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Convexly Compact Random Attractors

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Abstract: *Our work consists of some types of convexly compact dissipativity for random dynamical systems, these types: point ($ck-$)**,** compact ($ck-$)**,** local ($ck-$), bounded ($ck-$). Weakly ($ck-$) and the relationship among them. Additionally, we give a description of the structure of **a** convexly compact global attractor (random Levinson center) for RDSs. Finally, we present a mathematical model and analysis for a stochastic reaction-diffusion system and find the convexly compact attractor of this model.*

Keywords: Compact random set, omega limit set, convexly compact dissipativity, convexly random Levinson Center, Convexly Compact Attractor, stochastic reaction-diffusion system.

1.Introduction:

Dissipative systems have an importance in many science such as physics and engineering. The hypothesis of dissipation yield results in essential constraint on behavior of its dynamic. Many papers devoted to the study the dissipativety of RDSs, see, for example, J. Huang [8], C. Kuehn [10], Y. Wang [11**].** A. Gu **[**7], L. Yuhong [17], A. Yasir, I. Kadhim [12, 13,14,15,16].

Stochastic environmental models (e.g. prey-predator interaction models or epidemic prevalence models) which include random effects (environmental or genetic noise). In such systems, stability and dissipation reflect the ability of the system to return to a state of equilibrium after disturbances, which is important for analyzing the survival or extinction of species. But when spaces are non-compact, the traditional concept of (dissipation) is not enough, and this is where convexly compact dissipation comes into play.

The use of convexly compact dissipation allows proving the existence of attraction groups (attractors) or non-random measurements (stationary measures) even in spaces that are not completely compact but convex compact (as in some spaces of functions).

2. Preliminaries:

This section contains the basic definitions and properties of RDSs, For more details, one can see [1], [2], [9].

Definition 2.1[9]: *Let* $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $\theta:\mathbb{T}\times\Omega\longrightarrow\Omega$ is measurable function satisfy the following

$$\theta_0 = \mathrm{id},\ \theta_t\circ\theta_s = \theta_{t+s} \text{ for all } t,s\in\mathbb{T}\ ;\ \text{and}\ \ \theta_t\mathbb{P} = \mathbb{P} \text{ for all } t\in\mathbb{T}.$$

A set $B\in\mathcal{F}$ is called **θ *−invariant*** if $\theta_t B = B$ for all $t\in\mathbb{T}$. An MDS θ called ergodic under $\mathbb{P}$ if for any θ −invariant set $B\in\mathcal{F}$ we have either $\mathbb{P}(B)=0$ or $\mathbb{P}(B)=1$.

Definition 2.2[9]: *Let* X be a topological space and $\mathbb{T}$ be a locally compact group. ***The random dynamical system (RDS)*** is a pair (θ,φ) involving an MDS θ and a cocycle φ over θ of continuous mappings of X , i.e. a measurable mapping

$$\varphi:\mathbb{T}\times\Omega\times X\longrightarrow X,\ (t,\omega,x)\longmapsto\varphi(t,\omega,x)\ \text{, such that}$$

(1) for every $t\in\mathbb{T}$ and $\omega\in\Omega$, the function $x\longmapsto\varphi(t,\omega,x)\equiv\varphi(t,\omega)x$ is continuous

(2) for all $t,s\in\mathbb{T}$ and $\omega\in\Omega$,the function $\varphi(t,\omega):=\varphi(t,\omega,\cdot)$ fulfill:

$$\varphi(0,\omega)=\mathrm{id},\ \varphi(t+s,\omega)=\varphi(t,\theta_s\omega)\circ\varphi(s,\omega)\ .$$

The property (ii) called cocycle property of φ.

Definition 2.3[9]: Consider the metric space (X, d).

(1) A random set is The set-valued function $\omega \longmapsto M(\omega) \neq \emptyset$ such that for any $x \in X$ the function

$$\omega \longmapsto \text{dist}_X(x, M(\omega))$$

is measurable. The random set M is called ***a random closed set*** if $M(\omega)$ is closed for each $\omega \in \Omega$ and any adjective applying on $M(\omega)$ is closed for each $\omega \in \Omega$ applied similarly on M .

(2) A random set $\{M(\omega)\}$ is said to be ***bounded*** if for some $x_0 \in X$ and some positive random variable $r(\omega)$ the following fulfill

$$M(\omega) \subset \{x \in X: d(x, x_0) \leq r(\omega)\} \quad \text{for all } \omega \in \Omega.$$

(3) A tempered random variable (t.r.v) is ***a measurable function*** $\varepsilon: \Omega \longrightarrow \mathbb{R}$ with

$$\lim_{t \longrightarrow +\infty} \frac{1}{|t|} \log|\varepsilon(\theta_t \omega)| = 0.$$

Definition 2.13 [9]: Let $M: \omega \to M(\omega)$ be a random set. The set-valued function

$$\omega \longmapsto \Gamma_M(\omega) := \cap \overline{\gamma_M^t(\omega)} = \cap \overline{\cup \varphi(\tau, \theta_{-\tau}\omega) M(\theta_{-\tau}\omega)}, t > 0, \tau \geq t$$

is called the ***omega-limit set*** of the trajectories starting from M.

Definition 2.4 [9]: Consider the RDS (θ, φ) . A set-valued mapping $\omega \longmapsto S(\omega)$is called ***forward invariant (backward invariant)*** whenever for all $t > 0$ and $\omega \in \Omega$ we have $\varphi(t, \omega) S(\omega) \subseteq S(\theta_t \omega)$(respectively, $S(\theta_t \omega) \subseteq \varphi(t, \omega) S(\omega)$).

Definition 2.5 [9]: A collection $\mathcal{U}$ of random sets is called a ***universe of sets*** if

(1) Every members of $\mathcal{U}$ is closed, and

(2) $\mathcal{U}$ is closed with respect to inclusions .

Definition 2.6 [9]: An ***absorbing random set*** for RDS (θ, φ) in the universe $\mathcal{U}$ is a random set A have the property that

if for every $M \in \mathcal{U}$ and for all ω there exists $t_0(\omega)$ with

$$\varphi(t, \theta_{-t}\omega) M(\theta_{-t}\omega) \subset A(\omega) \text{ for all } t \geq t_0(\omega), \omega \in \Omega.$$

Definition 2.8 [9]: A random closed set $\{M(\omega)\}$ from a universe $\mathcal{M}$ is called a ***random attractor*** of RDS (θ, φ) in $\mathcal{M}$ if $B(\omega)$ is proper subset of X for every $\omega \in \Omega$ and :

(1) B is an invariant set, i.e. $\varphi(t, \omega) B(\omega) = B(\theta_t \omega)$ for $t \geq 0$, $\omega \in \Omega$;

(2) B is an attracting in $\mathcal{U}$, i.e. for all $M \in \mathcal{U}$

$$\lim_{n \to +\infty} d_X\{\varphi(t, \theta_{-t}\omega) M(\theta_{-t}\omega), B(\omega)\} = 0, \omega \in \Omega$$

Where $d_X\{A \setminus B\} = sup_{x \in A} dis_X(x, B)$.

Definition 2.18[3]: Let (X, d) be a metric space, $K \subset X$ is ***precompact*** or ***totally bounded*** if every sequence in K admits a subsequence converges to a point of X.

Definition 2.10 [9]: Let $D : \omega \longmapsto D(\omega)$ be a multifunction. We call the multifunction

$$\omega \longmapsto \gamma_D^t(\omega) := \bigcup_{\tau \geq t} \varphi(\tau, \theta_{-\tau}\omega) D(\theta_{-\tau}\omega)$$

the tail (from the moment t) of the pullback trajectories emanating from D. If $D(\omega) = \{v(\omega)\}$ is a single valued function, then $\omega \longmapsto \gamma_v(\omega) \equiv \gamma_D^0(\omega)$ is said to be the ***(pull back) trajectory*** (or ***orbit***) emanating from v.

Definition 2.11[9]: Let (θ,φ) be an RDS. A random set A is said to be ***attract*** another random set B if $\mathbb{P}$ −almost surely

$$\lim_{t\to\infty} d(\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega),A(\omega))=0$$

Lemma 2.12 [9]: If A attracts B then $d(\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega),A(\omega))\longrightarrow 0, as\ n\to\infty$ in probability.

3. Convexly Compact Attractors

In 1967, Komlós proved that every norm-bounded sequence in $\mathbb{L}^1$ possesses a subsequence whose Cesàro means converge almost surely (a.s.). In this section, that classical result is revisited from the perspective of convex compactness. Emphasis is placed on the fact that, in contrast to Komlós' theorem—which employs equal (Cesàro) weights—convergence is here obtained through arbitrary convex combinations (cf. [6]).

Let A be a non-empty set. The set Fin(A) consisting of all non-empty finite subsets of A carries a natural structure of a partially ordered set when ordered by inclusion. Moreover, it is a directed set, since $\mathrm{D}_1,\mathrm{D}_2\subseteq \mathrm{D}_1\cup\mathrm{D}_1$ for any $\mathrm{D}_1,\mathrm{D}_1\in \mathrm{Fin(A)}$. We remind the reader that for a subset C of a Banach space X, conv C denotes the smallest convex subset of X containing C.

Definition 3.1. A convex set C of a Banach space X is said to be ***convexly compact*** if for any non-empty set A and any family $\{F_\alpha\}_{\alpha\in A}$ of closed and convex subsets of C, the condition

$$\forall\, D\in \mathrm{Fin(A)},\ \bigcap_{\alpha\in \mathrm{D}} F_\alpha\neq\emptyset \qquad (3.1)$$

implies

$$\bigcap_{\alpha\in \mathrm{A}} F_\alpha\neq\emptyset, \qquad (3.2)$$

In the absence of the additional condition that the sets $\{F_\alpha\}_{\alpha\in A}$ be convex, Definition 3.1 postulating the ***finite-intersection property*** for families of closed and convex sets would be equivalent to the classical definition of compactness. It is, therefore, any convex and compact subset of a topological vector space is convexly compact.

Example 3.2 (*Convex compactness without compactness*). Let L be a locally-convex topological vector space, and let L^* be the topological dual of L, endowed with some compatible topology τ, possibly different from the weak-* topology $\sigma(L^*,L)$. For a neighborhood N of 0 in L, define the set C in the topological dual L^* of L by

$$C=\{x^*\in L:\langle x,x^*\rangle\leq 1,\forall x\in N\}.$$

In other words, $C=N^\circ$ is the polar of N. By the Banach-Alaoglu Theorem, C is compact with respect to the weak-* topology $\sigma(L^*,L)$, but it may not be compact with respect to τ. On the other hand, let $\{F_\alpha\}_{\alpha\in A}$ be a non-empty family of convex and τ −closed subsets of C with the finite-intersection property (3.1). It is a classical consequence of the Hahn-Banach Theorem that the collection of closed and convex sets is the same for all topologies consistent with a given dual pair. Therefore, the sets $\{F_\alpha\}_{\alpha\in A}$ are $\sigma(L^*,L)$ −closed, and the relation (3.2) holds by the aforementioned $\sigma(L^*,L)$ compactness of C.

In the following we Characterize the notion of ***convexly compact*** in terms of generalized sequences.

Definition 3.3. Let $\{x_\alpha\}_{\alpha\in A}$ be a net in a Banach space X. A net $\{y_\beta\}_{\beta\in B}$ is said to be a subnet of convex combinations of $\{x_\alpha\}_{\alpha\in A}$ if there exists a mapping $D: B\longrightarrow Fin(A)$ such that

1. $y_\beta\in$ conv$\{x_\alpha : \alpha\in D(\beta)\}$ for each $\beta\in B$, and

2. for each $\alpha\in A$ there exists $\beta\in B$ such that $\alpha'\succcurlyeq\alpha$ for each $\alpha'\in\bigcup_{\beta'\succcurlyeq\beta} D(\beta')$.

Proposition 3.4. A closed and convex subset C of a Banach space X is convexly compact if and only if for any net $\{x_\alpha\}_{\alpha\in A}$ in C there exists a subnet $\{y_\beta\}_{\beta\in B}$ of convex combinations of $\{x_\alpha\}_{\alpha\in A}$ such that $y_\beta\longrightarrow y$ for some $y\in C$.

Definition 3.5. A closed and convex subset C of a Banach space X is ***relatively convexly compact*** if for any net $\{x_\alpha\}_{\alpha\in A}$ in C there exists a subnet $\{y_\beta\}_{\beta\in B}$ of convex combinations of $\{x_\alpha\}_{\alpha\in A}$ such that $y_\beta\longrightarrow y$ for some $y\in X$.

Remark 3.6. Every convexly compact set is relatively convexly compact.

Definition 3.7. A random set $A(\omega) \subseteq X$ is said to be bounded in probability if

$$\lim_{M\longrightarrow\infty} \sup_{x\in A(\omega)} \mathbb{P}\{\omega: \|x(\omega)\| \geq M\} = 0.$$

In the following theorem we will deal with the random set with the property that:

$$x \in C(\omega) \text{ if and only if } E[\|x(\omega)\|] < \infty.$$

Theorem 3.8. A closed and convex random set $C(\omega)$ in X is convexly compact if and only if it is bounded in probability.

Proof. $\Leftarrow$ Let C be a convex, closed and bounded-in-probability subset of X, and let $\{F_\alpha\}_{\alpha\in A}$ be a family of closed and convex subsets of $C(\omega)$ satisfying (3.1). For $D \in Fin(A)$ we define

$$G_D = \begin{cases} C(\omega), & D = \emptyset \\ \bigcap_{\alpha\in D} F_\alpha, & D \neq \emptyset \end{cases}.$$

and fix an arbitrary $x_D \in G_D$. With $\phi(x(\omega)) = 1 - \exp(-\|x(\omega)\|)$, we set

$$u_D = \sup\{\mathbb{E}[\|\phi(g)\|]: g \in conv\{x_{D'} : D' \supseteq D\}\},$$

so that $0 \leq u_D \leq 1$ and $u_{D_1} \geq u_{D_2}$, for $D_1 \subseteq D_2$. Seen as a net on the directed set $(Fin(A), \subseteq)$, $\{u_D\}_{D\in Fin(A)}$ is monotone and bounded, and therefore convergent, i.e., $u_D \longrightarrow u_\infty$, for some $u_\infty \in [0,1]$. Moreover, for each $D \in Fin(A)$ we can choose $g_D \in conv\{g_{D'} : D \subseteq D'\}$ so that

$$u_D \geq \gamma_D \triangleq \mathbb{E}[\phi(\|g_D\|)] \geq u_D - \frac{1}{\#D},$$

where $\#D$ denotes the number of elements in D. Clearly, $\gamma_D \longrightarrow u_\infty$. The reader is invited to check that simple analytic properties of the function ϕ are enough to prove the following statement:

for each $M > 0$ there exists $\varepsilon = \varepsilon(M) > 0$, such that

if $\|x_1 - x_2\| \geq \frac{1}{M}$ and $0 \leq \min\{\|x_1\|, \|x_2\|\} \leq M$, then

$$\phi\left(\frac{1}{2}\|x_1 + x_2\|\right) \geq \frac{1}{2}\phi(\|x_1\|) + \phi(\|x_2\|) + \varepsilon.$$

It follows that for any $D_1, D_2 \in Fin(A)$ we have

$$\varepsilon\mathbb{P}\left\{\|g_{D_1} - g_{D_2}\| \geq \frac{1}{M}, \min\{g_{D_1}, g_{D_2}\} \leq M\right\} \leq \mathbb{E}\left[\phi\left(\frac{1}{2}\|g_{D_1} + g_{D_2}\|\right)\right]$$
$$-\frac{1}{2}\left(\mathbb{E}[\phi(\|g_{D_1}\|)] + \mathbb{E}[\phi(\|g_{D_2}\|)]\right).$$

The random variable $\frac{1}{2}\|g_{D_1} + g_{D_2}\|$ belongs to $conv\{f_{D'}: D' \supseteq D_1 \cap D_2\}$,

$$\mathbb{E}\left[\phi\left(\frac{1}{2}\|g_{D_1} + g_{D_2}\|\right)\right] \leq u_{D_1\cap D_2}.$$

Consequently,

$$\varepsilon\mathbb{P}\left\{\|g_{D_1} - g_{D_2}\| \geq \frac{1}{M}, \min\{g_{D_1}, g_{D_2}\} \leq M\right\} \leq \eta_{D_1,D_2}$$

Where

$$\eta_{D_1,D_2} = u_{D_1\cap D_2} - \frac{1}{2}\left(u_{D_1} + u_{D_2}\right) + \frac{1}{2}\left(\frac{1}{\#D_1} + \frac{1}{\#D_2}\right).$$

Thanks to the boundedness in probability of the set C, for $\kappa > 0$, we can find $M = M(\kappa) > 0$ such that

$$M > 1/\kappa \text{ and } \mathbb{P}[f \geq M] < \kappa/2$$

for any $f \in C$. Furthermore, let $D(\kappa) \in Fin(A)$ be such that $u_\infty + \varepsilon(M)\kappa/4 \geq u_D \geq u_\infty$ for all $D \supseteq D(\kappa)$, and $\#D(\kappa) > 4/(\varepsilon(M)\kappa)$. Then, for $D_1, D_2 \supseteq D(\kappa)$ we have

$$\mathbb{P}\{\|g_{D_1} - g_{D_2}\| \geq \kappa\} \leq \mathbb{P}\left\{\|g_{D_1} - g_{D_2}\| \geq \frac{1}{M}, \min\{g_{D_1}, g_{D_2}\} \leq M\right\} + \mathbb{P}\{\min\{g_{D_1}, g_{D_2}\} \leq M\}$$

$$\leq \frac{1}{\varepsilon(M)}\left(u_{D_1 \cap D_2} - \frac{1}{2}(u_{D_1} + u_{D_2}) + \frac{1}{2}\left(\frac{1}{\#D_1} + \frac{1}{\#D_2}\right)\right) \leq \kappa.$$

In other words, $\{g_D\}_{D \in Fin(A)}$ is a Cauchy net in $\mathbb{L}^0_+$ which, by completeness, admits a limit $g_\infty \in \mathbb{L}^0_+$. By construction and convexity of the sets $F_\alpha, \alpha \in A$, we have $g_D \in F_\alpha$ whenever $D \supseteq \{\alpha\}$. By closedness of F_α, we conclude that $g_\infty \in F_\alpha$, and so, $g_\infty \in \bigcap_{\alpha \in A} F_\alpha$.

$\Rightarrow$ It remains to show that convexly compact sets in $\mathbb{L}^0_+$ are necessarily bounded in probability. Suppose, to the contrary, that $C \subseteq \mathbb{L}^0_+$ is convexly compact, but not bounded in probability.

Then, there exists a constant $\varepsilon \in (0, 1)$ and a sequence $\{f_n\}_{n \in \mathbb{N}}$ in C such that

$$\mathbb{P}\{f_n \geq n\} > \varepsilon, \text{ for all } n \in \mathbb{N}. \quad (3.3)$$

By Proposition 3.4, there exists a subnet $\{g_\beta\}_{\beta \in B}$ of convex combinations of $\{f_n\}_{n \in \mathbb{N}}$ which converges to some $g \in C$. In particular, for each $n \in \mathbb{N}$ there exists $\beta_n \in B$ such that g_β can be written as a finite convex combination of the elements of the set $\{f_m: m \geq n\}$, for any $\beta \succcurlyeq \beta_n$. Using (3.1) and Lemma 9.8.6., p. 205 [5], we get the following estimate

$$\mathbb{P}\left\{g_\beta \geq \frac{n\varepsilon}{2}\right\} > \frac{\varepsilon}{2}, \text{ for all } \beta \succcurlyeq \beta_n. \quad (3.4)$$

Therefore,

$$\mathbb{P}\left\{g_\beta \geq \frac{n\varepsilon}{2}\right\} \geq \mathbb{P}\left\{g_\beta \geq \frac{n\varepsilon}{2}\right\} - \mathbb{P}\left\{|g - g_\beta| \geq \frac{n\varepsilon}{4}\right\} > \frac{\varepsilon}{2} - \mathbb{P}\left\{|g - g_\beta| \geq \frac{n\varepsilon}{4}\right\} > \frac{\varepsilon}{4},$$

for all “large enough” $\beta \in B$. Hence, $\mathbb{P}\{g = +\infty\} > 0$. a contradiction with the assumption $g \in C$.

4. Convexly Compact Dissipative Random Dynamical Systems

In this section, we define convex compactness for dissipative RDSs and introduce defrant types of dissipation when the random set is convexly compact. Then the relations among this types of dissipativity are discussed.

Definition 4.1 The RDS (θ, φ) is said to be ***convexly compact dissipative*** if for any convexly compact random set $D(\omega)$ in X there is random set $K(\omega)$ in X so that

$$\lim_{t \to +\infty} \sup_{x \in D(\theta_{-t}\omega)} \inf_{y \in K(\omega)} \|\varphi(t, \theta_{-t}\omega)x - y\| = 0$$

Definition 4.2 The RDS (θ, φ) is said to be

(1) ***point (ck −)dissipative*** if for every $x \in X$, there is (convexly compact) random set $K(\omega)$ in X so that,

$$\lim_{t \to +\infty} \inf_{y \in K(\omega)} \|\varphi(t, \theta_{-t}\omega)x - y\| = 0.$$

(2) ***bounded (ck −)dissipative*** if for any bounded random set $B(\omega)$ in X there is (convexly compact) random set $K(\omega)$ in X so that

$$\lim_{t \to +\infty} \sup_{x \in B(\theta_{-t}\omega)} \inf_{y \in K(\omega)} \|\varphi(t, \theta_{-t}\omega)x - y\| = 0.$$

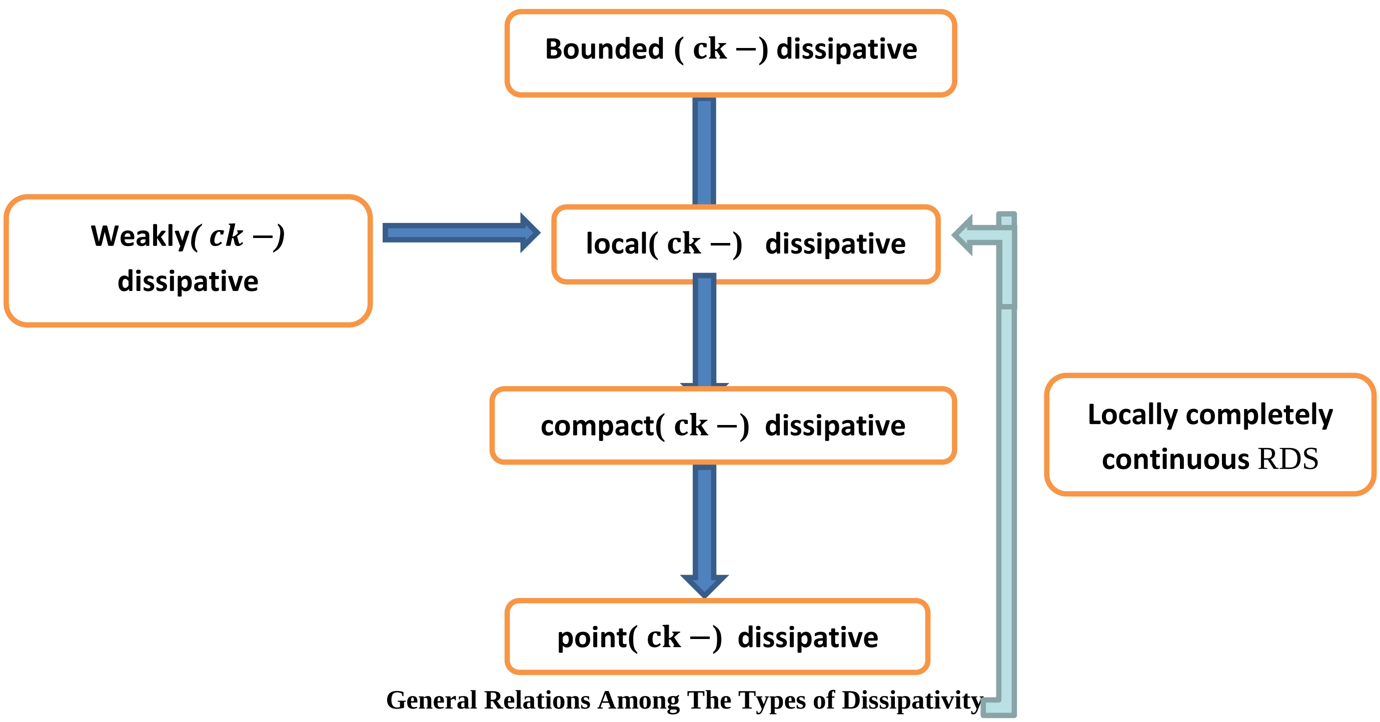
Definition 4.3: the RDS (θ, φ) is said to be

1. ***Locally completely continuous*** (***Locally ($ck-$) compact***) if for every random variable $p \in X^{\Omega}$, there exist a TRV $\delta = \delta(p,\omega) > 0$, $l = l(p) > 0$ such that $\varphi(l,\theta_{-l}\omega)M(\theta_{-l}\omega)$ is precompact, where

$$M(\omega) := B(p,\delta) = \{x \in X: \inf_{y\in p(\omega)}\|x-y\| < \delta(\omega)\}.$$

2. ***Weakly ($ck-$) dissipative*** whenever for every t.r.v $\varepsilon(\omega) > 0$ and every $x \in X$ there exists a nonempty convexly compact random set $K(\omega) \subseteq X$ such that, there exists $\tau = \tau(\varepsilon, x) > 0$ for $\varphi(\tau,\theta_{-\tau}\omega)x \in B(K(\omega),\varepsilon(\omega))$.

In this case, $K(\omega)$ will be called a ***weak ($ck-$) random attractor***.



General Relations Among The Types of Dissipativity

Definition 4.4. Consider RDS (θ,φ). A random set $M(\omega)$ is said to be ***convexly orbitally stable*** if for any tempered random variable ε and any non-negative number t, there exists tempered random variable δ such that

$$\inf_{y\in M(\omega)}\|x-y\| < \delta(\omega) \text{ implies } \inf_{y\in M(\omega)}\|\varphi(t,\theta_{-t}\omega)x-y\| < \varepsilon(\omega).$$

Definition 4.5. Let $K(\omega)$ be a convexly compact random set in X**.** We will call the set $L_X(\omega)$ defined by following equality:

$$L_X(\omega) := \Gamma_K(\omega) = \cap\{\varphi(t,\theta_{-t}\omega)K(\theta_{-t}\omega) | t \in T, \omega \in \Omega\}$$

the ***random convexly Levinson center*** of the convexly compact dissipative RDS (θ,φ).

Theorem 4.6. Consider the RDS $(\theta,\varphi)$is bounded in probability. The next statements are equivalent for every random set $\mathrm{B}(\omega) \subseteq X$:

1. the sequence $\{\varphi(t_k, \theta_{-t}\omega)x_k\}$ is random precompact for all sequence $t_k \to +\infty$ and $\{x_k\} \subseteq B(\theta_{-t}\omega)$, then
2. (a) $\Gamma_B(\omega) \neq \emptyset$ and convexly compact.
   (b) $\Gamma_B(\omega)$ is invariant, and

$$\lim_{t\to+\infty} \sup_{x\in B(\theta_{-t}\omega)} \|\varphi(t, \theta_{-t}\omega)x - \Gamma_B(\omega)\| = 0 \qquad (4.1)$$

3. For some convexly compact random set $\emptyset \neq K(\omega) \subseteq X$ we have

$$\sup_{x\in B(\theta_{-t}\omega)} \inf_{y\in K(\omega)} \|\varphi(t, \theta_{-t}\omega)x - y\| = 0 \qquad (4.2)$$

Proof: To prove (1) ⇒ (2). Let $\{x_k\} \subseteq B(\theta_{-t}\omega)$ where $t_k \to +\infty$. Then according to (1), the sequence $\{\varphi(t, \theta_{-t}\omega)x_k\}$ convergent. Assume $\bar{x} = \lim_{t\to+\infty} \varphi(t_k, \theta_{-t_k}\omega)x_k$.

Then $\bar{x} \in \Gamma_B(\omega)$, so, $\Gamma_B(\omega)$ is non-empty. Let us show that $\Gamma_B(\omega)$ is convexly compact. Let $\varepsilon_k \downarrow 0$ and $\{y_k\} \subseteq \Gamma_B(\omega)$, then there is $x_k \in B(\theta_{-t}\omega)$ and $t_k \geq k$ with $\|\varphi(t_k, \theta_{-t}\omega)x_k - \Gamma_B(\omega)\| < \varepsilon_k$.

According to condition (1), the sequence $\{\varphi(t_k, \theta_{-t_k}\omega)x_k\}$ is convexly precompact, and since $\varepsilon_k \downarrow 0$, so $\{y_k\}$ is convexly precompact. Easy to get the foreword invariance of $\Gamma_B(\omega)$ from the definition. To show $\Gamma_B(\omega)$ to be invariant it is enough to prove it is backward invariant. *Let* $y \in \Gamma_B(\omega)$ *and* $t \in T$. Hence *there is* $\{x_k\} \subseteq B(\theta_{-t}\omega)$ *and* $t_k \to +\infty$ such that

$y = \lim_{k\to+\infty} \varphi(t_k, \theta_{-t_k}\omega)x_k = \lim_{k\to+\infty} \varphi(t_k - t + t, \theta_{-t_k}\omega)x_k = \lim_{k\to+\infty} \varphi(t_k - t, \theta_{-t_k}\omega)x_k$.
As $t_k - t \longrightarrow +\infty$, according to condition (1), the sequence $\{\varphi(t_k - t, \theta_{-t_k}\omega)x_k\}$ can be considered convergent. Assume $y_k = \lim_{k\to+\infty} \varphi(t_k - t, \theta_{-t_k}\omega)x_k$. Then $y = \varphi(t, \theta_{-t}\omega)y_k$ and $y_t \in \Gamma_B(\omega)$, i.e., $y \in \varphi(t, \theta_{-t}\omega)\Gamma_B(\omega)$. The invariance of $\Gamma_B(\omega)$ is proved at the same way. Now
to prove (4.1) fulfill. Assume that (3.1) is invalid, then for some $\varepsilon_0 > 0$, $t_k \to +\infty$, *and* $x_k \in B$ such that

$$\left\|\varphi(t_k, \theta_{-t_k}\omega)x_k - \Gamma_B(\omega)\right\| \geq \varepsilon_0$$

According to condition (2), the sequence $\{\varphi(t_k, \theta_{-t_k}\omega)x_k\}$ convergent. Let $y = \lim_{t\to+\infty} \varphi(t_k, \theta_{-t_k}\omega)x_k$. Then $y \in \Gamma_B(\omega)$. Take $k \longrightarrow +\infty$ in (4.2), we get $y \notin \Gamma_B(\omega)$**.** This a contradiction and this end the proof of (1) ⇒(2).

It is evident that (2) ⇒ (3) and (3) ⇒ (1).

Corollary 4.7. Let $M(\omega) \subseteq X$ be nonempty random set and $\gamma_M^t(\omega)$ relatively convexly compact. We have $\Gamma_M(\omega) \neq \emptyset$, invariant and convexly compact, and

$$\lim_{t\to+\infty} \sup_{x\in M(\theta_{-t}\omega)} \inf_{y\in\Gamma_M(\omega)} \|\varphi(t, \theta_{-t}\omega)x - y\| = 0.$$

Proof. The proof follows directly from above theorem.

5. Convexly Compact Dissipativity and Asymptotic Behavior of Stochastic Reaction-Diffusion Systems

This section presents and analysis a mathematical model for a stochastic reaction-diffusion system. The focus is placed on proving the convexly compact dissipativity of the system and establishing the existence of a random attractor. In addition, a numerical simulation is provided as evidence of the system's asymptotic behavior.

5.1 Stochastic Reaction-Diffusion Model

Consider the stochastic partial differential equation (SPDE):

$$\partial u/\partial t = D\, \partial^2 u/\partial x^2 + u(1 - u) + \beta\, u\, \xi(t, x), for\ x \in [0, 1], t > 0,$$

with Dirichlet boundary conditions and initial condition $u(0, x) = 0.5$, such that
- D is the diffusion coefficient.
- $u(1 - u)$ is the logistic growth reaction term.
- $\beta u\, \xi(t, x)$ is the multiplicative stochastic noise.

5.2 Theoretical Analysis and Compact Dissipativity

We consider the Hilbert space $H = L^2([0,1])$ and define a random dynamical system (RDS) generated by the SPDE. We aim to show that the system has convexly compact dissipativity.

Step 1: Energy Estimate
Taking the $L^2 - norm$ of the solution and applying Itô's formula, we obtain:

$$E[||u(t)||^2] \leq E[||u(0)||^2]\, e^{\{-\alpha t\}} + C(\beta),$$

where $\alpha > 0$ and $C(\beta)$ is a constant depending on the noise intensity.
Step 2: Weak Compactness
Although L^2 is not strongly compact, bounded sets in L^2 are weakly relatively compact due to Banach-Alaoglu theorem. Thus, the absorbing set is weakly compact and convex, satisfying convexly compact dissipativity.
Step 3: Existence of Random Attractor
By standard theory (Crauel, Flandoli) [4], convexly compact dissipativity and measurability of the RDS guarantee the existence of a unique random attractor $A(\omega) \subset H$.

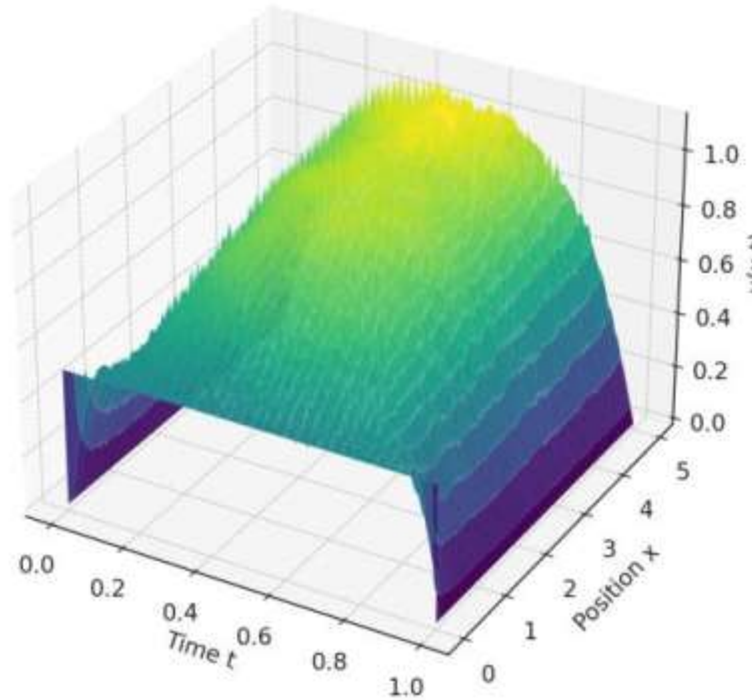
5.3 Numerical Simulation

We simulate the SPDE using an Euler-Maruyama method over the spatial domain [0,1] and time interval [0,5].
The resulting finite difference scheme is:

$$u_i^{\{n+1\}} = u_i^{\,n} + \Delta t \cdot [\, D \cdot (u_{\{i-1\}}^{\;n} - 2u_i^{\,n} + u_{\{i-1\}}^{\;n} / \,(\Delta x)^2 + u_i^{\,n}(1 - u_i^{\,n})] + \sqrt{\Delta t} \,\cdot\, \beta\, u_i^{\,n} \cdot \eta_i^{\,n}$$

Where,
$- u_i^{\,n}$: approximate solution at space point x_i and time t_n.
- Δt: time step, Δx: space step
- D: diffusion coefficient
- β: noise intensity
- $\eta_i^{\,n} \sim \mathcal{N}(0,1)$: standard normal random variable.
- Boundary condition: $u_i^{\,n} = u_N^{\,n} = 0$ (Dirichlet)
- Initial condition: $u_i^{\,0} = 0.5$ for all i

This scheme captures diffusion, nonlinear reaction, and stochastic fluctuations effectively.
The figure below illustrates the evolution of $u(x,t)$ and demonstrates that the solution stabilizes over time.
This supports the theoretical result of dissipativity.



Numerical Simulation of a Stochastic Reaction-Diffusion System

5.4 Conclusion

We have shown both analytically and numerically that the stochastic reaction-diffusion system exhibits convexly compact dissipativity. This ensures the existence of a random attractor and provides a robust framework for understanding the long-term behavior of the system under uncertainty.

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