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Concave and convex functions

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Abstract : *This research is a review of the topic of convex and concave functions. We presented some important theories and definitions related to this topic and presented some of the relationships that link these concepts.*

Keywords: convex set , concave function **,** GA-convex function

1. Introduction

A function that translates a convex subset of a vector space to the set of real numbers is called a convex function. There are numerous uses for functions' concavity and convexity in demonstrating inequality. In 2004, Cha studied formulas pertaining to convex function theorems and derived a number of significant inequalities that were then used to solve conditional extremum problems and prove other inequalities [2]. Xia used the concavity, convexity, and continuity of functions to derive Jensen's inequality in 2005 [7]. Song and Wan's investigation of GA-convex functions in 2010 yielded a more succinct Hadamard-type inequality for GA-convex functions [5]. Convexity, monotonicity, and non-negativity are some crucial characteristics of convex functions that aid in the derivation and demonstration of inequalities.

2. Basic definitions and theorems

Definition 2.1: [4] A set Y is convex where $Y \subset R^n$ if there is $y', y'' \in Y$ the straight line segment between y' and y'' belongs entirely to Y, in another way $\forall h \in [0,1]$ then

$$y^h = (1-h)y' + hy'' \in Y \ \forall h \in [0,1].$$

Definition 2. 2: [4] A function g defined on a convex set N is concave where $g : N \to R$, $N \subset R^n$ if

$$(1-h)g\left(y'\right) + hg\left(y''\right) \le g\left((1-h)y' + hy''\right),$$

where y' and $y'' \in N \ \forall h \in [0,1]$.

g is completely concave if

$$(1-h)g\left(y'\right) + hg\left(y''\right) < g\left((1-h)y' + hy''\right).$$

Definition 2. 3: [4] A function $g: N \to R$, $N \subset R^n$ is convex if

$$(1-h)g\left(y'\right) + hg\left(y''\right) \ge g\left((1-h)y' + hy''\right),$$

where y' and $y'' \in N \ \forall h \in [0,1]$.

g is completely convex if

$$(1-h)g\left(y'\right) + hg\left(y''\right) > g\left((1-h)y' + hy''\right).$$

Theorem 2.4: [4] Let functions $g_1, \dots, g_n$ are convex (concave) and $b1 > 0, \dots, bn > 0$, then $H = b1\, g_1 + \dots + bn\, g_n$ is convex (concave).

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Theorem 2.5: [4] A $\mathbb{C}^1$ function $g : N \to R$, $N \subset R^n$ is concave iff

$$g(z) - g(y) \le Dg(y)(z - y),$$

$\forall\ y, z \in N$, in other words

$$G(z) - g(y) \le \frac{\partial g}{\partial y_1}(y)(z_1 - y_1) + \cdots + \frac{\partial g}{\partial y_n}(y)(z_n - y_n),$$

and g is convex iff $g(z) - g(y) \ge Dg(y)(z - y)$.

Theorem 2. 6: [4] A function g defined on a convex set N is concave where $g : N \to R$, $N \subset R^n$ if

$$Dg(y^*)(z - y^*) \le 0$$

$\forall\ z \in N$, then y^* is a global maximizer of g on N.

Also , $\forall\ z \in N$ if

$$Dg(y^*)(z - y^*) \ge 0$$

and g is convex then y^* is a global minimizer of g on N.

Definition 2.7: [4] A function g(y) be quasi concave if $\mathcal{L}_\eta = \{y : g(y) \le \eta\}$ be a convex set $\forall$ constant η , where g(y) defined on $N \subset R^n$.

Also, g be quasi convex if $N_\eta = \{y : g(y) \ge \eta\}$ be a convex set $\forall$ constant η.

Definition 2. 8: [4] A function g(y) be quasi concave if

$$g(hy + (1 - h)z) \ge min(g(y), g(z))$$

$\forall y, z \in N$ and $h \in [0, 1]$.

And g is quasi convex if

$$g(hy + (1 - h)z) \le max(g(y), g(z)).$$

Theorem 2.9: [1] Let $Y, W \subseteq \mathcal{R}^n$ be convex sets and $s \in \Re$. Then the following sets are convex

1. $sY = \{q \in \mathcal{R}^n : there\ is\ y \in Y \ni q = sy\}$
2. $Y + W = \{q \in \mathcal{R}^n : there\ is\ \boldsymbol{y} \in Y\ and\ z \in W \ni q = y + z\}$
3. $Y \cap W$.

Definition 2.10: [1] $y \in \mathcal{R}^n$ be a convex combination of $y_1, \ldots, y_r \in \mathcal{R}^n$ if

$$y = \sum_{i=1}^{r} \delta^i y^i = \begin{bmatrix} \sum_{i=1}^{r} \delta^i y_1^i \\ \vdots \\ \sum_{i=1}^{r} \delta^i y_n^i \end{bmatrix}$$

$$where\ \delta^i \in [0, 1]\ and\ \sum_{i=1}^{r} \delta^i = 1\,.$$

Definition 2.11: [1] $\forall\, y, y' \in D\ and\ \sigma, \delta > 0\ ,\ \sigma y + \delta y' \in D\ then\ D \subseteq \mathcal{R}^n\ be\ a$ convex cone.

Definition 2.12 : [1] let $\boldsymbol{v} \in \mathcal{R}^n,\ v \neq 0\ , k \in \mathcal{R}$, the hyperplane span by $\boldsymbol{v}$ and k is the (n - 1) dimensional plane

$$H_{v,k} = \{y \in \mathcal{R}^n : v \cdot y = k\}.$$

The sets $\{y \in \mathcal{R}^n : v \cdot y \geq k\}$ *and* $\{y \in \mathcal{R}^n : v \cdot y \leq k\}$ be half-space above and the half-space below the hyperplane $H_{v,k}$.

Definition 2.13 : [1] Let $Y, W \subseteq \mathcal{R}^n$ *are* two nonempty sets

- There is $v \neq 0 \in \mathcal{R}^n$ and $k \in \mathcal{R}$ where $v \cdot y \geq k \geq v \cdot z$ iff Y and W are separated via a hyperplane $H_{v,k}$, for all $y \in Y, z \in W$.
- There is $v \neq 0 \in \mathcal{R}^n$ and $k \in \mathcal{R}$ where $v \cdot y > k > v \cdot z$ iff Y and W are strictly separated via a hyperplane $H_{v,k}$, for all $\boldsymbol{y} \in Y, \boldsymbol{z} \in W$.

Definition 2.14 : [1] Let $Y \subseteq \mathcal{R}^n$ such that $Y \neq 0$.

- There is $\boldsymbol{v} \neq \boldsymbol{0} \in \mathcal{R}^n$ where $v \cdot y \geq v \cdot y^*$ iff Y is supported at y^* for all $y \in Y$
- There is $v \neq 0 \in \mathcal{R}^n$ where $v \cdot y > v \cdot y^*$ iff Y is strictly supported at y^* for all $y \in Y$, $y \neq y^*$

Theorem 2.15 : [1] $\hbar yp\ g$ is convex iff g is concave , such that

$$\hbar yp\ g = \{(\boldsymbol{y}, z) \in \mathcal{R}^{n+1} : \boldsymbol{y} \in \mathcal{R}^n \text{ and } z \leq g(\boldsymbol{y})\}.$$

Theorem 2.16: [1] the following are satisfies
1. g be convex.
2. $ep\imath\ g$ be a convex set , where $ep\imath\ g = \{(\boldsymbol{y}, z) \in \mathcal{R}^{n+1} : \boldsymbol{y} \in \mathcal{R}^n \text{ and } z \geq g(\boldsymbol{y})\}$.
3. $\forall\ y', y'' \in K, y^\sigma = \sigma y' + (1-\sigma)y''$,

$$g(y^\sigma) \leq \sigma g(y') + (1-\sigma) g(y'').$$

Theorem 2.17 : [8] If $g(y)$ is concave downwards or upwards in $[\partial, f]$, then

$$g(\sigma y_1 + \delta y_2) \leq (or \geq) \sigma g(y_1) + \delta g(y_2).$$

Where (x) is concave downwards or upwards in $[\partial, \mathrm{f}]$ if

$$g(y) \leq (or \geq) \frac{y_2 - y}{y_2 - y_1} g(y_1) + \frac{y - y_1}{y_2 - y_1} g(y_2).$$

Definition 2.18 : [8] g(y) is concave down or concave up on $[\partial, \mathrm{f}]$ if

$$g(\sigma y_1 + \delta y_2) \leq (or \geq) \sigma g(y_1) + \delta g(y_2)$$

where $y_1, y_2 \in [\partial, \mathrm{f}], \sigma > 0, \delta > 0$ $\sigma + \delta = 1$.

Lemma 2.19 : [8] For any $y_1, y_2, \ldots, y_n \in [\partial, \mathrm{f}]$ there is

$$g\left(\frac{y_1 + y_2 + \cdots + y_n}{n}\right) \leq or \geq \frac{g(y_1) + g(y_2) + \cdots + g(y_n)}{n},$$

where g(y) is convex downwards or upwards on $[\partial, \mathrm{f}]$.

Definition 2.20 : [6] $\forall\ y_1, y_2 \in J \in (0, +\infty)$ and $h \subseteq (0,1)$, there is

$$g(y_1^h y_2^{1-h}) \leq \mathrm{h} g(y_1) + (1 - \mathrm{h}) g(y_2).$$

Then g(y) be a GA-sub convex function on J.

Theorem 2.21: [8] If $g(y)$ be a GA-convex function on $(\partial, f) \in (0, +\infty)$, then $g(e^y)$ be a G A-sub convex function on $(ln\partial, lnf)$, $\forall\ y_1, y_2 \in (\partial, f)$ and $h \in (0,1)$.
Theorem 2.22: [8] If g(y) be a GA-concave on $J \in (0, +\infty)$, $y_j \in J, \lambda_j \in \mathcal{R}$ $(j = 1,2, \ldots, r)$, $\lambda_1 + \lambda_2 + \cdots + \lambda_r = 1$ then

$$g\ (y_1^{\lambda_1}\ y_2^{\lambda_2}\ \dots\ y_k^{\lambda_k})\ \le\ \lambda_1 g(y_1)\ +\ \lambda_2 g(y_2)\ +\ \cdots +\lambda_k\ g(y_k)(\sum_{j=1}^{k}\lambda_j\ =\ 1\ ,\lambda_j\ > 0).$$

Theorem 2.23: [3] If $g: [\partial, f] \to (0, +\infty)$ be a GA-Concave, then

$$\left(\frac{1}{e}\left(\frac{f^f}{\partial^\partial}\right)\frac{1}{f-\partial}\right) \le \frac{1}{f-\partial}\int_\partial^f g(y)\ dy\ \le\ \left(\frac{1}{lnf - ln\partial} - \frac{\partial}{\partial - f}\right) g(\partial) + \left(\frac{f}{\partial - f} - \frac{1}{lnf - ln\partial}\right) g(f).$$

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