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An approximation of Complicated Functions

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***Abstract** : In this review the functions and their relationship to the best approximation, which gives us a correct judgment on the smoothness of functions and how to find their derivatives and their importance in solving many scientific problems in partial differential equations and the theory of approximation are offered . The most important theories are studied such as Ditzian-Totik, the direct and inverse theory in approximation, etc., especially the difficult wave functions.*

Keywords: Classical Smoothness, Ditzian-Totik Smoothness, Wavelet Approximation, Jackson's Theorem, K-Functional.

1. Introduction

In the [5], Ditzian, Z., & Totik have been presented a basic study on the best degree of approximation and smoothness of functions. developed that study in 2015 , see[7]. Studies continued for various the examples and spaces like L_p, Besov, and Sobolev, notice the following sources ([1],[3],[4],[6],[10],[11]) .

These theories and results have contributed to solving many problems, such as solving partial differential equations, image compression, frequency and signal analysis, and their applications in medicine, physics, astronomy, and mathematics.

Wavelet analysis shows functions as a series of basic functions that are focused on specific areas, which makes it good for studying complex local patterns. However, the actual performance of these transforms in representing or approximating a particular function strongly depends on the smoothness of the original function. This attribute makes moduli of smoothness play a key role in determining the rate of decay of wavelet coefficients and, consequently, the effectiveness of the approximation process, see([1],[2],[8],[9]).

2. Essential Definitions Relate to the Moduli of Smoothness

The classical modulus of smoothness of order r, for the function $f \in L_p(\mathbb{R})$is defined as follows: $\omega_r(f,t)_p = \sup_{0<h\le t} \|\Delta_h^r(f,.)\|_p$, where $\Delta_h^r f(x) = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} f(x+jh)$, it is the symmetric difference of order r. This measure reflects how much the function changes as it passes through a specific value, indicating its smoothness. The smaller the value of $\omega_r(f,t)_p$, the smoother and easier the function is to approximate, see[5].

Many types of moduli for smoothness have appeared, such as

The Ditzian-Totik modulus of smoothness is defined using the weigh function $\varphi(x) = \sqrt{1-x^2}$ on the domain $[-1,1]$, as in the source[5].

$\omega_r^{\varphi}(f,t)_p = \sup_{0<h\le t} \left\|\Delta_{h\varphi(x)}^r(f,.)\right\|_p$ .It is effective for approximation in finite intervals. It is also suitable for functions that exhibit changing behavior near their extremes.

In addition to many other types, like the generalised modulus of smoothness, the spectral modulus, the fractional modulus of smoothness, the local modulus of smoothness, and the k-mixed modulus of smoothness, see ([3],[4],[5],[6],[10],[11]).

In Sobolev spaces, the modulus of smoothness can be expressed in terms of partial derivatives. The modulus of smothness also plays a significant role in Besov and Triebel–Lizorkin spaces, which are essential in computational applications, including signal and image processing, see([3],[4]).

3. Compare with K-Functional

The K-Functional is an alternative measurement tool that is defined as follows:

$$\mathrm{K}(\mathrm{f},\delta) = \mathrm{K}(\mathrm{f},\delta,\mathrm{X}_1,\mathrm{X}_2) := \inf_{\mathrm{g}\in\mathrm{X}_2}\left\{\|\mathrm{f}-\mathrm{g}\|_{\mathrm{X}_1} + \delta\|\mathrm{g}\|_{\mathrm{X}_2}\right\}, \delta \ge 0,$$

where $\mathrm{X}_1, \mathrm{X}_2$ are two Banach spaces, with $\mathrm{X}_2 \subset \mathrm{X}_1$.

It was defined by Peter See [3]. Many researchers have tried to find a relate between the K-Functionl and the modulus of smoothness by using different spaces, such as $L_p, 0 \le p < \infty$ spaces and continuous spaces, like as in([3],[5]).

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Prove that for any function $f \in L_p$ and order r there are positive constants C_1, C_2, such that for every $t > 0$

$$C_1 K_r(f,t)_p \leq \omega_r(f,t)_p \leq C_2 K_r(f,t)_p .$$

The same result was achieved in [5] using the Ditzian-Totik modulus of smoothness and the K-Functionl with weigh function $\varphi(x) = \sqrt{1-x^2}$; see also ([3],[6],[7]).

4. Important Applications and Theories

4.1 Jackson's Theorem [10]:

If $f \in L_p[-1,1]$ and has smoothness of order r, then the degree of best approximation with polynomial of order n can be estimated using the modulus of smoothness

$E_n(f)_p \leq C\omega_r(f,t)_p$, where $E_n(f)_p = \inf_{\wp_n \in P} \|f - \wp_n\|_p$ the degree of best approximation.

4.2 Stechkin's Theorem [3],[5]:

If we give modulus of smoothness such that $\omega_r(f,t)_p \leq Mt^{\alpha}$, then we have $E_n(f)_p \leq Ct^{-\alpha}$.

4.3 Equivalence with K-functional [3]:

There is an equivalence between the modulus of smoothness and the K-Functional in the L_p spaces, such that $\omega_r(f,t)_p \approx K_r(f,t)_p$.

4.4 Ditzian–Timan–Nikolskii type Theorem [5]:

It links partial modulus of smoothness in certain directions with the ability to approximate multidimensional functions $E_n(f) \asymp \omega_{r_1}^{(1)}(f,t) + \cdots + \omega_{r_d}^{(d)}(f,t)$. This is particularly relevant when dealing with multivariable functions.

4.5 Velychko–Durr Type Theorem [12]:

Let $f \in L_p(\mathbb{R}^d), 1 \leq p < \infty$, and let $\{S_n(f)\}$ be a sequence of approximants such that

$\|f - S_n(f)\|_p \leq Cn^{-s}$, for all $n \in \mathbb{N}, C > 0$, and real number $s > 0$.

Then the function $f \in Besov\ space\ B_{p,q}^s(\mathbb{R}^d)$ (with appropriate q), and the modulus of smoothness of order $r > s$ satisfies :

$\omega_r(f,t)_p \leq C_1 t^s, \forall 0 < t \leq 1$, with a constant C_1depending on f and p.

5. Wavelet Approximation

5.1 The theoretical relationship between smoothness and wavelet approximation

This section discusses the theoretical relationship between smoothness and wavelet approximation, see [1],[2]. Let us assume that $f \in L^2(\mathbb{R})$ is representable by a wavelet sequence with basis functions $\chi_{j,k}(x)$. Then $f(x) = \sum_{j,k} c_{j,k}\chi_{j,k}(x)$, where the vanishing rate of the coefficients $c_{j,k}$ depends on the smoothness of the function f. For example, if $f \in C^r$, then $|c_{j,k}| \leq C2^{-j((r+1)/2)}$, the smaller the modulus of smoothness $\omega_r(f,t)$ is, the smaller the wavelet coefficients are, and thus a more efficient representation is achieved.

5.2 Approximation and Averaging

This observation is used to determine how well f can be approximated by cutting the wave series with a certain order J. If the function is smooth (i.e., $\omega_r(f,t) \to 0$), a small number of wave coefficients can accurately represent it, thereby enhancing the effectiveness of compression and approximation. This approximation rate is associated with results of the following type:

$\|f - f_J\|_2 \leq C\omega_r(f, 2^{-J})$, where f_J is the approximation of f using levels up to J,like as [1],[2].

5.3 Practical Applications

-Image Compression (JPEG2000): Wavelet transforms are used to efficiently approximate images by eliminating small parameters, which are estimated through smoothness.

-Noise Removal: Wavelet denoising algorithms rely on separating noise from the signal by analyzing the local smoothness coefficient.

-Approximation of numerical solutions to partial differential equations (PDEs): We use wavelets to approximate solutions, with the degree of smoothness of the solution controlling the distribution of levels in the approximation, see([1],[2],[8],[9]).

5.4 Compared to Jackson-type

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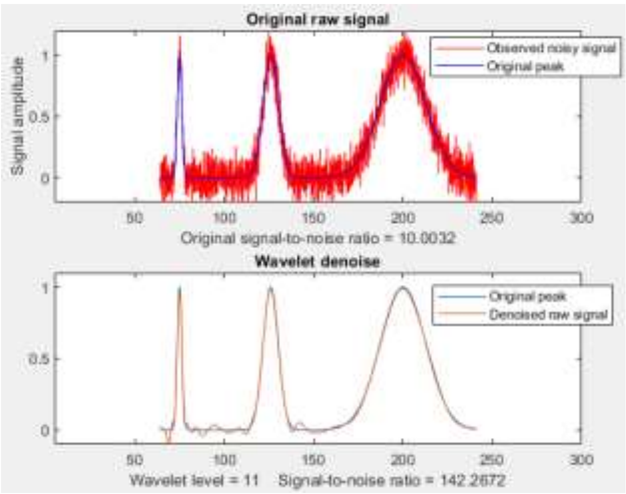
Although Jackson-type results are conventional in L_p spaces, the wavelet approximation offers:

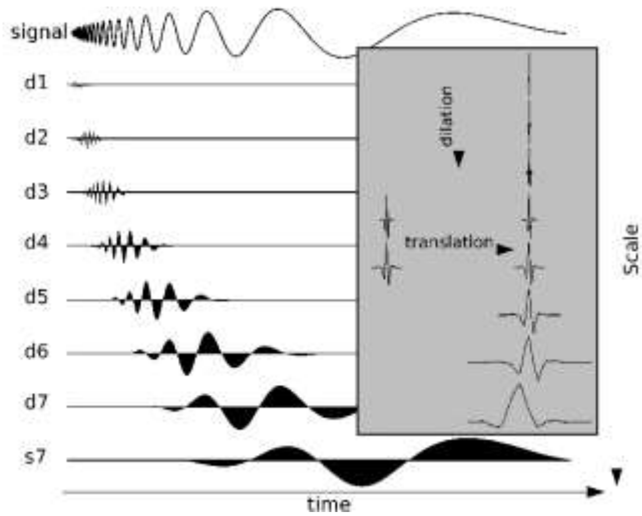
-A local representation that allows the analysis of the function's behavior at different locations,

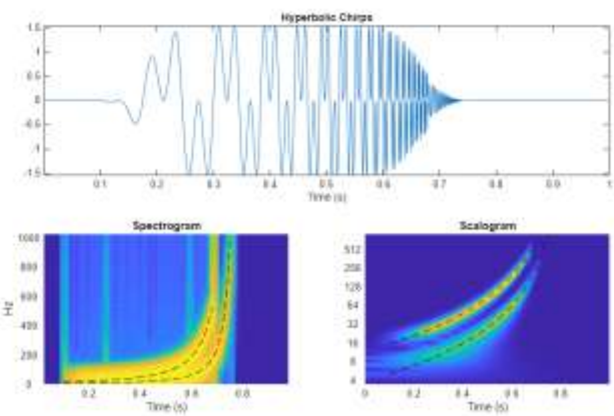
-Compressibility that depends directly on the local smoothness measure,

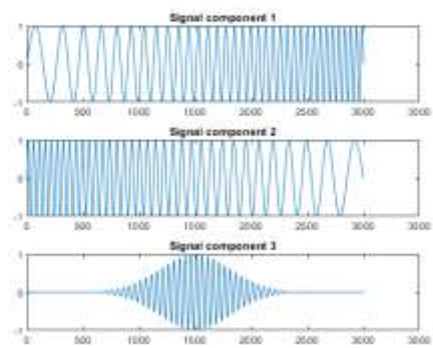
-The system performs more effectively when handling non-smooth functions or those with singularities, like as ([1],[2],[8],[9])

Below, we will list some diagrams that illustrate the relationship between smoothness units and wave approximation, taken from scientific sources,see([8],[9]).









The graphs indicate that the wavelet coefficients decrease faster when dealing with smooth functions, showing a clear link between how smooth a function is and how well it can be approximated. The Lipschitz index, also known as the smoothness measure, indicates that the smoother the signal, the fewer coefficients are required for accurate representation.

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