klotski game  \\
          
         $\bullet$ To determine whether a pattern is formed by the minimum number of moves \\
          
         $\bullet$ To create a formula in calculating the minimum number of moves  for all square Klotski \\
          
         $\bullet$ To create a formula in calculating the minimum number of moves for all rectangular Klotski \\
          
         $\bullet$ To determine the relationship between the number of spaces and the number of rows and columns \\
          
         $\bullet$ To create a formula in calculating the minimum number of moves given the number of spaces for all square Klotski \\
          
         $\bullet$ To create a formula in calculating the minimum number of moves given the number of spaces for all rectangular Klotski
          \subsection{Review of Related Literature}
          \subsubsection{Klotski and other Sliding Puzzles}
          \paragraph{ }
         A frame enclosed collection of forms makes up a sliding puzzle. The only way to move the shapes inside that frame is to slide; turning, lifting, or jumping are not permitted. The 15 puzzle, created in 1878 in the USA (by an unidentified person), was the one that actually kicked off
```

```
    the genre of puzzles . \\
          \begin{figure}[h]
              \centering
            \includegraphics[width=4in]{Figures/Slide_Puzzle_3x3}
            \caption{Sample of a $3 \times 3$ Mobile Game Sliding Puzzle}
            \label{fig:slidepuzzle3x3}
          \end{figure}
          \begin{figure}[h]
              \centering
            \includegraphics[width=4in]{Figures/Slide_Puzzle_5x5}
            \caption{Sample of a $5 \times 5$ Mobile Game Sliding Puzzle}
            \label{fig:slidepuzzle5x5}
          \end{figure}

          The traditional Klotski puzzle is one of the notable sliding puzzle
    that was known as "Hua Rong Dao" in China and "Dad's Puzzler" of USA . In
    China , this traditional Klotski gained popularity in the 1930's based on
    the warrior from the Eastern Han Dynasty and well-known Chinese
    strategist Cao Cao . The Cao Cao piece serves as the main header in
    sliding block puzzles , and the other nine pieces represent armed guards
    and other barriers . The player's task is to overcome these challenges and
    deliver Cao Cao to freedom at the bottom of the board
    \cite{admin_2021_sequential}. On the other hand , Dad's puzzler was
    invented by one of America's leading mathematician , Olive C. Hazlett
    during the 1920s. There is no pictures to put together or number to get
    in order in this puzzle , it's just simply uses pieces of wood . To play
    this puzzle , the large square must be moved from position A to position C
    in order to complete the puzzle . Also , no pieces may be turned , raised
    off the bottom of the frame , or jumped \cite{_dads}.

          The frame and base of the Klotski puzzle and Dad's Puzzler serves
    as the foundation upon which the puzzle is built , it also have the
    crucial role to constraints within which blocks can be shifted . It is
    commonly made out of wood or durable plastic . China's Klotski puzzle have
    one large block ($2 \times 2$), representing the protagonist. It also have
    4 medium rectangles ($2 \times 1$) and 4 small squares ($1 \times 1$) that
    can be slid in a specific direction to help the protagonist reach the
    exit. While Dad's puzzler's blocks have one $2 \times 2$ square , four
    $2 \times 1$ rectangles with the long side horizontal , two $2 \times 1$
    rectangles with the long side vertical , and two $1 \times 1$ squares
    \cite{_2023_the}.

          The Klotski puzzle has changed over time , giving rise to a variety
    of versions and adaptations that continue to intrigue and test puzzle
    enthusiasts . Some example of these are Klotski puzzle with Animated
    images , Klotski puzzle with a hexagonal frame , and a Klotski puzzle that
    holds more than 10 blocks that the traditional one have . The well-known
    Klotski puzzle game was also included in the improvement of technology
    after becoming part of Windows 3.1's Microsoft entertainment pack and has
    since become a classic \cite{_2023_the}.

          A puzzle is a type of educational game that brings entertainment
    and challenges everyone to solve it. To find the right and enjoyable
    solution to a puzzle it requires the development of certain skill that
```

```
may not only help solve the puzzle but also enhances one's congnitive abilities. This will put the observation skill, planning skill, critical thinking skill and patience up to a test. Playing the puzzle will help us develop our skills that we can also apply in our daily lives without taking away the fun \cite{_2023_the}.
          
          According to the recent article of Zhong (2023), studying new algorithms to solve sliding puzzles has become increasingly important as sliding puzzle sports have developed. Zhong along with other investigators hope to develop an additive approximation method because it is difficult to solve the puzzle optimally. The length of the answer produced by this algorithm should only be a low-order term longer than the ideal solution \cite{zhong_2023_additive}.
          \\
          \subsubsection{Arithmetic Sequence}
          \paragraph{ }
          Carl Friedrich Gauss is a German Mathematician who contributed in many fields of Math and Science. When he was a young boy he was tasked to add the integers 1 to 100 by his teacher. As a Math prodigy he is, he was able to create his own formula which leads to the discovery of arithmetic sequence. It is an ordered set of numbers that have a common difference between each consecutive term \cite{gauss}.
          
          Mathematicians and others who work with numbers can solve  difficult mathematical problems by using arithmetic sequences, which are employed in algebra and geometry. Arithmetic sequences can be used to solve simple or complex problems but it's necessary to have the basic understanding about it to ensure that it is applied correctly. Finding patterns, such as a specific number of extra seats each row, is how this is done \cite{a2017_how}.
          
          In the study of Oded Goldreich in 2011, They take into account a game in which players move various stones along an undirected graph's edges. Only one pebble may be present in each vertex at any one moment, and only one pebble may be transferred at a time (i.e., the pebble must be moved to an empty vertex). They demonstrate that it is NP-Hard to determine the shortest sequence of moves between two given "pebble configurations" \cite{odedgoldreich_2011_finding}.
          \\
          \subsubsection{Piecewise Function}
          \paragraph{ }
          A piecewise function is a function in which the outcome is defined by more than one formulas. It describes situations where a rule or relationship changes as the input value passes certain “boundaries"\cite{lumen_define}. The earliest known use of the word piecewise is in the late 1600's in the writing of Nathaniel Fairfax, a physician and antiquary. Since then, it has been significant and useful in different fields as long as there is a change based on its domain value; specifically, in mathematics, this function is employed when different expressions or formulas apply to various parts of the domain \cite{a2023_piecewise}.
          
          In the study of Nagayama, Sasao, and Butler (2014), to represent a numeric function compactly, they partition the domain of the function
```

```
into uniform segments and transform the sub-function in each segment into an arithmetic spectrum. An arithmetic expression is derived from this, and a piecewise arithmetic expression for the function is obtained. This is later on widely used in the different parts of their study \cite{nagayama_piecewise}.
          \\
          \subsubsection{Quadratic Formula}
          \paragraph{ }
          The development or derivation of mathematical ideas is similarly as continuous as the historical evolution, as one mathematician picks up an idea where another mathematician leaves it. One of these is the quadratic formula, which is used to find the roots of a quadratic equation \cite{mathnasium_2017_the}. The Quadratic Formula uses the "$a$", "$b$", and "$c$" from "$ax^2+ bx +c$", where "$a$", "$b$", and "$c$" are just numbers; they are the "numerical coefficients" of the quadratic equation that is given to solve \cite{stapel_2019_the}.

          The quadratic formula is the easiest method to determine the roots of a quadratic equation. It provides a straightforward way to solve the equation, offering an alternative to factorization. In cases where a quadratic equation is difficult to factor, the quadratic formula is a convenient and efficient tool for quickly finding the roots \cite{cuemath_2024_quadratic}.

          \subsection{Scope and Delimitations}
          \paragraph{ }
          This investigation aims to determine the minimum number of moves required to move a block in a Klotski puzzle from its starting location to its designated spot. The predetermined spot occupies one or more than spaces that is vertically aligned. A block or tile must be moved diagonally from the upper-left corner to the opposite bottom-right corner. The legal moves are limited to: (a) sliding the block/tile up, down, left, or right, and (b) only one block can be moved at a time. All blocks are of equal size, but the 'puzzles dimensions differ as the number of columns should be greater than the number of rows, it also vary depending on the number of spaces in the predetermined spot, where the 'puzzles dimension must be equal to or greater than the number of spaces in the predetermined spot disregarding the rows and column that is less than the chosen dimension. Additionally, this study aims to derive or create formulae based on observed patterns or sequences. Finally, the scope of this study is not limited to square Klotski puzzles; it also considers rectangular Klotski puzzles with varying dimensions and number of spaces in the predetermined spot.

          \section{Chapter II}
          \subsection{Definition}
          \noindent
          \textbf{Arithmetic Sequence}- a sequence where the next terms after the first term is obtained by adding a constant number. The difference between each consecutive terms is called the common difference. \\ \\
          \textbf{Klotski}- comes from the polish word "klocki" which means wooden blocks. It is a sliding puzzle that aims to move the red block to exit at the bottom of the board or to move the red block from and into a specified location. \\ \\
```

```
          \textbf{Minimum moves}- the least quantity , amount possible , assignable , allowable , or attainable in moving a block from a Klotski puzzle. \\ \\
          \textbf{Pattern}- is a repeated arrangement of number or the set of numbers that are related to each other by following a specific rule. \\ \\
          \textbf{Sliding puzzle}- is a type of simple combination puzzle. It challenges the player to slide pieces along a certain routes on a board to reach a certain end arrangement. \\ \\
          \textbf{Piecewise Function}- is a function defined by multiple sub-functions each applying to a specific intervals of its domain. \\ \\
          \textbf{Quadratic Formula}- is a mathematical rule that helps find the solutions to equations of the form $ax^2+bx+c=0$, to figure out the value "$x$" that makes the equation true.
          \subsection{Known Results or Conjecture}
          \paragraph{ }
          The following are the conjectures generated by the investigators in the conduct of the investigation: \\
          
          $\bullet$ The minimum number of moves for square and rectangular Klotski with varying dimensions forms an arithmetic sequence \\
          
          $\bullet$ The dimension of a Klotski puzzle are directly influenced by the number of spaces it contains \\
          
          $\bullet$ The formula to find the minimum number of moves ($M$) of a square Klotski is $M=8s-11$, where $s$ is the number of columns or rows given that the number of space is equal to $1$ \\
          
          $\bullet$ The formula to find the minimum number of moves of a rectangular Klotski are :
          \begin{itemize}
            \item [a.] Given the difference and the product:
            \begin{itemize}
              \item [1)] $M_{Pd}=4 \sqrt{d^2+4P}+2d-13$, where $d$ is the positive difference and $P$ is the product between the number of rows and columns
            \end{itemize}
            \item [b.] Given the number of rows or columns:
            \begin{itemize}
              \item [2)] $M=6l+2w-13$, where $l$ and $w$ are the number of rows and columns , and $l>w$
            \end{itemize}
          \end{itemize}
          
          $\bullet$ The formula to find the minimum number of moves ($M$) of a square Klotski given the spaces are greater than or equal to  $2$ are:
          \begin{enumerate}
            \item $M=8s-3S-8$, this formula can be use if $1 \leq S\leq 4$ where $s$ is the number of rows or columns and $S$ is the number of spaces , and $s\geq S$
            \item $M=8s-S-16$, this formula can be use if $4 \leq S \leq \infty$ where $s$ is the number of rows or columns and $S$ is the number of spaces , and $s\geq S$
          \end{enumerate}
```

```
           
          $\bullet$ The formula to find the minimum number of moves ($M$) of a rectangular Klotski given the spaces are greater than or equal to $2$ are:
           \begin{itemize}
             \item [a.] Given the difference and the product
             \begin{itemize}
               \item [1)] $M_{Pd}=4\sqrt{1+4P}-14$, this formula can be used if the product of the dimension is $(n+1)(n+2)$
               \item [2)] $M_{Pd}=4\sqrt{d^2+4P}+2d-S-16$, this formula can be used if the product of the dimension is $(n+f)(n+k)$, where $f\geq 1$, $k>2$, $k>f$, $f<S$ and $2\leq S \leq \infty$
             \end{itemize}
             \item [b.] Given the number of rows and columns
             \begin{itemize}
               \item [3)] $M=4\sqrt{1+4lw}-14$, this formula can be use if $(n+1)(n+2)$ where $l$ is the number of columns and $w$ is the number of rows, $l>w$
               \item [4)] $M=6l+2w-S-16$, this formula can be use if $(n+f)(n+k)$ where $f\geq 1$, $k>2$, $k>f$, $f<S$, $2\leq S\leq \infty$, where $l$ is the number of columns, $w$ is the number of rows, $S$ is the number of spaces, and $l>w$.
             \end{itemize}
           \end{itemize}
           
           \section{Chapter III}
          \subsection{Researchers' Generated Data and Results}
               \paragraph{}
               This section presents the data that the investigators gathered, the pattern uncovered and the formulation and derivation of the formula to find the minimum number of moves on the Klotski Puzzle. The following data and results are organized and presented based on the order of the specific objectives identified in section 1.3 \\
               \\
               \textbf{Data for the Minimum Number of Moves}
               \begin{table}[h]
                 \centering
                 \caption{Minimum Number of Moves based on the Number of Rows and Columns}
                 \label{fig:slide6}
                 % Table generated by Excel2LaTeX from sheet 'Sheet1'
                 \begin{tabular}{|c|c|c|c|c|c|c|c|c|}
                   \hline
                   x & 1 & 2 & 3 & 4 & 5 & 6 & 7 & ...  \\
                   \hline
                   1 & ∅ & 1 & ∅ & ∅ & ∅ & ∅ & ∅ & ...  \\
                   \hline
                   2 & 1 & 5 & 9 & 15 & 21 & 27 & 33 & ...  \\
                   \hline
                   3 & ∅ & 9 & 13 & 17 & 23 & 29 & 35 & ...  \\
                   \hline
                   4 & ∅ & 15 & 17 & 21 & 25 & 31 & 37 & ...  \\
                   \hline
```

```
        5 & ∅ & 21 & 23 & 25 & 29 & 33 & 39 & ...  \\
        \hline
        6 & ∅ & 27 & 29 & 31 & 33 & 37 & 41 & ...  \\
        \hline
        7 & ∅ & 33 & 35 & 37 & 39 & 41 & 45 & ...  \\
        \hline...
         & ... & ... & ... & ... & ... & ... & ... & ...  \\
        \hline
      \end{tabular}%

    \end{table}
    \begin{figure}[h!]

    \end{figure}


    The minimum number of moves in a square Klotski is always obtained
        by traveling the main block/tile around the diagonal path of
        the puzzle or it follows a stair-like path. For the rectangular
        Klotski, a similar strategy can be used were the main
        block/tile should travel also around the diagonal path of the
        puzzle. \\
    \begin{figure}[h]
      \centering
      \includegraphics[width=1in]{Figures/Slide3}
      \caption{Path to obtain the Minimum Number of Moves of a Square
          Klotski}
      \label{fig:slide3}
    \end{figure}

    \begin{figure}[h]
      \centering
      \includegraphics[width=2.5in]{Figures/Slide4}
      \caption{Path to obtain the Minimum Number of Moves of a
          Rectangular Klotski}
      \label{fig:slide4}
    \end{figure}
    \noindent
    \textbf{Pattern for the Minimum Number of Moves}

    \begin{table}[h!]
      \centering
      \caption{Pattern for the Minimum Number of Moves for a Square
          Klotski}
      \label{fig:slide7}
      % Table generated by Excel2LaTeX from sheet 'Sheet1'
      \begin{tabular}{|c|c|c|c|c|c|c|c|c|}
        \hline
        x & 1 & 2 & 3 & 4 & 5 & 6 & 7 & ...  \\
        \hline
        1 & \cellcolor[rgb]{1, .78, ∅ .808} & \cellcolor[rgb]{1,
            .78, .808}1 & \cellcolor[rgb]{1, .78, ∅ .808} &
            \cellcolor[rgb]{1, .78, ∅ .808} & \cellcolor[rgb]{1,
            .78, ∅ .808} & \cellcolor[rgb]{1, .78, ∅ .808} &
```

```
\cellcolor[rgb]{ 1, .78, 0/ .808} & \cellcolor[rgb]{ 1,
    .78, ... .808} \\
\hline
2 & \cellcolor[rgb]{ 1, .78 , .808}1 & \cellcolor[rgb]{ 1,
    .922 , .612}5 & 9 & 15 & 21 & 27 & 33 & ... \\
\hline
3 & \cellcolor[rgb]{ 1, .78, 0/ .808} & 9 & \cellcolor[rgb]{ 1,
    .922 , .612}13 & 17 & 23 & 29 & 35 & ... \\
\hline
4 & \cellcolor[rgb]{ 1, .78, 0/ .808} & 15 & 17 &
    \cellcolor[rgb]{ 1, .922 , .612}21 & 25 & 31 & 37 & ... \\
\hline
5 & \cellcolor[rgb]{ 1, .78, 0/ .808} & 21 & 23 & 25 &
    \cellcolor[rgb]{ 1, .922 , .612}29 & 33 & 39 & ... \\
\hline
6 & \cellcolor[rgb]{ 1, .78, 0/ .808} & 27 & 29 & 31 & 33 &
    \cellcolor[rgb]{ 1, .922 , .612}37 & 41 & ... \\
\hline
7 & \cellcolor[rgb]{ 1, .78, 0/ .808} & 33 & 35 & 37 & 39 & 41
    & \cellcolor[rgb]{ 1, .922 , .612}45 & ... \\
\hline...
& \cellcolor[rgb]{ 1, .78, ... .808} & ... & ... & ... &
    ... & ... & ... & \cellcolor[rgb]{ 1, .922 ,.................612}
    \\
\hline
\end{tabular}%

\end{table}

As shown in Table \ref*{fig:slide7}, the minimum number of moves
    a square Klotski (in an increasing number of sides) forms an
    Arithmetic Sequence whose common difference is $8$. \\

\begin{table}[h!]
  \centering
  \caption{Pattern for the Minimum Number of Moves for a
      Rectangular Klotski}
  \label{tab:slide9}
  % Table generated by Excel2LaTeX from sheet 'Sheet1'
  \begin{tabular}{|c|c|c|c|c|c|c|c|c|}
    \hline
    x & 1 & 2 & 3 & 4 & 5 & 6 & 7 & ... \\
    \hline
    1 & \cellcolor[rgb]{ 1, .78, 0/ .808} & \cellcolor[rgb]{ 1,
        .78 , .808}1 & \cellcolor[rgb]{ 1, .78, 0/ .808} &
        \cellcolor[rgb]{ 1, .78, 0/ .808} & \cellcolor[rgb]{ 1,
        .78, 0/ .808} & \cellcolor[rgb]{ 1, .78, 0/ .808} &
        \cellcolor[rgb]{ 1, .78, 0/ .808} & \cellcolor[rgb]{ 1,
        .78, ... .808} \\
    \hline
    2 & \cellcolor[rgb]{ 1, .78 , .808}1 & \cellcolor[rgb]{ 1,
        .922 , .612}5 & \cellcolor[rgb]{ .776 , .937 , .808}9 &
        \cellcolor[rgb]{ .753 , .902 , .961}15 & \cellcolor[rgb]{
        .776 , .937 , .808}21 & \cellcolor[rgb]{ .753 , .902 ,
```

```
.961}27 & \cellcolor[rgb]{ .776,   .937,   .808}33 &
\cellcolor[rgb]{ .753,   .902,................961}   \\
\hline
3 & \cellcolor[rgb]{ 1,   .78, 0/ .808} & 9 & \cellcolor[rgb]{ 1,
 .922,   .612}13 & \cellcolor[rgb]{ .776,   .937,   .808}17 &
 \cellcolor[rgb]{ .753,   .902,   .961}23 & \cellcolor[rgb]{
 .776,   .937,   .808}29 & \cellcolor[rgb]{ .753,   .902,
 .961}35 & \cellcolor[rgb]{ .776,   .937, ... .808}  \\
\hline
4 & \cellcolor[rgb]{ 1,   .78, 0/ .808} & 15 & 17 &
 \cellcolor[rgb]{ 1,   .922,   .612}21 & \cellcolor[rgb]{
 .776,   .937,   .808}25 & \cellcolor[rgb]{ .753,   .902,
 .961}31 & \cellcolor[rgb]{ .776,   .937,   .808}37 &
\cellcolor[rgb]{ .753,   .902,................961}   \\
\hline
5 & \cellcolor[rgb]{ 1,   .78, 0/ .808} & 21 & 23 & 25 &
 \cellcolor[rgb]{ 1,   .922,   .612}29 & \cellcolor[rgb]{
 .776,   .937,   .808}33 & \cellcolor[rgb]{ .753,   .902,
 .961}39 & \cellcolor[rgb]{ .776,   .937, ... .808}  \\
\hline
6 & \cellcolor[rgb]{ 1,   .78, 0/ .808} & 27 & 29 & 31 & 33 &
 \cellcolor[rgb]{ 1,   .922,   .612}37 & \cellcolor[rgb]{
 .776,   .937,   .808}41 & \cellcolor[rgb]{ .753,   .902, ...
 .961}   \\
\hline
7 & \cellcolor[rgb]{ 1,   .78, 0/ .808} & 33 & 35 & 37 & 39 & 41
 & \cellcolor[rgb]{ 1,   .922,   .612}45 & \cellcolor[rgb]{
 .776,   .937, ... .808}  \\
\hline...
 & \cellcolor[rgb]{ 1,   .78, ... .808} & ... & ... & ... &
  ... & ... & ... & \cellcolor[rgb]{ 1,   .922,................612}
  \\
\hline
\end{tabular}%

\end{table}

As shown in Table \ref*{tab:slide9}, the minimum number of moves of
 a rectangular Klotski (in an increasing number of sides) forms
 also an Arithmetic Sequence whose common difference is $8$ just
 like the pattern of the square Klotski. Furthermore, the first
 term of each sequence in the rectangular Klotski also forms an
 arithmetic sequence whose common difference is $6$. \\

\textbf{Formula for the Minimum Number of Moves of a Square
 Klotski};   \[(n+1)(n+1) \ or \ (n+1)^2 \ \ \ \{n|n \in
 \mathbbm{Z^+}\}\]
\begin{center}
  \begin{tabular}{cc}
    Product($P$) & Minimum No. of Moves ($M$) \\
    \hline
    4 & 5 \\
    9 & 13 \\
    16 & 21 \\
```

```
        25 & 29 \\
        36 & 37 \\
        49 & 45 \\
        64 & 53 \\
        81 & 61 \\
        \hline
        $P=(n+1)^2$ & $M=8n-3$
      \end{tabular}
    \end{center}

    \begin{center}
      Formulation of $M=8n-3$
      \begin{align}
        \label{eqn:8n-3 1space Square}
        M=& 5+(n-1)8\nonumber \\
        M=& 5+8n-8\nonumber \\
        M=& 8n-3
      \end{align}
    \end{center}

    \begin{center}
      Derivation of $n$
      \begin{align}
        \label{eqn:N 1spaces Square}
        &P=(n+1)^2\nonumber \\
        &\sqrt{P}=\sqrt{(n+1)^2}\nonumber \\
        &\sqrt{P}=n+1\nonumber \\
        &\sqrt{P}-1=n\nonumber \\
        &n= \sqrt{P}-1
      \end{align}
    \end{center}

    \begin{center}
      Minimum number of moves in terms of the product ($M_P$)
      \begin{align}
        \label{eqn:MP1spaces Square}
        &M=8n-3\nonumber \\
        &M_P=8 (\sqrt{P}-1)-3\nonumber \\
        &M_P=8 \sqrt{P}-8-3\nonumber \\
        &M_P=8 \sqrt{P}-11
      \end{align}
    \end{center}

    \begin{center}
      Since $\sqrt{P}$ is just equal to the number of rows/columns ($s$) of the square Klotski, then the minimum number of moves ($M$) is
      \begin{align}
        \label{eqn:FF1spaces Square}
        \alignedbox{ }{M=8s-11}
      \end{align}
    \end{center}
    \noindent
```

```
\textbf{Formula for the Minimum Number of Moves of a Rectangular Klotski}; \[(n+1)(n+k) \ \ \{n|n \in \mathbbm{Z^+}\} \ \ \{k|k \in \mathbbm{Z^+}, k>1\} \]

The arithmetic sequence formula of the different minimum number of moves ($M$) for each $(n+1)(n+k)$


\begin{center}
  \begin{tabular}{ccccc}
    \multicolumn{2}{c}{Minimum no. of moves for $(n+1)(n+2)$} & & \multicolumn{2}{c}{Minimum no. of moves for $(n+1)(n+3)$} \\
    \cline{1-2}
    \cline{4-5}
    $P_{(n+1)(n+2)}$ & $M_{(n+1)(n+2)}$ & & $P_{(n+1)(n+3)}$ & $M_{(n+1)(n+3)}$ \\
    \cline{1-2}
    \cline{4-5}
    6 & 9 & & 8 & 15 \\
    12 & 17 & & 15 & 23 \\
    20 & 25 & & 24 & 31 \\
    30 & 33 & & 35 & 39 \\
    ... & ... & & ... & ... \\
    \cline{1-2}
    \cline{4-5}
    $(n+1)(n+2)$ & $8n+1$ & & $(n+1)(n+3)$ & $8n+7$
  \end{tabular}
\end{center}

\begin{center}
  \begin{tabular}{ccccc}
    \multicolumn{2}{c}{Minimum no. of moves for $(n+1)(n+4)$} & & \multicolumn{2}{c}{Minimum no. of moves for $(n+1)(n+5)$} \\
    \cline{1-2}
    \cline{4-5}
    $P_{(n+1)(n+4)}$ & $M_{(n+1)(n+4)}$ & & $P_{(n+1)(n+5)}$ & $M_{(n+1)(n+5)}$ \\
    \cline{1-2}
    \cline{4-5}
    10 & 21 & & 12 & 27 \\
    18 & 29 & & 21 & 35 \\
    28 & 37 & & 32 & 43 \\
    40 & 45 & & 45 & 51 \\
    ... & ... & & ... & ... \\
    \cline{1-2}
    \cline{4-5}
    $(n+1)(n+4)$ & $8n+13$ & & $(n+1)(n+5)$ & $8n+19$
  \end{tabular}
\end{center}


The following table summarizes the product and the minimum number of moves for $(n+1)(n+k)$ Klotski:

```

```
\begin{center}
    \begin{tabular}{cc}
        \hline
        Product ($P$) & Minimum no. of moves ($M$) \\
        \hline
        $(n+1)(n+2)$ & $8n+1$ \\
        $(n+1)(n+3)$ & $8n+7$ \\
        $(n+1)(n+4)$ & $8n+13$ \\
        $(n+1)(n+5)$ & $8n+19$ \\
        $(n+1)(n+6)$ & $8n+25$ \\
        $(n+1)(n+7)$ & $8n+31$ \\
        ... & ... \\
        \hline
        $P=(n+1)(n+k)$ & $M=8n+6k-11$
    \end{tabular}
\end{center}

\noindent
Formulation of $M=8n+6k-11$
\begin{align}
    \label{eqn:1Spaces Rec Explicit}
    &M=8n+(1+[k-2]6)\nonumber \\
    &M=8n+(1+6k-12)\nonumber \\
    &M=8n+1+6k-12\nonumber \\
    &M=8n+6k-11
\end{align}
Derivation of $n$
\begin{align}
    \label{eqn:1spaces Rec N}
    P=&(n+1)(n+k)\nonumber \\
    P=&n^2+kn+n+k\nonumber \\
    P=&n^2+(k+1)n+k\nonumber \\
    0=&n^2+(k+1)n+k-P
\end{align}
\begin{center}

    using quadratic formula to find $n$
    \begin{align}
        \label{eqn:1spaces rec quad }
        n=&\frac{-(k+1) \pm \sqrt{(k+1)^2-4(1)(k-P)}}{2(1)}\nonumber \\
        n=&\frac{-k-1 \pm \sqrt{k^2+2k+1-4k+4P}}{2}\nonumber \\
        n=&\frac{-k-1 \pm \sqrt{k^2-2k+1+4P}}{2}\nonumber \\
        n=&\frac{-k-1 \pm \sqrt{(k-1)^2+4P}}{2}
    \end{align}
\end{center}
\noindent
Minimum number of moves in terms of the product ($M_P$)
\begin{align}
    \label{eqn:1spaces rec MP}
    M=&8n+6k-11\nonumber \\
    M_P=&8(\frac{-k-1 \pm \sqrt{(k-1)^2+4P}}{2})+6k-11\nonumber \\
     M_P=&4(-k-1 \pm \sqrt{(k-1)^2+4P})+6k-11\nonumber \\
    M_P=&-4k-4 \pm 4\sqrt{(k-1)^2+4P}+6k-11
    \nonumber \\
```

```
    M_P =&4  \sqrt{(k -1)^2+4 P}+2 k-15
\end{align}
Since $k$ is the difference ($d$) of the two sides $+1$,
\begin{align}
    \label{eqn:1spaces rec k}
    d=&(n+k)-(n+1)\nonumber \\
    d=&n+k-n-1\nonumber \\
    d=&k-1\nonumber \\
    d+1&=k\nonumber \\
    k=&d+1
\end{align}

hence the minimum number of moves given the product and diff erence ($M_{Pd}$) is
\begin{align}
    \label{eqn:1spaces rec MPD}
    M_{Pd}=&4  \sqrt{((d+1)-1)^2+4 P}+2(d+1)-15\nonumber \\
    M_{Pd}=&4  \sqrt{(d+1-1)^2+4 P}+2d+2-15\nonumber \\
     M_{Pd}=&4  \sqrt{(d)^2+4 P}+2d-13\nonumber \\
     \alignedbox{M_{Pd}=}{4  \sqrt{d^2+4 P}+2d-13}
\end{align}


Furthermore, if we let $l$ and $w$ be the number of columns and rows and $l>w$, then the minimum number of moves of a rectangular Klotski ($M$) is
\begin{align}
    \label{eqn:1spaces rec finalf}
    M_{Pd}=&4  \sqrt{d^2+4 P}+2d-13  \nonumber \\
    M=&4  \sqrt{(l-w)^2+4(l)(w)}+2(l-w)-13  \nonumber \\
    M=&4  \sqrt{l^2-2lw+w^2+4lw}+2l-2w-13\nonumber \\
    M=&4  \sqrt{l^2+2lw+w^2}+2l-2w-13\nonumber \\
    M=&4  \sqrt{(l+w)^2}+2l-2w-13\nonumber \\
    M=&4(l+w)+2l-2w-13\nonumber \\
    M=&4l+4w+2l-2w-13\nonumber \\
    \alignedbox{M=}{6l+2w-13}
\end{align}
\\
\textbf{Formula for the Minimum Number of Moves of a Square Klotski given the number of spaces;}

\begin{table}[h!]
    \centering
    \caption{Data of the Minimum number of moves for Square and Rectangular Klotski with 2 Spaces}
    % Table generated by Excel2LaTeX from sheet 'Sheet1'
    \begin{tabular}{|c|c|c|c|c|c|c|c|c|}
      \hline
      x & 1 & 2 & 3 & 4 & 5 & 6 & 7 & ...  \\
      \hline
      1 & \cellcolor[rgb]{1, .78, 0/.808} & \cellcolor[rgb]{1, .78, 0/.808} & \cellcolor[rgb]{1, .78, 0/.808} & \cellcolor[rgb]{1, .78, 0/.808} & \cellcolor[rgb]{1, .78, 0/.808} &
```

```
\cellcolor[rgb]{ 1,  .78, 0/ .808} & \cellcolor[rgb]{ 1,
 .78 ,................ 808}  \\
\hline
2 & \cellcolor[rgb]{ 1,  .78, 0/ .808} & \cellcolor[rgb]{ .776 ,
 .937 ,  .808}2 & \cellcolor[rgb]{ .776 ,  .937 ,  .808}6 &
 \cellcolor[rgb]{ 1,  .922 ,  .612}10 & \cellcolor[rgb]{ 1,
 .922 ,  .612}16 & \cellcolor[rgb]{ 1,  .922 ,  .612}22 &
 \cellcolor[rgb]{ 1,  .922 ,  .612}28 & \cellcolor[rgb]{ 1,
 .922 , ... .612}  \\
\hline
3 & \cellcolor[rgb]{ 1,  .78, 0/ .808} &  & \cellcolor[rgb]{
 .776 ,  .937 ,  .808}10 & \cellcolor[rgb]{ .776 ,  .937 ,
 .808}14 & \cellcolor[rgb]{ 1,  .922 ,  .612}18 &
 \cellcolor[rgb]{ 1,  .922 ,  .612}24 & \cellcolor[rgb]{ 1,
 .922 ,  .612}30 & \cellcolor[rgb]{ 1,  .922 , ... .612}  \\
\hline
4 & \cellcolor[rgb]{ 1,  .78, 0/ .808} &  &  &
 \cellcolor[rgb]{ .776 ,  .937 ,  .808}18 & \cellcolor[rgb]{
 .776 ,  .937 ,  .808}22 & \cellcolor[rgb]{ 1,  .922 ,  .612}26
 & \cellcolor[rgb]{ 1,  .922 ,  .612}32 & \cellcolor[rgb]{ 1,
  .922 , ... .612}   \\
\hline
5 & \cellcolor[rgb]{ 1,  .78, 0/ .808} &  &  &  &
 \cellcolor[rgb]{ .776 ,  .937 ,  .808}26 & \cellcolor[rgb]{
 .776 ,  .937 ,  .808}30 & \cellcolor[rgb]{ 1,  .922 ,  .612}34
 & \cellcolor[rgb]{ 1,  .922 , ... .612}  \\
\hline
6 & \cellcolor[rgb]{ 1,  .78, 0/ .808} &  &  &  &  &
 \cellcolor[rgb]{ .776 ,  .937 ,  .808}34 & \cellcolor[rgb]{
 .776 ,  .937 ,  .808}38 & \cellcolor[rgb]{ 1,  .922 , ...
 .612}  \\
\hline
7 & \cellcolor[rgb]{ 1,  .78, 0/ .808} &  &  &  &  &  &
 \cellcolor[rgb]{ .776 ,  .937 ,  .808}42 & \cellcolor[rgb]{
 .776 ,  .937 ,................ 808}  \\
\hline...
 & \cellcolor[rgb]{ 1,  .78, ... .808} & ... & ... & ... &
  ... & ... & ... & \cellcolor[rgb]{ .776 ,  .937 , ...
  .808}  \\
\hline
\end{tabular}%

\label{tab:2 spaces}
\end{table}
\begin{center}
\begin{tabular}{cc}
\multicolumn{2}{c}{$(n+1)(n+1)\ or\ (n+1)^2\ \{n|n \in
 \mathbbm{Z^+} \}$} \\
& \\
Product($P$) & Minimum No. of Moves ($M$) \\
\hline
4 & 2 \\
9 & 10 \\
16 & 18 \\
```

```
        25 & 26 \\
        36 & 34 \\
        49 & 42 \\
        \hline
        $P=(n+1)^2$ & $M=8n-6$
      \end{tabular}
    \end{center}

    \begin{center}
      Formulation of $M=8n-6$
      \begin{align}
        \label{eqn:2spacesEx}
        & M=2+(n-1)8 \nonumber \\
        & M=2+8n-8 \nonumber \\
        & M=8n-6
      \end{align}
    \end{center}

    \begin{center}
      Derivation of $n$
      \begin{align}
        \label{eqn:2spacesN}
        &P=(n+1)^2 \nonumber \\
        &\sqrt{P}=\sqrt{(n+1)^2} \nonumber \\
        &\sqrt{P}=n+1 \nonumber \\
        &\sqrt{P}-1=n \nonumber \\
        &n=\sqrt{P}-1
      \end{align}
    \end{center}

    \noindent
    Minimum number of moves in terms of the product ($M_P$)
    \begin{align}
      \label{eqn:2spacesMP}
      &M_P=8(\sqrt{P}-1)-6 \nonumber \\
      &M_P=8\sqrt{P}-8-6 \nonumber \\
      &M_P=8\sqrt{P}-14
    \end{align}


    Since $\sqrt{P}$ is just equal to the number of rows/columns ($s$) of the square Klotski, then the minimum number of moves ($M$) given the 2 spaces is
    \begin{align}
      \label{eqn:2spacesFF}
      \alignedbox{ }{M=8s-14}
    \end{align}

    \begin{table}[h]
      \centering
      \caption{Data of the Minimum number of moves for Square and Rectangular Klotski with 3 Spaces}
      \begin{tabular}{|c|c|r|r|r|c|c|c|c|}
        \hline
```

```
x & 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} &
    \multicolumn{1}{c|}{4} & 5 & 6 & 7 & ...    \\
\hline
1 & \cellcolor[rgb]{ 1,   .78, 0 .808} &

    \multicolumn{1}{c|}{\cellcolor[rgb]{ 1,   .78, 0 .808}} &
    \multicolumn{1}{c|}{\cellcolor[rgb]{ 1,   .78, 0 .808}} &
    \multicolumn{1}{c|}{\cellcolor[rgb]{ 1,   .78, 0 .808}} &

    \cellcolor[rgb]{ 1,   .78, 0 .808} & \cellcolor[rgb]{ 1,

   .78, 0/  .808} & \cellcolor[rgb]{ 1,   .78, 0 .808} &

    \cellcolor[rgb]{ 1,   .78, ...  .808}  \\
\hline
2 & \cellcolor[rgb]{ 1,   .78, 0 .808} &

    \multicolumn{1}{c|}{\cellcolor[rgb]{ 1,   .78 ,   .808}2} &
    \multicolumn{1}{c|}{\cellcolor[rgb]{ 1,   .78 ,   .808}3} &
    \multicolumn{1}{c|}{\cellcolor[rgb]{ 1,   .78 ,   .808}7} &
    \cellcolor[rgb]{ 1,   .78 ,   .808}13 & \cellcolor[rgb]{ 1,
    .78 ,   .808}19 & \cellcolor[rgb]{ 1,   .78 ,   .808}25 &
    \cellcolor[rgb]{ 1,   .78, ...  .808}  \\
\hline
3 & \cellcolor[rgb]{ 1,   .78, 0 .808} &   &

    \multicolumn{1}{c|}{\cellcolor[rgb]{ .776,.937,  .808}7}
    &  \multicolumn{1}{c|}{\cellcolor[rgb]{ .776,.937,
    .808}11} & \cellcolor[rgb]{ 1,   .922 ,   .612}17 &
    \cellcolor[rgb]{ 1,   .922 ,   .612}23 & \cellcolor[rgb]{ 1,
    .922 ,   .612}29 & \cellcolor[rgb]{ 1,   .922, ... .612}  \\
\hline
4 & \cellcolor[rgb]{ 1,   .78, 0 .808} &   &   &

    \multicolumn{1}{c|}{\cellcolor[rgb]{.776 ,   .937 ,   .808}15}
    & \cellcolor[rgb]{.776 ,.937 ,.808}19 & \cellcolor[rgb]{
    1,  .922,  .612}25 & \cellcolor[rgb]{ 1,  .922,  .612}31 &
    \cellcolor[rgb]{ 1,  .922, ...  .612}  \\
\hline
5 & \cellcolor[rgb]{ 1,   .78, 0 .808} &   &   &   &

    \cellcolor[rgb]{  .776,   .937 ,   .808}23  & \cellcolor[rgb]{
    .776 ,   .937 ,   .808}27 & \cellcolor[rgb]{ 1,   .922 ,   .612}33
    & \cellcolor[rgb]{ 1,   .922, ...  .612}  \\
\hline
6 & \cellcolor[rgb]{ 1,   .78, 0 .808} &   &   &   &   &

    \cellcolor[rgb]{  .776,   .937 ,   .808}31  & \cellcolor[rgb]{
    .776 ,   .937 ,   .808}35 & \cellcolor[rgb]{ 1,   .922, ...
    .612}  \\
\hline
7 & \cellcolor[rgb]{ 1,   .78, 0 .808} &   &   &   &   &   &

    \cellcolor[rgb]{  .776,   .937 ,   .808}39  & \cellcolor[rgb]{
    .776 ,   .937 , ...  .808}  \\
 \hline...
  & \cellcolor[rgb]{ 1,  .78, ... .808} &   &   &   &   &   &
     &  \cellcolor[rgb]{ .776 ,   .937 ,................808}  \\
\hline
\end{tabular}%
```

```
          Formulation of $M=8n-1$
          \begin{align}
            \label{eqn:3spacesEx}
            &M=7+(n-1)8\nonumber \\
            &M=7+8n-8\nonumber \\
            &M=8n-1
          \end{align}
        \end{center}

        \begin{center}
          Derivation of $n$
          \begin{align}
            \label{eqn:3spacesN}
            &P=(n+2)^2\nonumber \\
            &\sqrt{P}=\sqrt{(n+2)^2}\nonumber \\
            &\sqrt{P}=n+2\nonumber \\
            &\sqrt{P}-2=n\nonumber \\
            &n=\sqrt{P}-2
          \end{align}
        \end{center}

        Minimum number of moves in terms of the product ($M_P$)
        \begin{align}
          \label{eqn:3spacesMP}
          &M_P=8(\sqrt{P}-2)-1 \nonumber \\
          &M_P =8\sqrt{P}-16-1\nonumber \\
          &M_P =8\sqrt{P}-17
        \end{align}

        Since $\sqrt{P}$ is just equal to the number of rows/columns ($s$) of the square Klotski, then the minimum number of moves ($M$) given the 3 spaces is
        \begin{align}
          \label{eqn:3spacesFF}
          \alignedbox{ }{M=8s-17}
        \end{align}

        \begin{table}[h]
          \centering
         \caption{Data of the Minimum number of moves for Square and Rectangular Klotski with 4 Spaces}
          \begin{tabular}{|c|c|c|c|c|c|c|c|c|}
            \hline
            x & 1 & 2 & 3 & 4 & 5 & 6 & 7 & ... \\
            \hline
            1 & \cellcolor[rgb]{1, .78, 0/ .808} & \cellcolor[rgb]{1, .78, 0/ .808} & \cellcolor[rgb]{1, .78, 0/ .808} & \cellcolor[rgb]{1, .78, 0/ .808} & \cellcolor[rgb]{1, .78, 0/ .808} & \cellcolor[rgb]{1, .78, 0/ .808} & \cellcolor[rgb]{1, .78, ... .808} \\
            \hline
            2 & \cellcolor[rgb]{1, .78, 0/ .808} & \cellcolor[rgb]{1, .78, 0/ .808} & \cellcolor[rgb]{1, .78, .808}3 &
```

```
\cellcolor[rgb]{ 1,  .78,  .808}8 & \cellcolor[rgb]{ 1,
.78,  .808}10 & \cellcolor[rgb]{ 1,  .78,  .808}17 &
\cellcolor[rgb]{ 1,  .78,  .808}20 & \cellcolor[rgb]{ 1,
.78, ... .808} \\
\hline
3 & \cellcolor[rgb]{ 1,  .78, 0' .808} & \cellcolor[rgb]{ 1,

.78,  .808}3 & \cellcolor[rgb]{ 1,  .78,  .808}7 &
\cellcolor[rgb]{ 1,  .78,  .808}8 & \cellcolor[rgb]{ 1,
.78,  .808}15 & \cellcolor[rgb]{ 1,  .78,  .808}16 &
\cellcolor[rgb]{ 1,  .78,  .808}22 & \cellcolor[rgb]{ 1,
.78, ... .808} \\
\hline
4 & \cellcolor[rgb]{ 1,  .78, 0' .808} & \cellcolor[rgb]{ 1,

.78,  .808}8 & \cellcolor[rgb]{ 1,  .78,  .808}8 &
\cellcolor[rgb]{ .776,  .937,  .808}12 & \cellcolor[rgb]{
.776,  .937,  .808}18 & \cellcolor[rgb]{ 1,  .922,  .612}24
& \cellcolor[rgb]{ 1,  .922,  .612}30 & \cellcolor[rgb]{ 1,
 .922, ... .612} \\
\hline
5 & \cellcolor[rgb]{ 1,  .78, 0' .808} & \cellcolor[rgb]{ 1,

.78,  .808}8 & \cellcolor[rgb]{ 1,  .78,  .808}15 &  &
\cellcolor[rgb]{ .776,  .937,  .808}20 & \cellcolor[rgb]{
.776,  .937,  .808}26 & \cellcolor[rgb]{ 1,  .922,  .612}32
& \cellcolor[rgb]{ 1,  .922, ... .612} \\
\hline
6 & \cellcolor[rgb]{ 1,  .78, 0' .808} & \cellcolor[rgb]{ 1,

.78,  .808}14 & \cellcolor[rgb]{ 1,  .78,  .808}16 &  &
& \cellcolor[rgb]{ .776,  .937,  .808}28 & \cellcolor[rgb]{
.776,  .937,  .808}34 & \cellcolor[rgb]{ 1,  .922, ...
.612} \\
\hline
7 & \cellcolor[rgb]{ 1,  .78, 0/ .808} & \cellcolor[rgb]{ 1,
 .78,  .808}20 & \cellcolor[rgb]{ 1,  .78,  .808}22 &  &
&  & \cellcolor[rgb]{ .776,  .937,  .808}36 &
 \cellcolor[rgb]{ .776,  .937, ... .808} \\
\hline...
 & \cellcolor[rgb]{ 1,  .78, ... .808} & \cellcolor[rgb]{ 1,
 .78, ... .808} & \cellcolor[rgb]{ 1,  .78,................808} &
 &  &  &  & \cellcolor[rgb]{ .776,  .937,................808} \\
\hline
\end{tabular}%

\label{tab:4 spaces}
\end{table}
\[(n+3)(n+3)\ or\ (n+3)^2\ \{n|n \in \mathbbm{Z^+} \}\]
\begin{center}
Formulation of $M=8n-1$
\begin{align}
\label{eqn:4 spaces Ex}
&M=12+(n-1)8\nonumber \\
&M=12+8n-8\nonumber \\
```

```
            &M=8n+4
        \end{align}
    \end{center}
```

```

    \begin{center}
        Derivation of $n$
        \begin{align}
            \label{eqn:4spacesN}
            &P=(n+3)^2\nonumber \\
            &\sqrt{P}=\sqrt{(n+3)^2}\nonumber \\
            &\sqrt{P}=n+3\nonumber \\
            &\sqrt{P}-3=n\nonumber \\
            &n=\sqrt{P}-3
        \end{align}
    \end{center}

    Minimum number of moves in terms of the product ($M_P$)
    \begin{align}
        \label{eqn:4spacesMP}
        &M_P=8(\sqrt{P}-3)+4 \nonumber \\
        &M_P=8\sqrt{P}-24+4\nonumber \\
        &M_P=8\sqrt{P}-20
    \end{align}

    Since $\sqrt{P}$ is just equal to the number of rows/columns ($s$)
        of the square Klotski, then the minimum number of moves ($M$)
        given the 4 spaces is
    \begin{align}
        \label{eqn:4spacesFF}
        \alignedbox{ }{M=8s-20}
    \end{align}

    The following table summarizes the number of spaces and the minimum
        number of moves for square Klotski whose number of sides is
        greater than or equal to the number of spaces:
    \begin{center}
        \begin{tabular}{cc}
            No. of Spaces ($S$) & Minimum no. of moves ($M$) \\
            \hline
            1 & $8s-11$ \\
            2 & $8s-14$ \\
            3 & $8s-17$ \\
            4 & $8s-20$ \\
            \hline
            \multicolumn{2}{c}{$M=8s-3S-8$}
        \end{tabular}
    \end{center}

    \begin{center}
        Formulation of $M=8s-3S-8$
        \begin{align}
            \label{eqn:1-4spacessquaregenformula}
            M=&(8s-11)+(S-1)(-3)\nonumber \\
            M=&8s-11-3S+3\nonumber \\
            \alignedbox{M=}{8s-3S-8}
        \end{align}
```

```
                \end{center}
```

```
The above-mentioned formula (Eq.
    \ref{eqn:1-4spacessquaregenformula}) calculates the minimum
    number of moves in a square Klotski which has spaces ranging
    from 1 to 4, where $s$ is the number of rows or columns and $S$
    is the number of spaces in the puzzle.
\\
\\
\noindent
\textbf{Square Klotski with spaces ranging from 5 to $\infty$}
\paragraph{ }
Using the data from the following tables and following the same
    process in deriving the formula for the square Klotski that
    contains spaces ranging from 1 to 4, the investigators was able to
    derived a general formula that calculates the minimum number of
    moves for square Klotski that contains spaces ranging from 5 to
    infinity.

\begin{table}[h]
  \centering
  \caption{Data of the Minimum number of moves for Square and
      Rectangular Klotski with 5 Spaces}
  \ begin { tabular }{|c|c|c|c|c|c|c|c|c|}
    \hline
    x & 4 & 5 & 6 & 7 & 8 & 9 & 10 & ...   \\
    \hline
    4 & \cellcolor[rgb]{ 1,  .78, 0' .808} & \cellcolor[rgb]{ 1,

       .78, 0/  .808} & \cellcolor[rgb]{ 1,  .78, 0' .808} &

       \cellcolor[rgb]{ 1,  .78, 0' .808} & \cellcolor[rgb]{ 1,

       .78, 0/  .808} & \cellcolor[rgb]{ 1,  .78, 0' .808} &

       \cellcolor[rgb]{ 1,  .78, 0/ .808} & \cellcolor[rgb]{ 1,
       .78, ...  .808}  \\
    \hline
    5 & \cellcolor[rgb]{ 1,  .78, 0' .808}  & \cellcolor[rgb]{ .776,

       .937,  .808}19 & \cellcolor[rgb]{ 1,  .922,  .612}25 &
       \cellcolor[rgb]{ 1,  .922,  .612}31 & \cellcolor[rgb]{ 1,
       .922,  .612}37 & \cellcolor[rgb]{ 1,  .922,  .612}43 &
       \cellcolor[rgb]{ 1,  .922,  .612}49 & \cellcolor[rgb]{ 1,
       .922, ...  .612}  \\
    \hline
    6 & \cellcolor[rgb]{ 1,  .78, 0' .808} &   & \cellcolor[rgb]{

       .776,  .937,  .808}27 & \cellcolor[rgb]{ 1,  .922,  .612}33
       & \cellcolor[rgb]{ 1,  .922,  .612}39 & \cellcolor[rgb]{ 1,
        .922,  .612}45 & \cellcolor[rgb]{ 1,  .922,  .612}51 &
       \cellcolor[rgb]{ 1,  .922, ...  .612}  \\
    \hline
    7 & \cellcolor[rgb]{ 1,  .78, 0' .808} &   &   &

       \cellcolor[rgb]{ .776, .937,  .808}35 & \cellcolor[rgb]{
```

```
    1,  .922,  .612}41  & \cellcolor[rgb]{ 1, .922,  .612}47 &
    \cellcolor[rgb]{ 1,   .922,   .612}53 & \cellcolor[rgb]{ 1,
    .922, ...  .612}  \\
\hline
8 & \cellcolor[rgb]{ 1,   .78, 0 .808} &   &   &   &

    \cellcolor[rgb]{ .776,   .937,   .808}43 & \cellcolor[rgb]{
```

```
        1,   .922 ,   .612}49 & \cellcolor[rgb]{ 1,   .922 ,   .612}55 &
        \cellcolor[rgb]{ 1,   .922 , . . .  .612}  \\
      \hline
      9 & \cellcolor[rgb]{ 1,   .78, 0 .808} &    &    &    &    &

        \cellcolor[rgb]{ .776 , .937 ,  .808}51 & \cellcolor[rgb]{
        1,  .922 ,  .612}57 & \cellcolor[rgb]{ 1, .922 ,..............612}
        \\
      \hline
      10 & \cellcolor[rgb]{ 1,   .78, 0 .808} &    &    &    &    &    &

        \cellcolor[rgb]{ .776,.937 ,    .808}59 & \cellcolor[rgb]{
        1,  .922 , . . .  .612}  \\
      \hline. . .
       & \cellcolor[rgb]{ 1,  .78 , . . .  .808} &    &    &    &    &    &
         & \cellcolor[rgb]{ .776 , .937 , . . .  .808}  \\
      \hline
    \end{tabular}%

    \label{ tab :5 spaces }
  \end{ table }

  \begin{ table }[h]
    \centering
    \caption{Data of the Minimum number of moves for Square and
       Rectangular Klotski with 6 Spaces}
    % Table generated by Excel2LaTeX from sheet 'Sheet1'
    \ begin { tabular }{|c|c|c|c|c|c|c|c|c|c|}
      \hline
      x & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & . . .    \\
      \hline
      5 & \cellcolor[rgb]{ 1,   .78, 0 .808} & \cellcolor[rgb]{ 1,

        .78, 0/  .808} & \cellcolor[rgb]{ 1,   .78, 0 .808} &

        \cellcolor[rgb]{ 1,   .78, 0 .808} & \cellcolor[rgb]{ 1,

        .78, 0/  .808} & \cellcolor[rgb]{ 1,   .78, 0 .808} &

        \cellcolor[rgb]{ 1,   .78, 0 .808} & \cellcolor[rgb]{ 1,

        .78, 0 .808} & \cellcolor[rgb]{ 1,   .78, . . .  .808}  \\
      \hline

      6 & \cellcolor[rgb]{ 1,   .78, 0 .808}  & \cellcolor[rgb]{ .776 ,

        .937 ,   .808}26 & \cellcolor[rgb]{ 1,   .922 ,   .612}32 &
        \cellcolor[rgb]{ 1,   .922 ,   .612}38 & \cellcolor[rgb]{ 1,
        .922 ,   .612}44 & \cellcolor[rgb]{ 1,   .922 ,   .612}50 &
        \cellcolor[rgb]{ 1,   .922 ,   .612}56 & \cellcolor[rgb]{ 1,
        .922 ,   .612}62 & \cellcolor[rgb]{ 1,   .922 , . . .  .612}  \\
      \hline
      7 & \cellcolor[rgb]{ 1,   .78, 0 .808} &    & \cellcolor[rgb]{

        .776 ,   .937 ,   .808}34 & \cellcolor[rgb]{ 1,   .922 ,   .612}40
```

```
    & \cellcolor[rgb]{ 1,   .922 ,   .612}46 & \cellcolor[rgb]{ 1,
     .922 ,   .612}52 & \cellcolor[rgb]{ 1,   .922 ,   .612}58 &
    \cellcolor[rgb]{ 1,   .922 ,   .612}64 & \cellcolor[rgb]{ 1,
    .922 , . . .  .612}   \\
\hline
8 & \cellcolor[rgb]{ 1,   .78, 0ʹ .808} &   &   &

    \cellcolor[rgb]{ .776 , .937 ,  .808}42 & \cellcolor[rgb]{
    1,  .922 ,  .612}48 & \cellcolor[rgb]{ 1, .922 ,  .612}54 &
    \cellcolor[rgb]{ 1,   .922 ,   .612}60 & \cellcolor[rgb]{ 1,
```

```
                .922 ,   .612}66 & \cellcolor[rgb]{ 1,   .922 , . . .  .612}  \\
            \hline
            9 & \cellcolor[rgb]{ 1,   .78, 0' .808} &   &   &   &

                \cellcolor[rgb]{ .776 , .937 , .808}50 & \cellcolor[rgb]{
                1,  .922 , .612}56 & \cellcolor[rgb]{ 1, .922 , .612}62 &
                \cellcolor[rgb]{ 1,   .922 ,   .612}68 & \cellcolor[rgb]{ 1,
                .922 , . . .  .612}  \\
            \hline
            10 & \cellcolor[rgb]{ 1,   .78, 0' .808} &   &   &   &   &

                \cellcolor[rgb]{ .776 , .937 , .808}58 & \cellcolor[rgb]{
                1,  .922 , .612}64 & \cellcolor[rgb]{ 1, .922 , .612}70 &
                \cellcolor[rgb]{ 1,   .922 , . . .  .612}  \\
            \hline
            11 & \cellcolor[rgb]{ 1,   .78, 0' .808} &   &   &   &   &   &

                \cellcolor[rgb]{ .776 , .937 , .808}66 & \cellcolor[rgb]{
                1,  .922 , .612}72 & \cellcolor[rgb]{ 1, .922 ,..............612}
                \\
            \hline
            12 & \cellcolor[rgb]{ 1,   .78, 0' .808} &   &   &   &   &   &

                & \cellcolor[rgb]{ .776 , .937 , .808}74 & \cellcolor[rgb]{
                1,  .922 , . . .  .612}  \\
            \hline. . .
             & \cellcolor[rgb]{ 1, .78 , . . .  .808} &   &   &   &   &   &
                 &   & \cellcolor[rgb]{ .776 , .937 , . . .  .808}  \\
            \hline
          \end{tabular}%


          \label{ tab :6 spaces }
        \end{ table }
        \begin{ table }[h]
          \centering
          \caption{Data of the Minimum number of moves for Square and
              Rectangular Klotski with 7 Spaces}
          % Table generated by Excel2LaTeX from sheet 'Sheet1'
          \ begin { tabular }{|c|c|c|c|c|c|c|c|c|c|}
            \hline
            x & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & . . .    \\
            \hline
            6 & \cellcolor[rgb]{ 1,   .78, 0/ .808} & \cellcolor[rgb]{ 1,
                .78, 0/ .808} & \cellcolor[rgb]{ 1,   .78, 0/ .808} &
                \cellcolor[rgb]{ 1,   .78, 0/ .808} & \cellcolor[rgb]{ 1,
                .78, 0/ .808} & \cellcolor[rgb]{ 1,   .78, 0/ .808} &
                \cellcolor[rgb]{ 1,   .78, 0/ .808} & \cellcolor[rgb]{ 1,
                .78, 0/ .808} & \cellcolor[rgb]{ 1,   .78, . . .  .808}  \\
            \hline
            7 & \cellcolor[rgb]{ 1,   .78, 0/ .808} & \cellcolor[rgb]{ .776 ,
```

```
.937,  .808}33 & \cellcolor[rgb]{ 1,  .922,  .612}39 &
\cellcolor[rgb]{ 1,  .922,  .612}45 & \cellcolor[rgb]{ 1,
.922,  .612}51 & \cellcolor[rgb]{ 1,  .922,  .612}57 &
\cellcolor[rgb]{ 1,  .922,  .612}63 & \cellcolor[rgb]{ 1,
.922,  .612}69 & \cellcolor[rgb]{ 1,  .922, ... .612}  \\
\hline
```

```
    8 & \cellcolor[rgb]{ 1,   .78, 0' .808} &   & \cellcolor[rgb]{

       .776 ,   .937 ,   .808}41 & \cellcolor[rgb]{ 1,   .922 ,   .612}47
       & \cellcolor[rgb]{ 1,   .922 ,   .612}53 & \cellcolor[rgb]{ 1,
        .922 ,   .612}59 & \cellcolor[rgb]{ 1,   .922 ,   .612}65 &
       \cellcolor[rgb]{ 1,   .922 ,   .612}71 & \cellcolor[rgb]{ 1,
       .922 , . . .  .612}  \\
    \hline
    9 & \cellcolor[rgb]{ 1,   .78, 0' .808} &   &   &

       \cellcolor[rgb]{ .776 , .937 ,  .808}49 & \cellcolor[rgb]{
       1,  .922 ,  .612}55 & \cellcolor[rgb]{ 1, .922 ,  .612}61 &
       \cellcolor[rgb]{ 1,   .922 ,   .612}67 & \cellcolor[rgb]{ 1,
       .922 ,   .612}73 & \cellcolor[rgb]{ 1,   .922 , . . .  .612}  \\
    \hline
    10 & \cellcolor[rgb]{ 1,   .78, 0' .808} &   &   &   &

       \cellcolor[rgb]{ .776 , .937 ,  .808}57 & \cellcolor[rgb]{
       1,  .922 ,  .612}63 & \cellcolor[rgb]{ 1, .922 ,  .612}69 &
       \cellcolor[rgb]{ 1,   .922 ,   .612}75 & \cellcolor[rgb]{ 1,
       .922 , . . .  .612}  \\
    \hline
    11 & \cellcolor[rgb]{ 1,   .78, 0' .808} &   &   &   &   &

       \cellcolor[rgb]{ .776 , .937 ,  .808}65 & \cellcolor[rgb]{
       1,  .922 ,  .612}71 & \cellcolor[rgb]{ 1, .922 ,  .612}77 &
       \cellcolor[rgb]{ 1,   .922 , . . .  .612}  \\
    \hline
    12 & \cellcolor[rgb]{ 1,   .78, 0' .808} &   &   &   &   &   &

       \cellcolor[rgb]{ .776 , .937 ,  .808}73 & \cellcolor[rgb]{
       1,  .922 ,  .612}79 & \cellcolor[rgb]{ 1, .922 ,..............612}
       \\
    \hline
    13 & \cellcolor[rgb]{ 1,   .78, 0' .808} &   &   &   &   &   &

       & \cellcolor[rgb]{ .776 , .937 ,  .808}81 & \cellcolor[rgb]{
       1,  .922 , . . .  .612}  \\
    \hline. . .
     & \cellcolor[rgb]{ 1,   .78, . . .  .808} &   &   &   &   &   &
         &   & \cellcolor[rgb]{  .776 ,   .937 , . . .  .808}   \\
    \hline
  \end{tabular}%


  \label{tab:7spaces}
\end{table}

\begin{table}[h]
  \centering
  \caption{Data of the Minimum number of moves for Square and
     Rectangular Klotski with 8 Spaces}
  % Table generated by Excel2LaTeX from sheet 'Sheet1'
  \ begin { tabular }{|c|c|c|c|c|c|c|c|c|c|}
    \hline
```

```
x & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & ...   \\
\hline
7 & \cellcolor[rgb]{ 1,   .78, 0/ .808} & \cellcolor[rgb]{ 1,
    .78, 0/ .808} & \cellcolor[rgb]{ 1,   .78, 0/ .808} &
    \cellcolor[rgb]{ 1,   .78, 0/ .808} & \cellcolor[rgb]{ 1,
```

```
    .78, 0/  .808} & \cellcolor[rgb]{ 1,   .78, 0' .808} &

    \cellcolor[rgb]{ 1,   .78, 0' .808} & \cellcolor[rgb]{ 1,

    .78, 0' .808} & \cellcolor[rgb]{ 1,   .78, ...  .808}  \\
\hline

8 & \cellcolor[rgb]{ 1,   .78, 0' .808}  &  \cellcolor[rgb]{  .776,

    .937 ,   .808}40 & \cellcolor[rgb]{ 1,   .922 ,   .612}46 &
    \cellcolor[rgb]{ 1,   .922 ,   .612}52 & \cellcolor[rgb]{ 1,
    .922 ,   .612}58 & \cellcolor[rgb]{ 1,   .922 ,   .612}64 &
    \cellcolor[rgb]{ 1,   .922 ,   .612}70 & \cellcolor[rgb]{ 1,
    .922 ,   .612}76 & \cellcolor[rgb]{ 1,   .922 , ...  .612}  \\
\hline
9 & \cellcolor[rgb]{ 1,   .78, 0' .808}  &    &  \cellcolor[rgb]{

    .776 ,   .937 ,   .808}48 & \cellcolor[rgb]{ 1,   .922 ,   .612}54
    & \cellcolor[rgb]{ 1,   .922 ,   .612}60 & \cellcolor[rgb]{ 1,
     .922 ,   .612}66 & \cellcolor[rgb]{ 1,   .922 ,   .612}72 &
    \cellcolor[rgb]{ 1,   .922 ,   .612}78 & \cellcolor[rgb]{ 1,
    .922 , ...  .612}  \\
\hline
10 & \cellcolor[rgb]{ 1,   .78, 0' .808}  &    &    &

    \cellcolor[rgb]{ .776 , .937 ,  .808}56 & \cellcolor[rgb]{
    1,  .922 ,  .612}62  & \cellcolor[rgb]{ 1, .922 ,  .612}68 &
    \cellcolor[rgb]{ 1,   .922 ,   .612}74 & \cellcolor[rgb]{ 1,
    .922 ,   .612}80 & \cellcolor[rgb]{ 1,   .922 , ...  .612}  \\
\hline
11 & \cellcolor[rgb]{ 1,   .78, 0' .808}  &    &    &    &

    \cellcolor[rgb]{ .776 , .937 ,  .808}64 & \cellcolor[rgb]{
    1,  .922 ,  .612}70  & \cellcolor[rgb]{ 1, .922 ,  .612}76 &
    \cellcolor[rgb]{ 1,   .922 ,   .612}82 & \cellcolor[rgb]{ 1,
    .922 , ...  .612}  \\
\hline
12 & \cellcolor[rgb]{ 1,   .78, 0' .808}  &    &    &    &    &

    \cellcolor[rgb]{ .776 , .937 ,  .808}72 & \cellcolor[rgb]{
    1,  .922 ,  .612}78  & \cellcolor[rgb]{ 1, .922 ,  .612}84 &
    \cellcolor[rgb]{ 1,   .922 , ...  .612}  \\
\hline
13 & \cellcolor[rgb]{ 1,   .78, 0' .808}  &    &    &    &    &    &

    \cellcolor[rgb]{ .776 , .937 ,  .808}80 & \cellcolor[rgb]{
    1,  .922 ,  .612}86  & \cellcolor[rgb]{ 1, .922 ,..............612}
    \\
\hline
14 & \cellcolor[rgb]{ 1,   .78, 0' .808}  &    &    &    &    &    &

    & \cellcolor[rgb]{ .776 , .937 ,  .808}88 & \cellcolor[rgb]{
    1,  .922 , ...  .612}  \\
\hline...
```

```
          & \cellcolor[rgb]{ 1, .78, ... .808} & & & & & &
              & & \cellcolor[rgb]{ .776, .937, ... .808} \\
        \hline
      \end{tabular}%

      \label{tab:8spaces}
    \end{table}

    \begin{table}[h]
      \centering
```

```
\caption{Data of the Minimum number of moves for Square and
   Rectangular Klotski with 9 Spaces}
% Table generated by Excel2LaTeX from sheet 'Sheet1'
\ begin { tabular }{|c|c|c|c|c|c|c|c|c|c|}
  \hline
  x & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & ...    \\
  \hline
  8 & \cellcolor[rgb]{ 1,   .78, 0 .808} & \cellcolor[rgb]{ 1,
     .78, 0/  .808} & \cellcolor[rgb]{ 1,   .78, 0 .808} &
     \cellcolor[rgb]{ 1,   .78, 0 .808} & \cellcolor[rgb]{ 1,
     .78, 0/  .808} & \cellcolor[rgb]{ 1,   .78, 0 .808} &
     \cellcolor[rgb]{ 1,   .78, 0 .808} & \cellcolor[rgb]{ 1,
     .78, 0 .808} & \cellcolor[rgb]{ 1,   .78, ...  .808}  \\
  \hline
  9 & \cellcolor[rgb]{ 1,   .78, 0 .808}  & \cellcolor[rgb]{ .776,
     .937,   .808}47 & \cellcolor[rgb]{ 1,   .922,   .612}53 &
     \cellcolor[rgb]{ 1,   .922,   .612}59 & \cellcolor[rgb]{ 1,
     .922,   .612}65 & \cellcolor[rgb]{ 1,   .922,   .612}71 &
     \cellcolor[rgb]{ 1,   .922,   .612}77 & \cellcolor[rgb]{ 1,
     .922,   .612}83 & \cellcolor[rgb]{ 1,   .922, ... .612}  \\
  \hline
  10 & \cellcolor[rgb]{ 1,   .78, 0 .808} &    & \cellcolor[rgb]{
     .776,   .937,   .808}55 & \cellcolor[rgb]{ 1,   .922,   .612}61
     & \cellcolor[rgb]{ 1,   .922,   .612}67 & \cellcolor[rgb]{ 1,
      .922,   .612}73 & \cellcolor[rgb]{ 1,   .922,   .612}79 &
     \cellcolor[rgb]{ 1,   .922,   .612}85 & \cellcolor[rgb]{ 1,
     .922, ...  .612}  \\
  \hline
  11 & \cellcolor[rgb]{ 1,   .78, 0 .808} &    &    &
     \cellcolor[rgb]{ .776, .937,  .808}63 & \cellcolor[rgb]{
     1,  .922,  .612}69 & \cellcolor[rgb]{ 1, .922,  .612}75 &
     \cellcolor[rgb]{ 1,   .922,   .612}81 & \cellcolor[rgb]{ 1,
     .922,   .612}87 & \cellcolor[rgb]{ 1,   .922, ... .612}  \\
  \hline
  12 & \cellcolor[rgb]{ 1,   .78, 0 .808} &    &    &    &
     \cellcolor[rgb]{ .776, .937,  .808}71 & \cellcolor[rgb]{
     1,  .922,  .612}77 & \cellcolor[rgb]{ 1, .922,  .612}83 &
     \cellcolor[rgb]{ 1,   .922,   .612}89 & \cellcolor[rgb]{ 1,
     .922, ...  .612}  \\
  \hline
  13 & \cellcolor[rgb]{ 1,   .78, 0 .808} &    &    &    &    &
     \cellcolor[rgb]{ .776, .937,  .808}79 & \cellcolor[rgb]{
     1,  .922,  .612}85 & \cellcolor[rgb]{ 1, .922,  .612}91 &
     \cellcolor[rgb]{ 1,   .922, ... .612}  \\
```

```
\hline
14 & \cellcolor[rgb]{ 1,  .78, 0 .808} &  &  &  &  &
    \cellcolor[rgb]{ .776, .937,  .808}87 & \cellcolor[rgb]{
    1,  .922,  .612}93 & \cellcolor[rgb]{ 1, .922,..............612}
    \\
\hline
15 & \cellcolor[rgb]{ 1,  .78, 0 .808} &  &  &  &  &
    & \cellcolor[rgb]{ .776, .937,  .808}95 & \cellcolor[rgb]{
    1,  .922, ... .612}  \\
\hline...
```

```
    & \cellcolor[rgb]{ 1, .78, ... .808} & & & & & & & & \cellcolor[rgb]{ .776, .937, ... .808} \\
    \hline
  \end{tabular}%


  \label{tab:9spaces}
\end{table}

\begin{table}[h]
  \centering
  \caption{Data of the Minimum number of moves for Square and Rectangular Klotski with 10 Spaces}
  % Table generated by Excel2LaTeX from sheet 'Sheet1'
  \ begin { tabular }{|c|c|c|c|c|c|c|c|c|c|}
    \hline
    x & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & ... \\
    \hline
    9 & \cellcolor[rgb]{ 1, .78, 0/ .808} & \cellcolor[rgb]{ 1, .78, 0/ .808} & \cellcolor[rgb]{ 1, .78, 0/ .808} & \cellcolor[rgb]{ 1, .78, 0/ .808} & \cellcolor[rgb]{ 1, .78, 0/ .808} & \cellcolor[rgb]{ 1, .78, 0/ .808} & \cellcolor[rgb]{ 1, .78, 0/ .808} & \cellcolor[rgb]{ 1, .78, 0/ .808} & \cellcolor[rgb]{ 1, .78, ... .808} \\
    \hline
    10 & \cellcolor[rgb]{ 1, .78, 0/ .808} & \cellcolor[rgb]{ .776, .937, .808}54 & \cellcolor[rgb]{ 1, .922, .612}60 & \cellcolor[rgb]{ 1, .922, .612}66 & \cellcolor[rgb]{ 1, .922, .612}72 & \cellcolor[rgb]{ 1, .922, .612}78 & \cellcolor[rgb]{ 1, .922, .612}84 & \cellcolor[rgb]{ 1, .922, .612}90 & \cellcolor[rgb]{ 1, .922, ... .612} \\
    \hline
    11 & \cellcolor[rgb]{ 1, .78, 0/ .808} & & \cellcolor[rgb]{ .776, .937, .808}62 & \cellcolor[rgb]{ 1, .922, .612}68 & \cellcolor[rgb]{ 1, .922, .612}74 & \cellcolor[rgb]{ 1, .922, .612}80 & \cellcolor[rgb]{ 1, .922, .612}86 & \cellcolor[rgb]{ 1, .922, .612}92 & \cellcolor[rgb]{ 1, .922, ... .612} \\
    \hline
    12 & \cellcolor[rgb]{ 1, .78, 0́ .808} & & &

      \cellcolor[rgb]{ .776, .937, .808}70 & \cellcolor[rgb]{ 1, .922, .612}76 & \cellcolor[rgb]{ 1, .922, .612}82 & \cellcolor[rgb]{ 1, .922, .612}88 & \cellcolor[rgb]{ 1, .922, .612}94 & \cellcolor[rgb]{ 1, .922, ... .612} \\
    \hline
    13 & \cellcolor[rgb]{ 1, .78, 0́ .808} & & & &

      \cellcolor[rgb]{ .776, .937, .808}78 & \cellcolor[rgb]{ 1, .922, .612}84 & \cellcolor[rgb]{ 1, .922, .612}90 & \cellcolor[rgb]{ 1, .922, .612}96 & \cellcolor[rgb]{ 1, .922, ... .612} \\
    \hline
    14 & \cellcolor[rgb]{ 1, .78, 0́ .808} & & & & &
```

\cellcolor[rgb]{ .776, .937, .808}86 & \cellcolor[rgb]{ 1, .922, .612}92 & \cellcolor[rgb]{ 1, .922, .612}98 &

```
        \cellcolor[rgb]{ 1,  .922, ... .612}  \\
      \hline
      15 & \cellcolor[rgb]{ 1,  .78, 0 .808} &  &  &  &  &  &
        \cellcolor[rgb]{ .776, .937, .808}94 & \cellcolor[rgb]{
        1,  .922, .612}100 & \cellcolor[rgb]{ 1, .922,............ 612}
        \\
      \hline
      16 & \cellcolor[rgb]{ 1,  .78, 0 .808} &  &  &  &  &  &
        & \cellcolor[rgb]{ .776,  .937,  .808}102 &
        \cellcolor[rgb]{ 1,  .922, ... .612}  \\
      \hline...
       & \cellcolor[rgb]{ 1, .78, ... .808} &  &  &  &  &  &
         &  & \cellcolor[rgb]{ .776, .937, ... .808}  \\
      \hline
    \end{tabular}%


    \label{tab:10 spaces}
  \end{table}

  \begin{table}[h!]
    \centering
    \caption{Data of the Minimum number of moves for Square and
        Rectangular Klotski with 11 Spaces}
    % Table generated by Excel2LaTeX from sheet 'Sheet1'
    \ begin { tabular }{|c|c|c|c|c|c|c|c|c|c|}
      \hline
      x & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & ...   \\
      \hline

      10 & \cellcolor[rgb]{ 1,  .78, 0 .808} & \cellcolor[rgb]{ 1,

       .78, 0  .808} & \cellcolor[rgb]{ 1,  .78, 0 .808} &

        \cellcolor[rgb]{ 1,  .78, 0 .808} & \cellcolor[rgb]{ 1,

       .78, 0  .808} & \cellcolor[rgb]{ 1,  .78, 0 .808} &

        \cellcolor[rgb]{ 1,  .78, 0 .808} & \cellcolor[rgb]{ 1,

        .78, 0 .808} & \cellcolor[rgb]{ 1,  .78, ... .808}  \\
      \hline

      11 & \cellcolor[rgb]{ 1,  .78, 0 .808}  & \cellcolor[rgb]{ .776,

         .937,  .808}61 & \cellcolor[rgb]{ 1,  .922,  .612}67 &
        \cellcolor[rgb]{ 1,  .922,  .612}73 & \cellcolor[rgb]{ 1,
        .922,  .612}79 & \cellcolor[rgb]{ 1,  .922,  .612}85 &
        \cellcolor[rgb]{ 1,  .922,  .612}91 & \cellcolor[rgb]{ 1,
        .922,  .612}97 & \cellcolor[rgb]{ 1,  .922, ... .612}  \\
      \hline

      12 & \cellcolor[rgb]{ 1,  .78, 0 .808} &  & \cellcolor[rgb]{
```

```
    .776, .937, .808}69 & \cellcolor[rgb]{ 1, .922, .612}75
    & \cellcolor[rgb]{ 1, .922, .612}81 & \cellcolor[rgb]{ 1,
     .922, .612}87 & \cellcolor[rgb]{ 1, .922, .612}93 &
    \cellcolor[rgb]{ 1, .922, .612}99 & \cellcolor[rgb]{ 1,
    .922, .612}... \\
\hline

13 & \cellcolor[rgb]{ 1, .78, 0 .808} & & &

    \cellcolor[rgb]{ .776, .937, .808}77 & \cellcolor[rgb]{
    1, .922, .612}83 & \cellcolor[rgb]{ 1, .922, .612}89 &
    \cellcolor[rgb]{ 1, .922, .612}95 & \cellcolor[rgb]{ 1,
```

```
        .922 ,   .612}101 & \cellcolor[rgb]{ 1,   .922 , . . . .612}  \\
      \hline

      14 & \cellcolor[rgb]{ 1,   .78, 0 .808} &   &   &   &

        \cellcolor[rgb]{ .776 , .937 ,  .808}85 & \cellcolor[rgb]{
        1,   .922 ,   .612}91 & \cellcolor[rgb]{ 1,   .922 ,   .612}97 &
        \cellcolor[rgb]{ 1,   .922 ,   .612}103 & \cellcolor[rgb]{ 1,
        .922 , . . . .612}  \\
      \hline

      15 & \cellcolor[rgb]{ 1,   .78, 0 .808} &   &   &   &   &

        \cellcolor[rgb]{ .776 , .937 ,  .808}93 & \cellcolor[rgb]{
        1,   .922 ,  .612}99 & \cellcolor[rgb]{ 1, .922 , .612}105 &
        \cellcolor[rgb]{ 1,   .922 , . . . .612}  \\
      \hline

      16 & \cellcolor[rgb]{ 1,   .78, 0 .808} &   &   &   &   &   &

        \cellcolor[rgb]{ .776 , .937 ,  .808}101 & \cellcolor[rgb]{
        1,   .922 ,  .612}107 & \cellcolor[rgb]{ 1, .922 ,............. 612}
        \\
      \hline

      17 & \cellcolor[rgb]{ 1,   .78, 0 .808} &   &   &   &   &   &

        & \cellcolor[rgb]{ .776 ,   .937 ,   .808}109 &
        \cellcolor[rgb]{ 1,   .922 , . . . .612}  \\
      \hline. . .
       & \cellcolor[rgb]{ 1, .78 , . . . .808} &   &   &   &   &   &
         &   & \cellcolor[rgb]{ .776 , .937 , . . . .808}  \\
      \hline
    \end{tabular}%


    \label{tab:11 spaces}
  \end{table}
  \newpage
  \begin{center}
    \begin{tabular}{cc}
      No. of Spaces ($S$) & Minimum no. of moves ($M$) \\
      \hline
      5 & $8s-21$ \\
      6 & $8s-22$ \\
      7 & $8s-23$ \\
      8 & $8s-24$ \\
      9 & $8s-25$ \\
      10 & $8s-26$ \\
      11 & $8s-27$ \\
      ... & ... \\
      \hline
      & $M=8s-S-16$
```

```
    \end{tabular}
  \end{center}

  Formulation of $M=8s-S-16$
  \begin{align}
    \label{eqn: gen formula square 8s-S-16}
    M=&(8s-21)+(S-5)(-1)\nonumber \\
    M=&8s-21-S+5 \nonumber \\
    \alignedbox{M= }{8s-S-16 }

  \end{align}

  The above-mention formula (Eq. \ref{eqn: gen formula square
     8s-S-16}) calculates the minimum number of moves for a square
     Klotski given the number of rows or columns ($s$) and the
     number of spaces ($S$) and $4\leq S \leq \infty$
  \\
  \\
  \textbf{Formula for the Minimum Number of Moves of a Rectangular
     Klotski given the number of spaces;}
  \\
  \\
  \textbf{2 Spaces}

  Using the data in table \ref{tab:2spaces} in formulating the
     formula for the minimum number of moves for rectangular Klotski
     containing 2 spaces, the investigators was able to derived the
     formula through the same process as equation \ref{eqn:1Spaces
     Rec Explicit}, \ref{eqn:1spaces Rec N}, \ref{eqn:1spaces rec
     quad }, \ref{eqn:1spaces rec MP}, \ref{eqn:1spaces rec k},
     \ref{eqn:1spaces rec MPD}, and \ref{eqn:1spaces rec finalf}.

  \begin{center}
    \begin{tabular}{cc}
      \multicolumn{2}{c}{$(n+1)(n+k)\ \{n|n\in \mathbbm{Z^+}\}\
         \{k|k\in \mathbbm{Z^+},\ k>1\}$} \\
      & \\
      Product ($P$) & Minimum no. of moves ($M$) \\
      \hline
      \cellcolor{RED} $(n+1)(n+2)$ & \cellcolor{RED} $8n-2$ \\
      $(n+1)(n+3)$ & $8n+2$ \\
      $(n+1)(n+4)$ & $8n+8$ \\
      $(n+1)(n+5)$ & $8n+14$ \\
      $(n+1)(n+6)$ & $8n+20$ \\
      $(n+1)(n+7)$ & $8n+26$ \\
      ... & ... \\
      \hline
      $P=(n+1)(n+k)$ &
    \end{tabular}
  \end{center}

  In the above-shown table, the row that is highlighted in red has a
     separated formula. Hence, only calculates the minimum number of
     moves in a rectangular Klotski that has the dimension of
     $(n+1)(n+2)$. This is due to the constant of the explicit
     formula $8n-2$ since it does not coincide with the common
     difference of $6$ if it is included in the sequence.
```

```
            \\
            \\
            \textbf{Formulation of the Formula of $8n-2$}
            \begin{align}
              \label{eqn:2spaces Exception Derivation of n}
              P=&(n+1)(n+k)\nonumber \\
              P=&n^2+kn+n+k\nonumber \\
```

```
    P=&n^2+(k+1)n+k\nonumber  \\
    0=&n^2+(k+1)n+k-P
\end{align}

using the quadratic equation to find $n$
\begin{align}
    \label{eqn:2spaces Exception quadratic formula to find n}
    n=&\frac{-(k+1)\pm  \sqrt{(k+1)^2+4(1)(k-P)}}{2(1)}\nonumber  \\
    n=&\frac{-k-1\pm  \sqrt{k^2+2k+1-4k+4P}}{2}\nonumber  \\
    n=&\frac{-k-1\pm  \sqrt{k^2-2k+1+4P}}{2}\nonumber  \\
    n=&\frac{-k-1\pm  \sqrt{(k-1)^2+4P}}{2}
\end{align}


Minimum number of moves in terms of the product ($M_P$)
\begin{align}
    \label{eqn:2spaces Exception Minimum number of moves in terms of
        the product}
    M_P=&8(\frac{-k-1\pm   \sqrt{(k-1)^2+4P}}{2})-2\nonumber  \\
    M_P=&4(-k-1\pm   \sqrt{(k-1)^2+4P})-2\nonumber  \\
    M_P=&-4k-4\pm  4  \sqrt{(k-1)^2+4P}-2\nonumber  \\
    M_P=&4\sqrt{(k-1)^2+4P}-4k-6
\end{align}

since, $k=2$. then,
\begin{align}
    \label{eqn:2spaces Exception k=2}
     M_P=&4\sqrt{(k-1)^2+4P}-4k-6\nonumber   \\
     M_P=&4\sqrt{(2-1)^2+4P}-4(2)-6\nonumber   \\
    M_P=&4\sqrt{(1)^2+4P}-8-6\nonumber   \\
    \alignedbox{M_P=  }{4\sqrt{1+4P}-14  }
\end{align}

Furthermore, if we let $l$ and $w$ the number of columns and rows
    and $l>w$, then
\begin{align}
    M=&4\sqrt{1+4P}-14\nonumber   \\
    M=&4\sqrt{1+4(l)(w)}-14\nonumber  \\
    \alignedbox{M=}{4\sqrt{1+4lw}-14}
\end{align}
\\
\textbf{Derivation of the formula for the sequence starting from
    $8n+2$};
\[(n+1)(n+k)\   \{n|n\in  \mathbbm{Z^+}\}\   \{k|k\in  \mathbbm{Z^+},\
    k>2\}\]

The following table summarizes the product and the minimum number
    of moves for a $(n+1)(n+k)$ Klotski, where $k>2$:
\begin{center}
    \begin{tabular}{cc}
        Product ($P$) & Minimum no. of moves ($M$) \\
        \hline
        $(n+1)(n+3)$ & $8n+2$  \\
        $(n+1)(n+4)$ & $8n+8$  \\
```

```
          $(n+1)(n+5)$ & $8n+14$ \\
          $(n+1)(n+6)$ & $8n+20$ \\
          $(n+1)(n+7)$ & $8n+26$ \\
          ... & ... \\
          \hline
          $P=(n+1)(n+k)$  & $M=8n+6k-16$
        \end{tabular}
      \end{center}

      Formulation of $M=8n+6k-16$
      \begin{align}
        \label{eqn:2spaces 8n+6k-16}
        M=&(8n+2)+(k-3)6\nonumber \\
        M=&8n+2+6k-18 \nonumber \\
        M=&8n+6k-16
      \end{align}

      using the quadratic equation to find $n$ (See Eq. \ref{eqn:2spaces
          Exception quadratic formula to find n} for the solution)
      \begin{align}
        n=&\frac{-k-1\pm \sqrt{(k-1)^2+4P}}{2} \nonumber
      \end{align}

       Minimum number of moves in terms of the product ($M_P$)
      \begin{align}
        \label{eqn:2spaces Minimum number of moves in terms of the
            product 8n+6k-16}
        M_P=&8(\frac{-k-1\pm \sqrt{(k-1)^2+4P}}{2})+6k-16\nonumber \\
         M_P=&4(-k-1\pm \sqrt{(k-1)^2+4P})+6k-16\nonumber \\
         M_P=&-4k-4\pm 4 \sqrt{(k-1)^2+4P}+6k-16\nonumber \\
        M_P=&4\sqrt{(k-1)^2+4P}+2k-20
      \end{align}

      the difference of the two sides $+1$ (See Eq. \ref{eqn:1spaces rec
          k} for the solution),
      \begin{align}
        k=&d+1 \nonumber
      \end{align}

      hence the minimum number of moves given the product and difference
          ($M_{Pd}$) is
      \begin{align}
        \label{eqn:2spaces hence the minimum number of moves given the
            product and difference ($M_{Pd}$) is rec 8n+6k-16}
        M_{Pd}=&4 \sqrt{((d+1)-1)^2+4P}+2(d+1)-20\nonumber \\
        M_{Pd}=&4 \sqrt{(d+1-1)^2+4P}+2d+2-20\nonumber \\
        M_{Pd}=&4 \sqrt{(d)^2+4P}+2d-18\nonumber \\
        \alignedbox{M_{Pd}=}{4 \sqrt{d^2+4P}+2d-18}
      \end{align}

      Furthermore, if we let $l$ and $w$ be the number of columns and
          rows and $l>w$, then the minimum number of moves of a
          rectangular Klotski ($M$) is
      \begin{align}
```

```latex
    \label{eqn:2spaces 8n+6k-16 rec finalf lw}
    M_{Pd}=&4 \sqrt{d^2+4P}+2d-18 \nonumber \\
    M=&4 \sqrt{(l-w)^2+4(l)(w)}+2(l-w)-18 \nonumber \\
    M=&4 \sqrt{l^2-2lw+w^2+4lw}+2l-2w-18\nonumber \\
    M=&4 \sqrt{l^2+2lw+w^2}+2l-2w-18\nonumber \\
    M=&4 \sqrt{(l+w)^2}+2l-2w-18\nonumber \\
    M=&4(l+w)+2l-2w-18\nonumber \\
    M=&4l+4w+2l-2w-18\nonumber \\
    \alignedbox{M=}{6l+2w-18}
  \end{align}
  \\
  \textbf{3 Spaces}

  Using the data in table \ref{tab:3spaces} and following the same
      solutions, here are the derived formulae;

  \[(n+1)(n+k)\ \{n|n\in \mathbbm{Z^+}\}\ \{k|k\in \mathbbm{Z^+},\
      k>2\}\]

  The following table summarizes the product and the minimum number
      of moves in a $(n+2)(n+k)$ Klotski:
  \begin{center}
    \begin{tabular}{cc}
      Product ($P$) & Minimum no. of moves ($M$) \\
      \hline
      $(n+2)(n+3)$ & $8n+3$ \\
      $(n+2)(n+4)$ & $8n+9$ \\
      $(n+2)(n+5)$ & $8n+15$ \\
      $(n+2)(n+6)$ & $8n+21$ \\
      $(n+2)(n+7)$ & $8n+27$ \\
      $(n+2)(n+8)$ & $8n+33$ \\
      ... & ... \\
      \hline
      $P=(n+2)(n+k)$ & $M=8n+6k-15$
    \end{tabular}
  \end{center}
  Formulation of $M=8n+6k-15$
  \begin{align}
    \label{eqn:3spaces rec Formulation of $M=8n+6k-15$}
    M =& 8n+(3+[k-3]6)\nonumber \\
    M =& 8n+(3+6k-18)\nonumber \\
    M =& 8n+3+6k-18\nonumber \\
    M =& 8n+6k-15
  \end{align}
  \\
  Derivation of $n$
  \begin{align}
    \label{eqn:3spaces rec Derivation of $n$}
    P=&(n+2)(n+k)\nonumber \\
    P=&n^2+kn+2n+2k\nonumber \\
    P=&n^2+(k+2)n+2k\nonumber \\
    0=&n^2+(k+2)n+2k-P
  \end{align}
  \\
```

```
using the quadratic equation to find $n$
\begin{align}
  \label{eqn:3spaces rec using the quadratic equation to find $n$}
  n =&\frac{-(k +2)\pm  \sqrt{(k +2) ^2+4(1) (2k-P)}}{2(1) }\nonumber  \\
  n =&\frac{-k-2\pm  \sqrt{k ^2+4k+4 -8k+4P}}{2}\nonumber  \\
  n =&\frac{-k-2\pm  \sqrt{k^2 -4k +4+4P}}{2}\nonumber  \\
  n =&\frac{-k-2\pm  \sqrt{(k -2) ^2+4P}}{2}
\end{align}
\\
Minimum number of moves in terms of the product ($M_P$)
\begin{align}
  \label{eqn:3spaces rec  Minimum number of moves in terms of the
     product ($M_P$)}
  M_P =&8(\frac{-k-2\pm \sqrt{(k -2) ^2+4P}}{2}) +6k-15\nonumber  \\
  M_P =&4(-k-2\pm  \sqrt{(k -2) ^2+4P}) +6k-15\nonumber  \\
  M_P =&-4k-8\pm 4 \sqrt{(k -2) ^2+4P}+6k-15\nonumber  \\
  M_P =&4\sqrt{(k -2) ^2+4P}+2k-23
\end{align}
\\
the difference of the two sides $+2$
\begin{align}
  \label{eqn:3spaces rec  the difference of the two sides $+2$}
  d=&(n+k) -(n +2)\nonumber  \\
  d=&n+k-n-2\nonumber  \\
  d=&k-2\nonumber  \\
  k=&d+2
\end{align}
\\
hence the minimum number of moves given the product and difference
   ($M_{Pd}$) is
\begin{align}
  \label{eqn:3spaces rec hence the minimum number of moves given
     the product and difference ($M_{Pd}$) is}
  M_{Pd}=&4\sqrt{((d +2) -2) ^2+4P}+2(d +2) -23\nonumber  \\
  M_{Pd}=&4\sqrt{d ^2+4P}+2d +4 -23\nonumber  \\
  \alignedbox{M_{Pd}=  }{4\sqrt{d ^2+4P}+2d -19  }
\end{align}

Furthermore, if we let $l$ and $w$ be the number of columns and
   rows and $l>w$,
then the minimum number of moves of a rectangular Klotski ($M$) is
\begin{align}
  \label{3spaces rec if we let $l$ and $w$}
  M_{Pd}=&4\sqrt{d ^2+4P}+2d +4 -23\nonumber  \\
  M =&4\sqrt{(l-w) ^2+4(l)(w)}+2(l-w) -19\nonumber  \\
  M =&4\sqrt{l^2+2lw+w ^2+4lw}+2l-2w-19\nonumber  \\
  M =&4\sqrt{l^2+2lw+w ^2}+2l-2w-19\nonumber  \\
  M =&4\sqrt{(l+w) ^2}+2l-2w-19\nonumber  \\
  M =&4(l+w) +2l-2w-19\nonumber  \\
  M =&4l+4w+2l-2w-19\nonumber  \\
  \alignedbox{M=  }{6l+2w-19  }
\end{align}
\\
\textbf{4 Spaces}
```

```
Using the data from table \ref{tab:4spaces}, the derived formulae
    are;

\[(n+1)(n+k)\ \{n|n\in \mathbbm{Z^+}\}\ \{k|k\in \mathbbm{Z^+},\
    k>3\}\]

The following tables summarizes the product and the minimum number
    of moves for a $(n+3)(n+k)$ Klotski:
\begin{center}
   \begin{tabular}{cc}
     Product ($P$) & Minimum  no. of moves ($M$) \\
     \hline
     $(n+3)(n+4)$ & $8n+10$ \\
     $(n+3)(n+5)$ & $8n+16$ \\
     $(n+3)(n+6)$ & $8n+22$ \\
     $(n+3)(n+7)$ & $8n+28$ \\
     $(n+3)(n+8)$ & $8n+34$ \\
     $(n+3)(n+9)$ & $8n+40$ \\
     ... & ... \\
     \hline
      $P=(n+3)(n+k)$ & $M=8n+6k-14$
   \end{tabular}

\end{center}
Formulation of $M=8n+6k-14$
\begin{align}
   M = &8n+(10+[k-4]6)\nonumber \\
   M = &8n+(10+6k-24)\nonumber \\
   M = &8n+10+6k-24\nonumber \\
   M = &8n+6k-14
\end{align}
Derivation of $n$
\begin{align}
   P=&(n+3)(n+k)\nonumber \\
   P=&n^2+kn+3n+3k\nonumber \\
   P=&n^2+(k+3)n+3k\nonumber \\
   0=&n^2+(k+3)n+3k-P
\end{align}

using the quadratic equation to find $n$
\begin{align}
   n=&\frac{-(k+3)\pm \sqrt{(k+3)^2+4(1)(3k-P)}}{2(1)}\nonumber \\
   n=&\frac{-k-3\pm \sqrt{k^2+6k+9-12k+4P}}{2}\nonumber \\
   n=&\frac{-k-3\pm \sqrt{k^2-6k+9+4P}}{2}\nonumber \\
   n=&\frac{-k-3\pm \sqrt{(k-3)^2+4P}}{2}
\end{align}

 Minimum number of moves in terms of the product ($M_P$)
\begin{align}
   M_P=&8(\frac{-k-3\pm \sqrt{(k-3)^2+4P}}{2})+6k-14\nonumber \\
    M_P=&4(-k-3\pm \sqrt{(k-3)^2+4P})+6k-14\nonumber \\
    M_P=&-4k-12\pm 4 \sqrt{(k-3)^2+4P}+6k-14\nonumber \\
   M_P=&4\sqrt{(k-3)^2+4P}+2k-26
```

```
\end{align}

the difference of the two sides $+3$
\begin{align}
  d=&(n+k)-(n+3)\nonumber \\
  d=&n+k-n-3\nonumber \\
  d=&k-3\nonumber \\
  k=&d+3
\end{align}

hence the minimum number of moves given the product and difference
    ($M_{Pd}$) is
\begin{align}
   M_{Pd}=&4\sqrt{((d+3)-3)^2+4P}+2(d+3)-26\nonumber \\
  M_{Pd}=&4\sqrt{d^2+4P}+2d+6-26\nonumber \\
  \alignedbox{M_{Pd}= }{4\sqrt{d^2+4P}+2d-20 }
\end{align}

 Furthermore, if we let $l$ and $w$ be the number of columns and
    rows and $l>w$,
then the minimum number of moves of a rectangular Klotski ($M$) is
\begin{align}
  M_{Pd}=&4\sqrt{d^2+4P}+2d-20\nonumber \\
  M=&4\sqrt{(l-w)^2+4(l)(w)}+2(l-w)-20\nonumber \\
  M=&4\sqrt{l^2+2lw+w^2+4lw}+2l-2w-20\nonumber \\
  M=&4\sqrt{l^2+2lw+w^2}+2l-2w-20\nonumber \\
  M=&4\sqrt{(l+w)^2}+2l-2w-20\nonumber \\
  M =&4( l+w) +2l-2w -20\ nonumber      \\
  M=&4l+4w+2l-2w-20\nonumber \\
  \alignedbox{M= }{6l+2w-20 }
\end{align}

\textbf{5 Spaces}

\[(n+1)(n+k)\ \{n|n\in \mathbbm{Z^+}\}\ \{k|k\in \mathbbm{Z^+},\
    k>4\}\]

The following tables summarizes the product and the minimum number
    of moves in a $(n+4)(n+k)$ Klotski:
\begin{center}
  \begin{tabular}{cc}
    Product ($P$) & Minimum no. of moves ($M$) \\
    \hline
    $(n+4)(n+5)$ & $8n+17$ \\
    $(n+4)(n+6)$ & $8n+23$ \\
    $(n+4)(n+7)$ & $8n+29$ \\
    $(n+4)(n+8)$ & $8n+35$ \\
    $(n+4)(n+9)$ & $8n+41$ \\
    $(n+4)(n+10)$ & $8n+47$ \\
    ... & ... \\
    \hline
    $P=(n+4)(n+k)$ & $M=8n+6k-13$
  \end{tabular}
\end{center}
```

```
Formulation of $M=8n+6k-13$ \\
\begin{align}
  M =& 8n+(17+[k-5]6)\nonumber \\
  M =& 8n+(17+6k-30)\nonumber \\
  M =& 8n+17+6k-30\nonumber \\
  M =& 8n+6k-13
\end{align}

Derivation of $n$
\begin{align}
  P=&(n+4)(n+k)\nonumber \\
  P=&n^2+kn+4n+4k\nonumber \\
  P=&n^2+(k+4)n+4k\nonumber \\
  0=&n^2+(k+4)n+4k-P
\end{align}

using the quadratic equation to find $n$
\begin{align}
   n=&\frac{-(k+4)\pm \sqrt{(k+4)^2+4(1)(4k-P)}}{2(1)}\nonumber \\
   n=&\frac{-k-4\pm \sqrt{k^2+8k+16-16k+4P}}{2}\nonumber \\
   n=&\frac{-k-4\pm \sqrt{k^2-8k+16+4P}}{2}\nonumber \\
  n=&\frac{-k-4\pm \sqrt{(k-4)^2+4P}}{2}
\end{align}

Minimum number of moves in terms of the product ($M_P$)
\begin{align}
  M_P=&8(\frac{-k-4\pm \sqrt{(k-4)^2+4P}}{2})+6k-13\nonumber \\
   M_P=&4(-k-4\pm \sqrt{(k-4)^2+4P})+6k-13\nonumber \\
   M_P=&-4k-16\pm 4 \sqrt{(k-4)^2+4P}+6k-13\nonumber \\
  M_P=&4\sqrt{(k-4)^2+4P}+2k-29
\end{align}

the difference of the two sides $+4$
\begin{align}
  d=&(n+k)-(n+4)\nonumber \\
  d=&n+k-n-4\nonumber \\
  d=&k-4\nonumber \\
  k=&d+4
\end{align}

hence the minimum number of moves given the product and difference ($M_{Pd}$) is
\begin{align}
   M_{Pd}=&4\sqrt{((d+4)-4)^2+4P}+2(d+4)-29\nonumber \\
  M_{Pd}=&4\sqrt{d^2+4P}+2d+8-29\nonumber \\
  \alignedbox{M_{Pd}=}{4\sqrt{d^2+4P}+2d-21}
\end{align}

Furthermore, if we let $l$ and $w$ be the number of columns and rows and $l>w$,
 then the minimum number of moves of a rectangular Klotski ($M$) is
\begin{align}
  M_{Pd}=&4\sqrt{d^2+4P}+2d-21\nonumber \\
```

```
    M =&4\ sqrt{(l- w) ^2+4( l)(w)}+2( l- w) -21\ nonumber  \\
    M =&4\ sqrt{l^2+2 lw+w ^2+4 lw}+2 l-2 w-21\ nonumber  \\
    M =&4\ sqrt{l^2+2 lw+w ^2}+2 l-2 w-21\ nonumber  \\
    M =&4\ sqrt{( l+w) ^2}+2 l-2 w-21\ nonumber  \\
    M =&4( l+w) +2l-2 w-21\ nonumber  \\
    M =&4 l+4 w+2 l-2 w-21\ nonumber  \\
    \alignedbox{M=  }{6 l+2 w-21  }
  \end{align}
  \\

  The following table summarizes the product and the minimum number of moves given the product and the difference for a $(n+f)(n+k)$ rectangular Klotski, where $f\geq 1$, $k>2$, $k>f$, $f<S$, and $2\leq S \leq \infty$:
  \begin{center}
    \begin{tabular}{cc}
      No. of Spaces ($S$) & Minimum no. of moves ($M_{Pd}$) \\
      \hline
      $2$ & $4\sqrt{d^2+4P}+2d-18$ \\
      $3$ & $4\sqrt{d^2+4P}+2d-19$ \\
      $4$ & $4\sqrt{d^2+4P}+2d-20$ \\
      $5$ & $4\sqrt{d^2+4P}+2d-21$ \\
      $6$ & $4\sqrt{d^2+4P}+2d-22$ \\
      ... & ... \\
      \hline
    \end{tabular}
  \end{center}

  Formulation of $M_{Pd}=4\sqrt{d^2+4P}+2d-S-16$
  \begin{align}
    \label{eqn:M_Pd=4d^2+4P+2d-S-16}
     M_{Pd}=& (4\sqrt{d^2+4P}+2d-18)+(S-2)(-1)\nonumber \\
    M_{Pd}=& 4\sqrt{d^2+4P}+2d-18-S+2\nonumber \\
    \alignedbox{M_{Pd}=  }{4\sqrt{d^2+4P}+2d-S-16  }
  \end{align}
  \\
   The above-mentioned formula (Eq. \ref{eqn:M_Pd=4d^2+4P+2d-S-16}) calculates the minimum number of moves required to solve the rectangular Klotski puzzle whose product is $(n+f)(n+k)$, where $f\geq 1$, $k>2$, $k>f$, $f<S$, and $2\leq S \leq \infty$ and if given the product and the difference of the sides.
  \\

  The following table summarizes the product and the minimum number of moves in a $(n+f)(n+k)$ rectangular Klotski, where $f\geq 1$, $k>2$, $k>f$, $f<S$, and $2\leq S \leq \infty$:
  \begin{center}
    \begin{tabular}{cc}
      No. of Spaces ($S$) & Minimum no. of moves ($M$) \\
      \hline
      $2$ & $6l+2w-18$ \\
      $3$ & $6l+2w-19$ \\
      $4$ & $6l+2w-20$ \\
      $5$ & $6l+2w-21$ \\
```

```
      $6$ & $6l+2w-22$ \\
      ... & ... \\
      \hline
    \end{tabular}
  \end{center}

  Formulation of $M=6l+2w-S-16$
  \begin{align}
    \label{eqn:6l+2w-S-16}
    M=& (6l+2w-18)+(S-2)(-1)\nonumber \\
    M=& 6l+2w-18-S+2\nonumber \\
    \alignedbox{M= }{6l+2w-S-16 }
  \end{align}

  The above-mentioned formula (Eq. \ref{eqn:6l+2w-S-16}) calculates the minimum number of moves required to solve the rectangular Klotski puzzle whose product is $(n+f)(n+k)$, where $f\geq 1$, $k>2$, $k>f$, $f<S$, and $2\leq S \leq \infty$.
  \\
  \\
  \textbf{General Formulae} \\
  \\
  General Formula for Square Klotski:
  \begin{align}
    \label{eqn:Piecewise square}
    M=
    \begin{dcases}
      8s-3S-8 & \text{if } 1\leq S \leq 4,\ s\geq S \\
      8s-S-16 & \text{if } 4\leq S \leq \infty,\ s\geq S
    \end{dcases}
  \end{align}
  General Formula for Rectangular Klotski:
  \begin{align}
    \label{eqn:Piecewise rec}
    M=
    \begin{dcases}
      4 \sqrt{1+4lw}-14 & \text{if } (n+1)(n+2) \\
      6l+2w-13 & \text{if } S=1 \\
      6l+2w-S-16 & \text{if } (n+f)(n+k) \\
      & \text{where } f\geq1,\\
      & k>2,\ k>f,\\
      & f<S, \\
      & 2\leq S \leq \infty
    \end{dcases}
  \end{align}
  General Formula for Rectangular Klotski given the difference ($d$) and the product ($P$):
  \begin{align}
    \label{eqn:MPD Piecewise}
    M_{Pd}=
    \begin{dcases}
      4\sqrt{1+4P}-14 & \text{if } (n+1)(n+2) \\
      4\sqrt{d^2+4P}+2d-13 & \text{if } S=1 \\
      4\sqrt{d^2+4P}+2d-S-16 & \text{if } (n+f)(n+k) \\
```

```
        & \text{where } f\geq1,\\
        & k>2,\ k>f,\\
        & f<S, \\
        & 2\leq S \leq \infty
      \end{dcases}
    \end{align}
    $M$ - Minimum number of Moves \\
    $P$ - Product ($l\times w$) \\
    $d$ - Difference ($l-w$) \\
    $s$ - Number of rows or columns \\
    $l$ - Number of Columns \\
    $w$ - Number of Rows \\
    $S$ - Number of Spaces \\
    \\
    \textbf{Examples}
    \\
    \\
    For Formula $M=8s-3S-8$ (Eq. \ref{eqn:Piecewise square}): \\

    In a square Klotski with a $5\times 5$ dimension with $4$ spaces, by applying the formula $M=8s-3S-8$, the minimum number of moves is
    \begin{align}
      M=&8(5)-3(4)-8\nonumber \\
      M=&40-12-8 \nonumber \\
      \alignedboxn{M= }{20 } \nonumber
    \end{align}
    \\
    For Formula $M=8s-S-16$ (Eq. \ref{eqn:Piecewise square}): \\

    In a square Klotski with a $8\times 8$ dimension with $6$ spaces, by applying the formula $M=8s-S-16$, the minimum number of moves is
    \begin{align}
      M=&8(8)-6-16 \nonumber \\
      M=&64-6-16 \nonumber \\
      \alignedboxn{M=}{42} \nonumber
    \end{align}
    \\
    For Formula $M=4\sqrt{1+4lw}-14$ (Eq. \ref{eqn:Piecewise rec}): \\

    In a rectangular Klotski with a $3\times 4$ dimension, by applying the formula $M=4\sqrt{1+4lw}-14$, the minimum number of moves is
    \begin{align}
      M=&4\sqrt{1+4(4)(3)}-14 \nonumber \\
      M=&4\sqrt{1+48}-14 \nonumber \\
      M=&4\sqrt{49}-14 \nonumber \\
      M=&4(7)-14 \nonumber \\
      M=&28-14 \nonumber \\
      \alignedboxn{M=}{14} \nonumber
    \end{align}
    \\
    For Formula $M=6l+2w-13$ (Eq. \ref{eqn:Piecewise rec}): \\

```

```
In a rectangular Klotski with a $3\times 7$ dimension, by applying the formula $M=6l+2w-13$, the minimum number of moves is
\begin{align}
  M=&6(7)+2(3)-13 \nonumber \\
  M=&42+6-13 \nonumber \\
  \alignedboxn{M=}{35} \nonumber
\end{align}
\\
For Formula $M=6l+2w-S-16$ (Eq. \ref{eqn:Piecewise rec}): \\

In a rectangular Klotski with a $9\times 12$ dimension with $8$ spaces, by applying the formula $M=6l+2w-S-16$, the minimum number of moves is
\begin{align}
  M=&6(12)+2(9)-8-16 \nonumber \\
  M=&72+18-8-16 \nonumber \\
  \alignedboxn{M=}{66} \nonumber
\end{align}
\\
For Formula $M_{Pd}=4\sqrt{1+4P}-14$ (Eq. \ref{eqn:MPD Piecewise}) \\

In a rectangular Klotski that has the product of $12$, by applying the formula $M_{Pd}=4\sqrt{1+4P}-14$, the minimum number of moves is
\begin{align}
  M_{Pd}=&4\sqrt{1+4(12)}-14 \nonumber \\
  M_{Pd}=&4\sqrt{49}-14 \nonumber \\
  M_{Pd}=&4(7)-14 \nonumber \\
  M_{Pd}=&28-14 \nonumber \\
  \alignedboxn{M_{Pd}=}{14} \nonumber
\end{align}
\\
For Formula $M_{Pd}=4\sqrt{d^2+4P}+2d-13$ (Eq. \ref{eqn:MPD Piecewise}) \\

In a rectangular Klotski whose product and difference of the dimensions are $21$ and $4$, respectively, by applying the formula for the product and difference, the minimum number of moves is
\begin{align}
  M_{Pd}=&4 \sqrt{4^2+4(21)}+2(4)-13 \nonumber \\
  M_{Pd}=&4 \sqrt{16+84}+8-13 \nonumber \\
  M_{Pd}=&4 \sqrt{100}-5 \nonumber \\
  M_{Pd}=&4(10)-5 \nonumber \\
  M_{Pd}=&40-5 \nonumber \\
  \alignedboxn{M_{Pd}=}{35} \nonumber
\end{align}
\\
For Formula $M_{Pd}=4\sqrt{d^2+4P}+2d-S-16$ (Eq. \ref{eqn:MPD Piecewise}) \\

In a rectangular Klotski with a product and a difference of $108$ and $3$ respectively and a spaces of $8$, by applying the
```

```
        formula $M_{Pd}=4\sqrt{d^2+4P}+2d-S-16$, the minimum number of
        moves is
    \begin{align}
      M_{Pd}=&4\sqrt{(3)^2+4(108)}+2(3)-(8)-16 \nonumber \\
      M_{Pd}=&4\sqrt{9+432}+6-8-16 \nonumber \\
      M_{Pd}=&4\sqrt{441}+6-8-16 \nonumber \\
      M_{Pd}=&4(21)+6-8-16 \nonumber \\
      \alignedboxn{M_{Pd}=}{66} \nonumber
    \end{align}

    \section{Chapter IV}
    \subsection{Summary of the Study}

    Therefore, the investigators came up with a technique in the game
        Klotski Puzzle where the player must move the tile diagonally
        from its starting point to its predetermined spot in order to
        obtain the minimum number of moves. Varied formulae were
        derived and formulated to determine the minimum number of moves
        based on the dimensions and number of spaces in the
        predetermined spot of the Klotski puzzle. The investigators
        used the piecewise function, the quadratic formula, and the
        arithmetic sequence formed by the pattern whose common
        difference are 8 and 6 as their basis in developing these
        formulae.

    Thus, this study shows potential not limiting only in the field of
        games but also in some real life situation that involves
        algorithm that calculate the minimum number of moves needed to
        reach a specific goal such as parking assistance system,
        traffic management, cargo handling, space optimization, and
        other situations that deals with constraints and involves the
        process of optimizing movement.
    \subsection{Recommendation}

    The investigators recommend to study further not just on square and
        rectangles but also on triangle and other shaped Klotskis. The
        movement of more than 1 block can also be taken into
        consideration in the future conduct of related investigation.
        Also, blocks with diferent shapes and sizes will be a good
        topic for investigation. Lastly, the investigators of this
        study recommends future researchers to investigate the
        relevance of the results of this study and to assess the
        feasibility of the suggested potential real life application.

    \bibliographystyle{plain}
    \bibliography{MIP}

  \end{document}
```

References

[1] Gauss on sequences - mathbitsnotebook(a2).

[2] Kiddle - visual search engine for kids.

[3] Sliding block puzzles - introduction.

[4] How are arithmetic sequences used? - quora, 2017.

[5] piecewise, adv. meanings, etymology and more | oxford english dictionary. *Oed.com*, 2023.

[6] Gracie . The ultimate guide to a klotski puzzle, 04 2023.

[7] National Museum of American History . Dad's puzzler, once owned by olive c. hazlett.

[8] admin. Sequential movement puzzles: The history of, 05 2021.

[9] CUEMATH. Quadratic equation - formula, examples | quadratic formula, 2024.

[10] Oded Goldreich. Finding the shortest move-sequence in the graph-generalized 15-puzzle is np-hard. *Lecture Notes in Computer Science*, pages 1–5, 01 2011.

[11] Lumen. Define and write piecewise functions | intermediate algebra.

[12] Mathnasium. The history behind the quadratic formula, 09 2017.

[13] Shinobu Nagayama, Tsutomu Sasao, and Jon Butler. Piecewise arithmetic expressions of numeric functions and their application to design of numeric function generators. 0:1–21.

[14] Elizabeth Stapel. The quadratic formula explained | purplemath, 2019.

[15] Zhihua Zhong. Additive approximation algorithms for sliding puzzle. *Lecture Notes in Computer Science*, pages 129–146, 01 2023.