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Some Properties of Sequences in Complex Valued - Metric Spaces

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Abstract: *This paper will discuss some properties of sequences in complex-valued metric spaces. Complete with several theorems and examples and their applications. In this paper we will prove that it is possible to construct a function such that the metric space over the real numbers is complete if and only if the metric space over the complex field is complete.*

Keywords : Metric Space, Sequence, Convergence

1. INTRODUCTION

Metric spaces were first introduced by Maurice Fréchet in 1906. He provided the basic concept of abstract spaces with metric properties, which were later formulated more formally by Felix Hausdorff in 1914. Fréchet focused on the concepts of convergence, continuity, and other concepts important in non-geometric spaces. This paper will develop the understanding of complex-valued metric spaces.

Definition 1. [2]
Let X be a non-empty set. An association $: X \times X \to R^+$, f is called the metric function on X if it meets the respective requirements $x, y, \in X$ then it applies :

a. $f(x,y) \geq 0$ and $f(x,y) = 0 \Leftrightarrow x = y$
b. $f(x,y) = f(y,x)$
c. $f(x,z) \leq f(x,y) + f(y,z)$

Partner (X, f) is called a metric space.

Example 1. [5]
Let X be a non-empty set. Defining a function

$$f(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

It is easy to prove that f is a metric function.Next, real-valued metric spaces are extended to complex-valued metric spaces. In 2011, Azam introduced complex-valued metric spaces.

Definition 2. [5, 7]
Suppose C is the set of complex numbers and $z_{1,} z_2 \in C$. A partial order is defined $\preccurlyeq$ on C which satisfies $z_1 \preccurlyeq z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$. Next, the partial order in complex numbers has the following properties. (Sintunavarat, 2012).

D1. $0 \preccurlyeq z_1 \prec z_2 \Leftrightarrow |z_1| < z_2$
D2. $z_1 \preccurlyeq z_2, z_2 \prec z_3$ maka $z_1 \prec z_3$
D3. $z \in C, \quad a, b \in R, \quad a \leq b$ maka $az \preccurlyeq bz$

Meanwhile (Azam 2011) defines complex valued metrics as follows.

Definition 3. [1]
Let X be a non-empty set. A function $f_c : X \times X \longrightarrow C$, It is said that a metric function is complex valued if for all $x, y, \in X$ fulfil :

$f_c - 1.$ $0 \preccurlyeq f_c(x,y)$.
$f_c - 2.$ $f_c(x,y) = 0 . \Leftrightarrow x = y$
$f_c - 3.$ $f_c(x,y) = f_c(y,x)$.
$f_c - 4.$ $f_c(x,z) \preccurlyeq f_c(x,y) + f_c(y,z)$.

Next is the couple (X, f_c) called complex-valued ministerial space

Example 2. [5]
Known $X \subseteq C$, defined an association $f_c : X \times X \longrightarrow C$, with $f_c(z_1, z_2) = |z_1 - z_2|$, Where $z_1 = a + bi$ and $z_2 = c + di$, for each $a, b, c, d \in R$.

Definition 4. [2]
Sequence (x_n) in metric space (X, f_c) called convergent to x if for each $0 \prec c \in C$ with $c = \varepsilon + \varepsilon i$ and $0 < \varepsilon \in R$, there is $K \in N$ applies $f_c(x_n, x) \prec c$, for each $n > K$.

Therem 1. [2]
For example, a sequence (x_n) in metric space (X, f_c) then the limit is single.

Definition 5. [6]
Let $S \subseteq X$ be, S is called complete if for each sequence (x_n) Cauchy on metric spaces (X, f_c) converge to $x \in X$.

2. MAIN RESULT

Complex-valued metric spaces are an extension of classical metric spaces, where the value of the "distance" between two points can be a complex number. This concept was first introduced by Azzam et al. in 2011 with a focus on the existence of union fixed points in this space.

Definition 6.
Let (X, f_c) be is a complex-valued metric space. Sequence (x_n) on X called convergent to $x \in X$ if for all

$c \in C,\ c \succ 0$ ada $K \in N$ sedemikian sehingga $f_c(x_n, x) \prec c$ for each $n > K$, written $\lim_{n\to\infty} x_n = x$.

Theorem 2.
Let (X, f_c) be complex valued metric spaces and (x_n) is the sequence in X then the sequence (x_n) converge to x if and only if $|f_c(x_n, x)| \prec 0$ converge to 0.
Proof.
$\Rightarrow$
Given any number $\varepsilon \in R, \varepsilon > 0$ choose a complex number $c = \frac{\varepsilon}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}} i$ so $c \in C.$ and $c \succ 0$. Since the sequence (x_n) convergent to x it means that there is $K \in N$ so for all $n \in N$ If $n > K$ then it applies $f_c(x_n, x) \prec c$ as a result $|f_c(x_n, x)| \prec |c| = \varepsilon$ this is meaningful $|f_c(x_n, x)| \to 0$ for $n \to \infty$.
$\Leftarrow$
Because $|f_c(x_n, x)| \to 0$ For $n \to \infty$ then there is $K \in N$, so for each $n > K$ applies $|f_c(x_n, x)| \prec |c|$. (x_n) convergent to x. ■

Definition 7.
Let (X, f_c) be is a space of complex-valued metrics and (x_n) is a sequence in X. Sequence (x_n) is called a Cauchy sequence if for each $n, m \in N$ there is $K \in N$ and every $0 \prec c \in C$, so it happens $f_c(x_n, x_m) \prec c$, with $n, m > K$.

Theorem 3.
Let (X, f_c) be is a space of complex valued tricks and (x_n) is a sequence X. Sequence (x_n) is called the Cauchy sequence if and only if $|f_c(x_n, x_{n+m})| \longrightarrow 0$.
Proof.
$\Rightarrow$
Let $\varepsilon \in R$, be with $\varepsilon > 0$, if selected :

$$c = \frac{\varepsilon}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}} i$$

so $c \in C,\ c \succ 0$. Since (x_n) is a Cauchy sequence, it means that there is $K \in N$ such that it is for all $n \in N$ with $n > K$, applies $f_c(x_n, x_{n+m}) \prec c$, so that $f_c(x_n, x_{n+m}) \prec c$ obtained $|f_c(x_n, x_{n+m})| \prec |c| < \varepsilon$ the result is obtained $|f_c(x_n, x_{n+m})| \longrightarrow 0$.
$\Leftarrow$
Let $c \in C$ be with $c \succ 0$. If $\varepsilon = |c|$ so $\varepsilon > 0$. Because $|f_c(x_n, x_{n+m})| \longrightarrow 0$. That means there is $K \in N$ so that $n > K$ applies $|f_c(x_n, x_{n+m})| < \varepsilon = |c|$ then it is obtained $f_c(x_n, x_{n+m}) \prec c$, so it's proven (x_n) Cauchy sequence. ■

Theorem. 4.
Let (x_n) be sequence in X and $h \in [0,1)$. If $a_n = |f_n(x_n, x_{n+1})|$ fulfil $a_n \le h a_{n-1}$ for each $n \in N$, so (x_n) Cauchy sequence.

Proof.
Because $h \in [0,1)$ then for each $n \in N$ applies $a_n \le h a_{n-1} \le h^2 a_{n-2} \le h^3 a_{n-3} \cdots \le h^n a_0$, for $n, m \in N$ with $m > n$ obtained :
$|f_n(x_n, x_m)| \le a_n + a_{n+1} + a_{n+2} + \cdots + a_{m-1}$
$\le h^n(1 + h + h^2 + \cdots + h^{m-n-1}) a_0$
$\le \frac{h^n}{1-h} a_0$. Because $h \in [0,1)$ and $\lim_{n\to\infty} h^n = 0$, so $|f_n(x_n, x_m)| \to 0$ and so (x_n) Cauchy sequence. ■

Definition 8.
Let (X, f_n) be is a complex-valued metric space and (x_n) sequence on X, (X, f_n) called complete on X if for each Cauhy sequence converges to X.

Theorem 5.
Let (X, f) be metric space and (X, f_C) complex valued metric space. Defined :

$$f_C(x, y) = f(x, y) + f(x, y)i$$

Metric space (X, f) complete if and only if (X, f_C) complete.

Proof
$\Rightarrow$
Known (X, f) complete metric space, will be proven (X, f_c) complete. Take any Cauchy sequence (x_n) on (X, f_c), will be proven (x_n) convergent sequence to (X, f_c). Given any $\varepsilon \in R,\ \varepsilon > 0$, choose a complex number $c = \varepsilon + \varepsilon i$. Because (x_n) Cauchy sequence on (X, f_c) then there is $K \in N$ so for each $n, m \in N$ with $n, m > K$ applies $f_c(x_n, x_m) \prec 0$, this is meaningful $f(x_n, x_m) < \varepsilon$ as a result (x_n) Cauchy sequence in (X, f) and because (X, f) metric space is complete then (x_n) convergen to $x \in X$. It will be shown next that it will converge to the same point at (X, f_n).
Take anything $c \in C$ with $c \succ 0$. For example $c = c_1 + c_2 i$ then it is obtained $c_1, c_2 \in R$ with $c_1, c_2 > 0$. Next because (x_n) Cauchy sequence in complete space (X, f) this means there is $K_1 \in N$ such that for each $n > K_1$ applies $f(x_n, x) < c_1$ (1)
For $c_2 > 0$ and (x_n) Cauchy sequence in complete space (X, f). This means there is $K_2 \in N$ such that if $n > K_2$ then it applies $f(x_n, x) < c_2$ (2)
Choose $K = \max\{K_1, K_2\}$ If $n \in N$ and $n > K$ then from Equation (1) and (2) applies $f(x_n, x) < c_1$ and $f(x_n, x) < c_2$ this is meaningful $f_c(x_n, x) \prec c$ the result is the Cauchy sequence (x_v) covergent on (X, f_c) so (X, f_c) complete.
$\Leftarrow$
Known (X, f_c) complete will be proven (X, f) complete metric space. Take any Cauchy sequence (x_n) in metric space (X, f) means there is $K_1 \in N$ so for each $n, m > K_1$ applies $f(x_n, x_m) < \varepsilon_1$. for each $0 < \varepsilon_1 \in R$. for $\varepsilon_2 > 0$ means there is $K_2 \in N$ so for each $n, m > K_2$ applies $f(x_n, x_m) < \varepsilon_2$.
Choose $K = \max\{K_1, K_2\}$ for anything $n \in N$ which fulfills $n > K$ applies $f(x_n, x_m) < \varepsilon_1$ and $f(x_n, x_m) < \varepsilon_2$ as a result $f_c(x_n, x_m) \prec c$. Means (x_n) barisan Cauchy sequence on (X, f_c) and because (X, f_c) complete, then the sequence is (x_n) convergent to $x \in (X, f_c)$. The

limit of the sequence will be shown (x_n) also $x \in (X, f)$. For that take any $0 < \varepsilon \in R$, choose a complex number $c = \varepsilon + \varepsilon i$, because (x_n) convergent on (X, f_c) then there is $K \in N$ so it applies $f_c(x_n, x) \prec c = \varepsilon + \varepsilon i$ then it can be obtained $f(x_n, x) < \varepsilon$ means the Cauchy sequence is convergent so that (X, f) complete

■

3. CONCLUSION

1. Let (X, f_c) be complex valued metric spaces and (x_n) is the sequence in X then the sequence (x_n) converge to x if and only if $|f_c(x_n, x)| \prec 0$ converge to 0.
2. Let (X, f_c) be is a space of complex valued tricks and (x_n) is a sequence X. Sequence (x_n) is called the Cauchy sequence if and only if $|f_c(x_n, x_{n+m})| \longrightarrow 0$.
3. Let (X, f) be metric space and (X, f_C) complex valued metric space. Defined :

$$f_C(x, y) = f(x, y) + f(x, y)i$$

Metric space (X, f) complete if and only if (X, f_C) complete..

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