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Some Results about Pseudo with BD-algebra

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Abstract: *In this paper, we introduce the notions of pseudo BD-algebra and bounded pseudo BD-algebra. Also, we give some theorems and relationships among them are debated.*

Keywords: *BD*-algebra, pseudo *BD*-algebra , bounded pseudo *BD*-algebra.

Introduction:

In 1966 by Y .Imai and K.Iseki introducee d the notion of BCKalgebra[12], In 1998, Y. B. Jun, E. H. Rogh and H. S. Kim introducee d the notion of a BH- ɑ lge brɑ , and the notion of ideal of a BH- ɑ lge brɑ [12]. In 2022, T.Bantaojai and et. cl. introducee d the notion of a *BD*-algebra [1,12]. In this paper, we define the concepts of pseudo *BD*-algebra and bounded pseudo *BD*-algebra. Also, we give some theorems and relationships among them are debated.

1. BD-algebra

In this part, we introduce the definition of *BD*-algebra and study some of proposition about it.

Definition 1.1[1.12].

A *BD*-algebra (BD-A) is a non-empty set ζ with a constant o and a binary " $\diamond$ " satisfying the following axioms hold $\forall \varepsilon, \eta \in \zeta$, if :

(1) $\varepsilon \diamond o = \varepsilon, \forall \varepsilon \in \zeta$,

(2)$(\varepsilon \diamond \eta) = o$ *and* $\eta \diamond \varepsilon = o$, *then* $\varepsilon = \eta$.

Remark **1.2[1.12].**

A *BD-A* can be (partially) order by $\varepsilon \leq \eta$ if and only if $(\varepsilon \diamond \eta) = 0$, $\forall \varepsilon, \eta \in \zeta$.

Proposition **1.3[1.12].**

In any *BD-A* $(\zeta;\diamond, o)$, the following hold: $\forall \varepsilon, \eta, \iota, \kappa \in \zeta$

(1) $\varepsilon \diamond \varepsilon = o$,

(2) $\varepsilon \diamond o = \varepsilon$,

(3)$(\varepsilon \diamond \eta) \diamond \iota = (\varepsilon \diamond \iota) \diamond \eta$,

(4)$(\varepsilon \diamond \eta) \diamond (\iota \diamond u) = (\varepsilon \diamond \iota) \diamond (\eta \diamond \kappa)$,

(5) $(\varepsilon \diamond (\varepsilon \diamond \eta)) \diamond \eta = o$,

(6) $(\varepsilon \diamond (\varepsilon \diamond \iota) \diamond \eta) \diamond \eta = o$,

Remark **1.4[1.12].**

Let $(\zeta;\diamond, o)$ be a *BD-A*, then

(1) If $\varepsilon \leq o, \forall \varepsilon \in \zeta$, then ζ contains only .

(2) If $\varepsilon \leq \eta$, then $\varepsilon * (\varepsilon * (\varepsilon * \eta)) = o, \forall \varepsilon, \eta \in \zeta$.

(3) If $\varepsilon \leq \eta$ such that $\varepsilon * \iota \leq \eta$, then $o \leq \iota, \forall \varepsilon, \eta, \iota \in \zeta$.

Definition **1.5[13].**

If $(\zeta;\diamond, o)$ is a *BD-A*, we call ζ is Bo if there is an element $e \in \zeta$ satisfying $\varepsilon \leq e$. for all $\varepsilon \in \zeta$, then e is called a unit (Un) of ζ.

Remark **1.6[13].**

In a bounded (Bo) Bo *BD-A* $(\zeta;\diamond, o)$ with Un e denoted $e \diamond \varepsilon$ by ε^*, for all $\varepsilon \in \zeta$.

Example **1.7[2].**Let $\zeta = \{0,1,2,3\}$ be a set with following table:

$\diamond$	**0**	**1**	**2**	**3**

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0	**0**	**0**	**0**	**0**
1	**1**	**0**	**0**	**0**
2	**2**	**0**	**0**	**0**
3	**3**	**3**	**3**	**0**

Therefore $(\zeta;\diamond,o)$ is a *BD-A*. Notice that ζ is Bo with Un 3.

***Remark* 1.8[13].**

The Un in Bo *BD-A* is not an unique and the following example shows this.

***Example* 1.9[13].**

Consider the following *BD-A* $\zeta=\{0,1,2\}$, with the following table:

$\diamond$	**0**	**1**	**2**
0	**0**	**0**	**0**
1	**1**	**0**	**0**
2	**2**	**0**	**0**

Notice that $(\zeta;\diamond,o)$ is Bo with two Uns 1,2.

***Remark* 1.10.**

In *BD-A*, we will study the Bo with one Un only.

***Proposition* 1.11[13].**

In Bo *BD-A*, $(\zeta;\diamond,o)$, $\exists\ \varepsilon,\eta\in\zeta$, the following are hold:

(1) $e^*=o, o^*=e$

(2) $\varepsilon^*\diamond\eta=\eta^*\diamond\varepsilon$

(3) $o\diamond\eta=o$

(4) $e^*\diamond\varepsilon=\text{o}$

(5) $\varepsilon^{**}\leq\varepsilon.$

2. Bounded Involutory BD-algebra

***Definition* 2.1.**

For a Bo *BD-A* $(\zeta;\diamond,o)$), if an element x satisfies $(\varepsilon^*)^*=\varepsilon$, then ζ is called an **involution (Inv)**. If every element of ζ is an Inv , we call ζ that is an involutory *BD-A*.

***Example* 2.2.**

Let $\zeta=\{0,1,2,3,4\}$ be a set with following table:

$\diamond$	**0**	**1**	**2**	**3**	**4**
0	**0**	**0**	**0**	**0**	**0**
1	**1**	**0**	**1**	**0**	**0**
2	**2**	**2**	**0**	**0**	**0**
3	**3**	**2**	**1**	**0**	**4**
4	**4**	**0**	**0**	**0**	**0**

Then $(\zeta;\diamond,o)$ is a Bo *BD-A* with Un 3, and ζ is an Inv.

***Proposition* 2.3.**

In Bo involutory *BD-A*, for $\varepsilon,\eta\in\zeta$, the following is hold $\varepsilon^*\leq\varepsilon^{***}$.

Proof:

$\varepsilon^*\diamond\varepsilon^{***}=(e\diamond\varepsilon)\diamond(e\diamond\varepsilon^{**})$, from Proposition 1.3(4), we will get

$\varepsilon^*\diamond\varepsilon^{***}=\big(e\diamond(e\diamond\varepsilon^{**})\big)\diamond\varepsilon=(e\diamond(e\diamond\varepsilon))\diamond\varepsilon)$, since ζ is an Inv, thus $\varepsilon^*\diamond\varepsilon^{***}=\big((e\diamond\varepsilon)\diamond(e\diamond\varepsilon)\big)=\varepsilon^*\diamond\varepsilon^*=o.$ ■

***Proposition* 2.4.**

Let $(\zeta;\diamond,o)$ is a Bo *BD-A*, $\exists\ \varepsilon,\eta\in\zeta$,then the following condition are equivalent:

(1) ζ is Inv;

(2) $\varepsilon\diamond\eta=\eta^*\diamond\varepsilon^*$;

(3)$\varepsilon\diamond\eta^*=\eta\diamond\varepsilon^*$;

(4) if $\varepsilon \leq \eta^{*}$then $\eta \leq \varepsilon^{*}$.

Proof:

(1)⇒(2) Since Xinvolutory, we have $\varepsilon^{**} = \varepsilon, \forall\ \varepsilon\ \in \zeta$, by Proposition 1.3(7). Implies that $\varepsilon \diamond \eta = \varepsilon^{**} \diamond \eta = \eta^{*} \diamond \varepsilon^{*}$. gives

(2)⇒(3) By (2), $\varepsilon \diamond \eta^{*} = \eta^{**} \diamond \varepsilon^{*}$and $\eta \diamond \varepsilon^{*} = \varepsilon^{**} \diamond \eta^{*}$, also By Proposition 1.11 (2), $\eta^{**} \diamond \varepsilon^{*} = \varepsilon^{**} \diamond \eta^{*}$, therefore $\varepsilon \diamond \eta^{*} = \eta \diamond \varepsilon^{*}$;

(3)⇒(4) If $\varepsilon \leq \eta^{*}$, then $\varepsilon \diamond \eta^{*} = o$. so $\varepsilon \diamond \eta^{*} = o$ by(3), therefore $\eta \leq \varepsilon^{*}$

(3)⇒(4), by Proposition1.11(5), we have $\varepsilon^{**} \leq \varepsilon$. Also it is clear that $\varepsilon^{*} \leq \varepsilon^{*}$, then (4) gives $\varepsilon \leq \varepsilon^{**}$, comparison gives $\varepsilon^{**} = \varepsilon, \forall \varepsilon \in \zeta$, therefore ζ is an Inv. ■

***Remark* 2.5.**

The following show that every *BD-A* can be extension to Bo *BD-A*

***Theorem* 2.6.**

Let $(\zeta;\diamond, o)$ be *BD-A* and $e \notin X$.We defined the operation $*'$ on

$$\bar{X} = \zeta \cup \{e\} \text{as follows} \quad x \diamond' y = \begin{cases} x \diamond y & \text{if } \varepsilon, \eta \in \zeta \\ 0 & \text{if } \varepsilon \in \zeta \text{ and } \eta = e \\ e & \text{if } \varepsilon = e \text{ and } \eta \in \zeta \\ 0 & \text{if } \varepsilon = \eta = e \end{cases}$$

Then $(\bar{X}, *', 0)$ is a Bo *BD-A* with Un e, and it is called the **Iseki's extension** (Is-ex)of $(\zeta;\diamond, o)$.

Proof:

Let $\varepsilon, \eta \in \bar{X}$, since ζ is *BD-A*, keep to prove $\zeta \cup \{e\}$ is *BD-A*

(1) $e \diamond o = e$ if $\varepsilon = e, \eta = o \in \zeta$

(2) $(e \diamond \varepsilon) \diamond \eta = 0$ *and* $\eta \diamond (e \diamond \varepsilon) = 0 \Rightarrow (e \diamond \eta) \diamond \varepsilon = 0$ *and* $e \diamond (\eta \diamond \varepsilon) = 0 \Rightarrow e \diamond (\varepsilon \diamond \eta) = 0$ *and* $e \diamond (\eta \diamond \varepsilon) = 0 \Rightarrow \varepsilon = \eta$, this show that $(\bar{X}, *', 0)$ is *BD-A* . It is clear that e is Un of $\bar{X}$. ■

***Proposition* 2.7.**

Let $(\zeta;\diamond, o)$ be a *BD-A* and $\bar{X}$ be the Is-ex of ζ. Then $\varepsilon^{*} = o$ or $\varepsilon^{*} = e,\ \forall\ x \in \bar{X}$.

Proof: The proof is clear by Theorem 2.6. ■

3. *Pseudo BD-algebra*

In this part, we introduce definitions of Ps *BD-A*, Bo *BD-A* and some of their properties.

***Definition* 3.1**

A pseudo (Ps)*BD-A* $(\zeta;\diamond, o)$ is anon empty set with a constant 0 and two binary operations $\diamond$ and # satisfying the following, $\exists\ \varepsilon, \eta, \iota \in \zeta$

(1) $\varepsilon \diamond o = \varepsilon \# o = \varepsilon$

(2) $(\varepsilon \diamond \eta) = o$, $(\varepsilon \# \eta) = o$ *and* $(\eta \diamond \varepsilon) = 0$, $(\eta \# \varepsilon) = 0$ implies $\varepsilon = \eta$.

***Remark* 3.2.**

In Ps *BD-A* $(\zeta, \diamond, \#, o)$, we can define a binary operation $\leq$ by $\varepsilon \leq \eta$ if and only if , $\varepsilon \diamond \eta = o$ and $\varepsilon \# \eta = 0,\ \forall\ \varepsilon, \eta \in \zeta$.

***Remark* 3.3.**

Every *BD-A* $(\zeta, \diamond, o)$ is a Ps *BD-A* $(\zeta, \diamond, \#, o)$ in a natural way. But the converse is not true as shown in the following example:

***Example* 3.4.**

Let $\zeta = \{0,1,2,3\}$ be a set with the following tables:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	1
3	3	3	0	0

#	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	3	0	3
3	3	3	0	0

Then $(\zeta, \diamond, o)$ and $(\zeta, \#, o)$ are not *BD-A*, since

$(2 \diamond 1) \diamond 3 = 1 \neq 0 = (2 \diamond 3) \diamond 1$ and $(2\#1)\#3 = 0 \neq 3 = (2\#3)\#1$.

It is easy to check that $(\zeta, \diamond, \#, o)$ is a Ps *BD-A* .

***Proposition* 3.5.**

Let $(\zeta, \diamond, \#, o)$ be a Ps BD-A. Then the following hold: for all $\varepsilon, \eta, \iota\ \in\ \zeta$.

1- $\varepsilon\ \diamond\ (\varepsilon \# \eta)\ \leq \eta, \varepsilon \# (\varepsilon\ \diamond\ \eta)\ \leq \eta$,

2- $\varepsilon\ \diamond\ \eta\ \leq \iota\ \Leftrightarrow\ \varepsilon \# \iota\ \leq\ \eta$,

3- $o\ \diamond\ (\varepsilon\ \diamond\ \eta)\ =\ (o \# \varepsilon) \# (o\ \diamond\ \eta)$,

4- $o \# (\varepsilon \# \eta)\ =\ (o\ \diamond\ \varepsilon)\ \diamond\ (o \# \eta)$,

5- $o\ \diamond\ \varepsilon\ =\ o \# \varepsilon$.

Proof.

1- We obtain $[\varepsilon\ \diamond\ (\varepsilon \# \eta)] \# \eta =\ (\varepsilon \# \eta)\ \diamond\ (\varepsilon \# \eta)\ =\ o$ and $[\varepsilon \# (\varepsilon\ \diamond\ \eta)]\ \diamond\ \eta\ =\ (\varepsilon\ \diamond\ \eta) \# (\varepsilon\ \diamond\ \eta)\ =\ 0$.

Hence $\varepsilon \diamond (\varepsilon \# \eta) \leq \eta \quad and \quad \varepsilon * (\varepsilon \# \eta) \leq \eta.$

2− $\varepsilon \diamond \eta \leq \iota \Leftrightarrow (\varepsilon \diamond \eta) \# \iota = o \Leftrightarrow (\varepsilon \# \iota) \diamond \eta = o \Leftrightarrow \varepsilon \# \iota \leq \eta.$

(3) and (4) , For any $\varepsilon, \eta \in \zeta$, we have

$(o \# \varepsilon) \# (o \diamond \eta) = \left[\left((\varepsilon \diamond \eta) \diamond (\varepsilon \diamond \eta)\right) \# \varepsilon\right] \# (o \diamond \eta) = \left[\left((\varepsilon \diamond \eta) \# \varepsilon\right) \diamond (\varepsilon \diamond \eta)\right] \# (o \diamond \eta)$

$= \left[\left((\varepsilon \# \varepsilon) \diamond \eta\right) \diamond (\varepsilon \diamond \eta)\right] \# (o \diamond \eta) = [(o \diamond \eta) \# (o \diamond \eta)] \diamond (\varepsilon \diamond \eta) = o \diamond (\varepsilon \diamond \eta)$

And

$(o \diamond \varepsilon) \diamond (o \# \eta) = [((\varepsilon \# \eta) \diamond (\varepsilon \# \eta)) \diamond \varepsilon] \diamond (o \# \eta) = [((\varepsilon \# \eta) * \varepsilon) * (\varepsilon \# \eta)] \diamond (o \# \eta)$

$= [((\varepsilon \diamond \varepsilon) \# \eta) \# (\varepsilon \# \eta)] \diamond (o \# \eta) = [(o \# \eta) \diamond (o \# \eta)] \# (\varepsilon \# \eta) = o \# (\varepsilon \# \eta).$

(5) For any $\varepsilon \in \zeta$, we obtain $o \diamond \varepsilon = (\varepsilon \# \varepsilon) \diamond \varepsilon = (\varepsilon \diamond \varepsilon) \# \varepsilon = 0 \# \varepsilon.$

***Definition* 3.6**

A Ps *BD-A* $(\zeta, \diamond, \#, o)$ is said to be Bo if there is an element $e \in \zeta$ satisfying $\varepsilon \leq e$, $\forall\, \varepsilon \in \zeta$, i.e. $\varepsilon \leq e \Leftrightarrow \varepsilon \diamond e = o$ and $\varepsilon \# e = o$, then is called a Un of *X*. A Ps *BD-A* with Un is called bounded.

***Remark* 3.7.**

In a Bo Ps *BD-A* $(\zeta, \diamond, \#, o)$, we denote $e * \varepsilon$ and $e \# \varepsilon$ by ε^* and $\varepsilon^\#$, respectively, for every $\varepsilon \in \zeta$.

***Example* 3.8.**

In Example 3.4, notice that ζ is Bo Ps *BD-A* with Un 2.

***Proposition* 3.9.**

In Bo Ps *BD-A* $(\zeta, \diamond, \#, o)$, then the following hold, $\exists \varepsilon, \eta \in \zeta$:

(1) $e^* = o = e^\#$

(2) $o^* = e = o^\#$

(3) $\varepsilon^* \# \eta = \eta^\# \diamond \varepsilon$

(4) $\varepsilon^* \# \eta^* = (\eta^*)^\# \diamond \varepsilon$

(5) $\varepsilon^\# \diamond \eta^\# = (\eta^\#)^* \# \varepsilon$

(6) $o \diamond \varepsilon = o = o \# \varepsilon$

(7) $e^* \# \varepsilon = o = e^\# \diamond \varepsilon$

(8) $(\varepsilon^*)^\# \leq$ □, $(\varepsilon^\#)^* \leq \varepsilon.$

Proof :

1. $e^* = e \diamond e = o = e \# e = e^\#,$
2. $o^* = e \diamond o = e = e \# o = o^\#,$
3. $\varepsilon^* \# \eta = (e \diamond \varepsilon) \# \eta = (e \# \eta) \diamond \varepsilon = \eta^\# \diamond \varepsilon,$
4. $\varepsilon^* \# \eta^* = (e \diamond \varepsilon) \# \eta^* = (e \# \eta^*) \diamond \varepsilon = (\eta^*)^\# \diamond \varepsilon,$
5. $\varepsilon^\# \diamond \eta^\# = (e \# \varepsilon) \diamond \eta^\# = (e \diamond \eta^\#) \# \varepsilon = (\eta^\#)^* \# \varepsilon,$
6. $Let\ \varepsilon \in \zeta, then\ o = (o \diamond \varepsilon)(\#) \# e\ (since\ U\ is\ bounded)$

$0 \diamond \varepsilon = (o \# e) \diamond \varepsilon\ Also, o = (o \# \varepsilon) \diamond e\ (since\ U\ is\ bounded)$

$= (o \diamond e) \# \varepsilon = o \# \varepsilon$

7. $e^* \# \varepsilon = o \# \varepsilon\ (by\ 1) = o (by\ 6) = o \diamond \varepsilon\ (by\ 6) = e^\# \diamond \varepsilon\ (by\ 1)$
8. $(\varepsilon^*)^\# \diamond \varepsilon = (e \# \varepsilon^*) \diamond \varepsilon = \left(e \# (e \diamond \varepsilon)\right) \diamond \varepsilon = o\,. Also,$

$(\varepsilon^\#)^* \# \varepsilon = (e \diamond \varepsilon^\#) \# \varepsilon = (e \diamond (e \# \varepsilon)) \# \varepsilon = o.$

Future Works

There are many avenues that one could explore. In this section, we state some of these open problems and conjectures.

1. Studying the Fuzzy Ps ideal of Ps BD-A.
2. Studying the Fuzzy complete Ps ideal (as a generalization of fuzzy Ps ideal) of Ps BD-A.
3. Studying the Fuzzy K-Ps ideal (as a generalization of fuzzy Ps ideal) of Ps BD-A.

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