International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 9 Issue 7 July - 2025, Pages: 46-55

On The Interval-valued Bifuzzy ψ-subalgebra of ψ-algebra

Nabaa Hasoon Jabir

Department of Mathematics
Faculty of Education for Girls,University of Kufa,
Iraq
nabaah.al-saedi@uokufa.edu.iq

Abstract: we will introduce a new notion called an interval-valued bifuzzy ψ-subalgebra of ψ-algebra, we described the relation between an interval-valued bifuzzy subset $\Omega = <\tilde{\mu}_{\Omega}, \tilde{\nu}_{\Omega}>$ of X and Lower $[s_1, s_2]$- Level of Ω, upper $[t_1,t_2]$-Level of Ω. Furthermore, we will present some results on image and pre-image. Also, we introduce the notion of image and pre-image of an interval-valued bifuzzy ψ-subalgebras of an ψ-algebra.

Keywords: ψ-algebra, ψ-subalgebra, fuzzy ψ-subalgebra, bifuzzy ψ-subalgebra, interval-valued bifuzzy ψ-subalgebra.

1. INTRODUCTION

In 1965, L.A. Zadeh introduced the notion of fuzzy subset, [1]. In 1976, K. Is´eki and S. Tanaka studied the notion of BCK-algebra, [2]. In 1991, O.G. Xi studied the notion of fuzzy BCK-algebra, [3]. In 2006, A.B. Saoid introduced fuzzy QS-algebra with interval-valued membership function, [4]. Also, T. Priya and T. Ramachandran introduced anti-fuzzy ideals of CI-algebra and its lower level cuts, [5]. Jun[6,7] studied the notion of cubic set as generalization of fuzzy set and interval-valued fuzzy set. In 2015, A.T. Hameed introduced the idea of SA-algebras. She stated some concepts related to it such as SA-subalgebra, SA-ideal, fuzzy SA-subalgebra and fuzzy SA-ideal of SA-algebra. She introduced the concept of homomorphisms on SA-algebra and fuzzy homomorphisms on SA-algebra, [9]. In 2023, A.T. Hameed and N.H. Jabir introduced the notion of ψ -subalgebra, bifuzzy ψ -subalgebra and they introduced the concept of homomorphisms on ψ -algebra and fuzzy homomorphisms on ψ -algebra. In this paper**,** the concepts of interval-valued bifuzzy ψ-subalgebra and how the image and the preimage of them under homomorphism of ψ-algebras are studied.

2. Preliminaries

In this section, we give some basic definitions and preliminaries proprieties of ψ-subalgebras of ψ -algebra such that we include some elementary aspects that are necessary for this paper.

Definition 2.1.([14]). Let $(X;+,-,0)$ be an algebra with two operations $(+)$ and $(-)$ and constant (0). X is called an ***ψ*-algebra** if it satisfies the following properties: for all $x, y, z \in X$,

(ψ_1) $x - x = 0,$

(ψ_2) $(0 - x) + x = 0,$

(ψ_3) $(x - y) - z = x - (z + y),$

(ψ_4) $(y + x) - (x - z) = y + z.$

In , we can define a binary relation ($\le$) by : $x \le y$ if and only if $x + y = 0$ and $x - y = 0$, $x, y \in X$.

Definition 2.2. [13].

Let $(X;+,-,0)$ be a ψ-algebra and let S be a nonempty set of X. S is called **a ψ- subalgebra of X** if $x + y \in S$ *and* $x - y \in S$, whenever $x, y \in S$.

Definition 2.3.[4].

Let X be a nonempty set, a fuzzy subset μ of X is a mapping $\mu: X \rightarrow [0,1]$.

Definition 2.4.[14].

For any $t \in [0,1]$ and a fuzzy subset μ in a nonempty set X, the set

$U(\mu, t) = \{x \in X \mid \mu(x) \geq t\}$ is called **an upper t-level cut of μ,** and the set $L(\mu, t) = \{x \in X \mid \mu(x) \leq t\}$ is called **a lower t-level cut of μ**.

Definition 2.5.[13].

Let $(X; +, -, 0)$ be a ψ-algebra, a fuzzy subset μ of X is called **a fuzzy ψ-subalgebra of X** if for all $x, y \in X$,

1- $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$ and
2- $\mu(x-y) \geq min\{\mu(x), \mu(y)\}$.

Definition 2.6. [14].

Let $(X; +, -, 0)$ be an ψ-algebra, a fuzzy subset μ of X is called **an anti-fuzzy ψ-subalgebra of X** if for all $x, y \in X$,

AFψS$_1$) $\mu(x+y) \leq max\{\mu(x), \mu(y)\}$,

AFψS$_2$) $\mu(x-y) \leq max\{\mu(x), \mu(y)\}$.

Proposition 2.7. [4].

Let μ be an anti-fuzzy subset of an ψ-algebra $(X; +, -, 0)$.

1- If μ is an anti-fuzzy ψ-subalgebra of , then it satisfies for any t∈[0, 1], $L(\mu, t) \neq \emptyset$ implies $L(\mu, t)$ is a ψ-subalgebra of X.

2- If $L(\mu, t)$ is a ψ-subalgebra of X, for all t∈[0, 1], $L(\mu, t) \neq \emptyset$, then μ is an anti-fuzzy ψ-subalgebra of X .

Proposition 2.8. [4].

Let μ be an anti-fuzzy subset of an ψ-algebra $(X; +, -, 0)$.

1- If μ is an anti-fuzzy ψ-ideal of , then it satisfies for any t∈[0, 1], $L(\mu, t) \neq \emptyset$ implies $L(\mu, t)$ is an ψ- ideal of X.

2- If $L(\mu, t)$ is an ψ- ideal of X, for all t∈[0, 1], $L(\mu, t) \neq \emptyset$, then μ is an anti-fuzzy ψ-ideal of X .

Definition 2.9. [12].

Let $f: (X; +, -, 0) \rightarrow (Y; +', -', 0')$be a mapping nonempty ψ-algebras X and Y respectively. If μ is anti-fuzzy subset of X, then the anti-fuzzy subset β of Y defined by:

$$f(\mu)(y) = \begin{cases} inf\{\mu(x): x \in f^{-1}(y)\} & if \ f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ 1 & otherwise \end{cases}$$

is said to be the image of μ under f.

Similarly if β is anti-fuzzy subset of , then the fuzzy subset $\mu = (\beta \circ f)$ of X (i.e the anti-fuzzy subset defined by $\mu(x) = \beta(f(x))$, for all

$x \in X$) is called the pre-image of β under f .

Now, we will recall the concept of bifuzzy subsets.

Definition 2.10. [6].

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be a bifuzzy subset of a ψ-algebra X. A is said to be **an bifuzzy ψ -subalgebra of X** if : for all x, y $\in$X,

(IFS$_1$) $\mu_A(x+y) \geq min\{\mu_A(x), \mu_A(y)\}$ and

$\mu_A(x-y) \geq min\{\mu_A(x), \mu_A(y)\}$.

(IFS$_2$) $\nu_A(x+y) \leq max\{\nu_A(x), \nu_A(y)\}$ and

$\nu_A(x-y) \leq max\{\nu_A(x), \nu_A(y)\}$.

i.e., μ_A is fuzzy ψ-subalgebra of ψ-algebra and ν_A is anti-fuzzy ψ-subalgebra of ψ-algebra.

Remark 2.11. [7].

An interval number is $\tilde{a} = [a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. Let I be a closed unit interval, (i.e., I = [0, 1]).

Let D[0, 1] denote the family of all closed subintervals of I = [0, 1] , that is, D[0, 1] = { $\tilde{a} = [a^-, a^+] \mid a^- \leq a^+$, for $a^-, a^+ \in$ I}

Now, we define what is known as refined minimum (briefly, rmin) of two element in D[0,1].

Definition 2.12. [3].

We also define the symbols ($\succcurlyeq$) , ($\preccurlyeq$) , (=) ,"rmin " and "rmax " in case of two elements in D[0, 1] . Consider two interval numbers (elements numbers)

$\tilde{a} = [a^-, a^+]$, $\tilde{b} = [b^-, b^+]$in D[0, 1] : Then

(1) $\tilde{a} \succcurlyeq \tilde{b}$ if and only if, $a^- \geq b^-$ and $a^+ \geq b^+$,

(2) $\tilde{a} \preccurlyeq \tilde{b}$ if and only if, $a^- \leq b^-$ and $a^+ \leq b^+$,

(3) $\tilde{a} = \tilde{b}$ if and only if, $a^- = b^-$ and $a^+ = b^+$,

(4) rmin $\{\tilde{a}, \tilde{b}\}$= [min $\{a^-, b^-\}$, min $\{a^+, b^+\}$],

(5) rmax $\{\tilde{a}, \tilde{b}\}$= [max $\{a^-, b^-\}$, max $\{a^+, b^+\}$],

Remark 2.13. [11].

Let $\tilde{0}$ = [0, 0] as its least element and $\tilde{1}$ = [1, 1] as its greatest element. Let $\tilde{a}_i \in$D[0, 1] where i $\in \Lambda$. We define $r\ inf_{i\in\Lambda}\tilde{a}$= $[r\ inf_{i\in\Lambda}a^-, r\ inf_{i\in\Lambda}a^+]$, $r\ sup_{i\in\Lambda}\tilde{a}$= $[r\ sup_{i\in\Lambda}a^-, r\ sup_{i\in\Lambda}a^+]$.

Definition 2.14. [10].

An **interval-valued fuzzy subset** $\widetilde{\mu}_A$ on subset is defined as $\widetilde{\mu}_A$ = {< x, [μ_A^- (x) , μ_A^+ (x)]>| x$\in X$} . Where μ_A^- (x) $\leq \mu_A^+$ (x), for all x$\in$ X. Then the ordinary fuzzy subsets μ_A^-: X $\to$ [0, 1] and μ_A^+: X $\to$ [0, 1] are called a **lower fuzzy subset and an upper fuzzy subset** of $\widetilde{\mu}_A$ respectively.

Let $\widetilde{\mu}_A$ (x) = [μ_A^- (x) , μ_A^+ (x)] , $\widetilde{\mu}_A$:X $\to$ D[0, 1], then A = {< x, $\widetilde{\mu}_A$ (x) >| x$\in$ X} .

Definition 2.15.[7].

Let $(X\,;+,-,0)$ be a nonempty set. A interval-valued bifuzzy set Ω in a structure $\Omega = \{< x, \tilde{\mu}_{\Omega}(x), \tilde{v}_{\Omega}(x) > | \, x \in X\}$, which is briefly denoted by $\Omega = <\tilde{\mu}_{\Omega}, \tilde{v}_{\Omega}>$, where $\tilde{\mu}_{\Omega} : X \rightarrow D[0, 1]$, $\tilde{\mu}_{\Omega}$ is an interval-valued fuzzy subset of X and $\tilde{v}_{\Omega} : X \rightarrow D[0, 1]$, $\tilde{v}_{\Omega}$ is an interval-valued fuzzy subset of X.

Proposition 2.16.[5].

Let $(X\,;+,-,0)$ be an ψ-algebra. An interval-valued bifuzzy subset $\Omega = <\tilde{\mu}_{\Omega}, \tilde{v}_{\Omega}>$ of . If for all $\tilde{t} \in D[0, 1]$ and $\tilde{s} \in D[0, 1]$, the set $\widetilde{U}\,(\Omega; \tilde{t}, \tilde{s})$ is an ψ-subalgebra of X, then Ω is an interval-valued bifuzzy ψ-subalgebra of X.

3. Interval-valued Bifuzzy ψ-subalgebras of ψ-algebra

In this section, we will introduce a new notion called interval-valued bifuzzy ψ -subalgebras of ψ-algebra and study several properties of it.

Definition 3.1.

Let $(X\,;+,-,0)$ be an ψ-algebra. An interval-valued bifuzzy subset $\Omega = < \tilde{\mu}_{\Omega}(x), \tilde{v}_{\Omega}(x) >$ of X is called **an interval-valued bifuzzy ψ-subalgebra of X** if, for all $x, y \in X$,

1- $\tilde{\mu}_{\Omega}(x+y) \succcurlyeq$ rmin$\{\tilde{\mu}_{\Omega}(x), \tilde{\mu}_{\Omega}(y)\}$, and

$\tilde{v}_{\Omega}(x+y) \preccurlyeq$ rmax$\{\tilde{v}_{\Omega}(x), \tilde{v}_{\Omega}(y)\}$.

2- $\tilde{\mu}_{\Omega}(x+y^{-}) \succcurlyeq$ rmin$\{\tilde{\mu}_{\Omega}(x), \tilde{\mu}_{\Omega}(y)\}$, and

$\tilde{v}_{\Omega}(x+y^{-}) \preccurlyeq$ rmax$\{\tilde{v}_{\Omega}(x), \tilde{v}_{\Omega}(y)\}$.

Example 3.2.

Let $X = \{0, 1, 2, 3\}$ in which the operations as in example $(+,-)$ be define by the following table:

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$-$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Then $(X; +, -, 0)$ is an ψ-algebra. Define an interval-valued bifuzzy subset $\Omega = < \tilde{\mu}_{\Omega}(x), \tilde{v}_{\Omega}(x) >$ of X of fuzzy subset μ: $X \rightarrow [0,1]$ by:

$$\tilde{\mu}_{\Omega}(x) = \begin{cases} [0.3,0.6] & \text{if } x = \{0,2\} \\ [0.1,0.4] & otherwise \end{cases}$$

and $\tilde{v}_{\Omega}(x) = \begin{cases} [0.2,0.3] & \text{if } x = \{0,2\} \\ [0.4,0.5] & otherwise \end{cases}$. The interval-valued bifuzzy subset $\Omega = < \tilde{\mu}_{\Omega}(x), \tilde{v}_{\Omega}(x) >$ is an interval-valued bifuzzy ψ-subalgebra of X.

Proposition 3.3.

Let $\Omega = < \tilde{\mu}_{\Omega}(x), \tilde{v}_{\Omega}(x) >$ be an interval-valued bifuzzy

ψ-subalgebra of an ψ-algebra $(X\,;+,-,0)$, then $\tilde{\mu}_\Omega(0) \succcurlyeq \tilde{\mu}_\Omega(x)$ and $\tilde{v}_\Omega(0) \preccurlyeq \tilde{v}_\Omega(x)$, for all $x \in X$.

Proof:

For all $x \in X$, we have

$$\tilde{\mu}_\Omega(0) = \tilde{\mu}_\Omega\,(x + x^-)$$

$$\succcurlyeq \text{rmin}\{\tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(x)\}$$

$$= \left[\min\{\mu_A^-(x), \mu_A^-(x)\}, \min\{\mu_A^+(x), \mu_A^+(x)\}\right]$$

$$= \left[\mu_A^-\,(x), \mu_A^+(x)\right] = \tilde{\mu}_\Omega\,(x).$$

Similarly, we can show that

$$\tilde{v}_\Omega\,(0) \preccurlyeq \text{rmax}\,\{\tilde{v}_\Omega\,(x), \tilde{v}_\Omega\,(x)\} = \tilde{v}_\Omega\,(x).\ \square$$

Proposition 3.4.

If an interval-valued bifuzzy subset $\Omega = <\tilde{\mu}_\Omega(x), \tilde{v}_\Omega(x)>$ of ψ-algebra $(X;+,-,0)$ is an interval-valued bifuzzy ψ-subalgebra, $\Omega(x+y) = \Omega\left(x + \left((y+0)+0\right)\right)$, for all $x, y \in X$.

Proof:

Let X be an ψ-algebra and $x, y \in X$, then we know that $y = (y+0)+0$. Hence, $\tilde{\mu}_\Omega(x+y) = \tilde{\mu}_\Omega\left(x + \left((y+0)+0\right)\right)$ and $\tilde{v}_\Omega(x+y) = \tilde{v}_\Omega\left(x + \left((y+0)+0\right)\right)$, therefore $\Omega(x+y) = \Omega\left(x + \left((y+0)+0\right)\right)$. □

Definition 3.5.

Let $(X\,;+,-,0)$ be an ψ-algebra. An interval-valued bifuzzy subset $\Omega = <\tilde{\mu}_\Omega, \tilde{v}_\Omega>$ of X, for $[t_1,t_2] \in D[0,1]$ and $[s_1, s_2] \in D[0,1]$ the set $U(\tilde{\mu}_\Omega\,|[t_1,t_2]) = \{x \in X|\ \tilde{\mu}_\Omega\,(x) \succcurlyeq [t_1,t_2]\}$is called **upper $[t_1,t_2]$-Level of** Ω and $L(\tilde{v}_\Omega|[s_1, s_2]) = \{x \in X\,|\tilde{v}_\Omega(x) \preccurlyeq [s_1,s_2]\}$is called **Lower $[s_1, s_2]$-Level of Ω.** And

$$\Omega\,(\tilde{t}, \tilde{s}) = \Omega([t_1,t_2], [s_1,s_2]) = U(\tilde{\mu}_\Omega\,|[t_1,t_2]) \cap L(\tilde{v}_\Omega|[s_1, s_2])$$

$$= \{x \in X|\ \tilde{\mu}_\Omega(x) \succcurlyeq [t_1, t_2] \text{ and } \tilde{v}_\Omega(x) \preccurlyeq [s_1, s_2]\}.$$

$$= \{\ x \in X \mid \tilde{\mu}_\Omega(x) \succcurlyeq \tilde{t},\ \tilde{v}_\Omega(x) \preccurlyeq \tilde{s}\}$$

$$= \tilde{U}(\Omega; \tilde{t}, \tilde{s}).$$

Proposition 3.6.

Let $(X\,;+,-,0)$ be an ψ-algebra. An interval-valued bifuzzy subset $\Omega = <\tilde{\mu}_\Omega\,,\, \tilde{v}_\Omega>$ of X. If Ω is an interval-valued bifuzzy ψ-subalgebra of X, then for any $\tilde{t}, \tilde{s} \in D[0,1]$, the set $\tilde{U}\,(\Omega; \tilde{t}, \tilde{s})$ is a ψ-subalgebra of X .

Proof:

Assume that $\Omega = <\tilde{\mu}_\Omega , \tilde{\nu}_\Omega>$ is an interval-valued bifuzzy ψ-subalgebra of X and let $\tilde{t}, \tilde{s} \in$ D[0, 1] be such that $\widetilde{U}\ (\Omega; \tilde{t}, \tilde{s}) \neq \emptyset$, and let $x , y \in X$ such that $x , y \in \widetilde{U}\ (\Omega; \tilde{t}, \tilde{s})$, then $\tilde{\mu}_\Omega(x) \succcurlyeq \tilde{t}$, $\tilde{\mu}_\Omega(y) \succcurlyeq \tilde{t}$ and $\tilde{\nu}_\Omega(x) \preccurlyeq \tilde{s}$, $\tilde{\nu}_\Omega(y) \preccurlyeq \tilde{s}$. Since Ω is an interval-valued bifuzzy ψ-subalgebra of X, we get

$\tilde{\mu}_\Omega\,(x + y) \succcurlyeq$ rmin{ $\tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(y)$} $\succcurlyeq \tilde{t}$ and $\tilde{\nu}_\Omega(x + y) \preccurlyeq$ rmax {$\tilde{\nu}_\Omega(x), \tilde{\nu}_\Omega(y)$} $\preccurlyeq \tilde{s}$.

Similarly, $\tilde{\mu}_\Omega(x + y^-) \succcurlyeq$ rmin{ $\tilde{\mu}_\Omega\,(x)$,$\tilde{\mu}_\Omega(y)$} $\succcurlyeq \tilde{t}$

and $\tilde{\nu}_\Omega(x - \mathrm{y}) \preccurlyeq$ rmax {$\tilde{\nu}_\Omega(x)$,$\tilde{\nu}_\Omega(y)$} $\preccurlyeq \tilde{s}$.

Hence the set $\widetilde{U}(\Omega; \tilde{t}, \tilde{s})$is a ψ-subalgebra of X. □

Proposition 3.7.

Let $(X\,; +, -, 0)$ be an ψ-algebra. An interval-valued bifuzzy subset $\Omega = <\tilde{\mu}_\Omega , \tilde{\nu}_\Omega>$ of X. If for all $\tilde{t} \in$ D[0, 1] and $\tilde{s} \in$ D[0, 1], the set $\widetilde{U}(\Omega; \tilde{t}, \tilde{s})$ is a ψ-subalgebra of X, then Ω is an interval-valued bifuzzy ψ-subalgebra of X.

Proof:

Suppose that $\widetilde{U}(\Omega; \tilde{t}, \tilde{s})$is an ψ-subalgebra of X and let $x , y \in X$ be

such that $\tilde{\mu}_\Omega(x + y) \prec$ rmin {$\tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(y)$}, and

$\tilde{\nu}_\Omega(x + y) \succ$ rmax {$\tilde{\nu}_\Omega(x), \tilde{\nu}_\Omega(y)$}.

Consider $\tilde{\beta} = 1/2\{\ \tilde{\mu}_\Omega(x + y)$ + rmin{$\tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(y)$}} and

$\tilde{\gamma} = 1/2\ \{\ \tilde{\nu}_\Omega(x + y)$ + rmax{$\tilde{\nu}_\Omega\,(x), \tilde{\nu}_\Omega(\,y)$}}.

We have $\tilde{\beta} \in$ D[0,1] and $\tilde{\gamma} \in$ D[0, 1], and

$\tilde{\mu}_\Omega\,(x + y) \prec \tilde{\beta} \prec$ rmin {$\tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(y)$ }, and

$\tilde{\nu}_\Omega\,(x + y) \succ \tilde{\gamma} \succ$ rmax {$\tilde{\nu}_\Omega\,(x)$,$\tilde{\nu}_\Omega(y)$}.

It follows that $x, y \in \widetilde{U}\ (\Omega; \tilde{t}, \tilde{s})$, and $(x + y) \notin \widetilde{U}(\Omega; \tilde{t}, \tilde{s})$.

This is a contradiction.

Hence $\tilde{\mu}_\Omega(x + y) \succcurlyeq$ rmin{ $\tilde{\mu}_\Omega\,(x), \tilde{\mu}_\Omega\,(y)$} $\succcurlyeq \tilde{t}$ and

$\tilde{\nu}_\Omega\,(x + y) \preccurlyeq$ rmax {$\tilde{\nu}_\Omega(x), \tilde{\nu}_\Omega\,(y)$} $\preccurlyeq \tilde{s}$.

Similarly, $\tilde{\mu}_\Omega(x + y^-) \succcurlyeq$ rmin{ $\tilde{\mu}_\Omega(x)$,$\tilde{\mu}_\Omega\ (y)$} $\succcurlyeq \tilde{t}$ and

$\tilde{\nu}_\Omega\,(x + y^-) \preccurlyeq$ rmax {$\tilde{\nu}_\Omega\,(x)$,$\tilde{\nu}_\Omega\,(y)$} $\preccurlyeq \tilde{s}$.

Therefore $\Omega = <\tilde{\mu}_\Omega , \tilde{\nu}_\Omega>$ is an interval-valued bifuzzy ψ-subalgebra of X. □

Theorem 3.8.

An interval-valued bifuzzy subset $\Omega = <\tilde{\mu}_\Omega , \tilde{\nu}_\Omega>$ is an interval-valued bifuzzy ψ-subalgebra of ψ-algebra X if and only if, ${\mu^-}_\Omega$ and ${\mu^+}_\Omega$ are fuzzy ψ-subalgebras of X and ${\nu^-}_\Omega$ and ${\nu^+}_\Omega$ are anti-fuzzy ψ-subalgebra of X.

Proof:

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 9 Issue 7 July - 2025, Pages: 46-55

Assume that Ω is an interval-valued bifuzzy ψ-subalgebra of X, for any $x, y \in X$,

$[\mu^-{}_\Omega(x+y), \mu^+{}_\Omega(x+y)] = \tilde{\mu}_\Omega(x+y) \succcurlyeq \text{rmin}\{\tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(y)\}$

$= \text{rmin}\{[\mu^-{}_\Omega(x), \mu^+{}_\Omega(x)], [\mu^-{}_\Omega(y), \mu^+{}_\Omega(y)]\}$

$= [\min\{\mu^-{}_\Omega(x), \mu^-{}_\Omega(y)\}, \min\{\mu^+{}_\Omega(x), \mu^+{}_\Omega(y)\}]$.

Thus $\mu^-{}_\Omega(x+y) \geq \min\{\mu^-{}_\Omega(x), \mu^-{}_\Omega(y)\}$, $\mu^+{}_\Omega(x+y) \geq \min\{\mu^+{}_\Omega(x), \mu^+{}_\Omega(y)\}$ and

$[v^-{}_\Omega(x+y), v^+{}_\Omega(x+y)] = \tilde{v}_\Omega(x+y) \preccurlyeq \text{rmax}\{\tilde{v}_\Omega(x), \tilde{v}_\Omega(y)\}$

$= \text{rmax}\{[v^-{}_\Omega(x), v^+{}_\Omega(x)], [v^-{}_\Omega(y), v^+{}_\Omega(y)]\}$

$= [\max\{v^-{}_\Omega(x), v^-{}_\Omega(y)\}, \max\{v^+{}_\Omega(x), v^+{}_\Omega(y)\}]$.

Thus $v^-{}_\Omega(x+y) \leq \max\{v^-{}_\Omega(x), v^-{}_\Omega(y)\}$, $v^+{}_\Omega(x+y) \leq \max\{v^+{}_\Omega(x), v^+{}_\Omega(y)\}$.

Similarly, $\mu^-{}_\Omega(x+y^-) \geq \min\{\mu^-{}_\Omega(x), \mu^-{}_\Omega(y)\}$, $\mu^+{}_\Omega(x+y^-) \geq \min\{\mu^+{}_\Omega(x), \mu^+{}_\Omega(y)\}$ and

$v^-{}_\Omega(x+y^-) \leq \max\{v^-{}_\Omega(x), v^-{}_\Omega(y)\}$, $v^+{}_\Omega(x+y^-) \leq \max\{v^+{}_\Omega(x), v^+{}_\Omega(y)\}$ and

Therefore $\mu^-{}_\Omega$ and $\mu^+{}_\Omega$ are fuzzy ψ-subalgebras of X, $v^-{}_\Omega$ and $v^+{}_\Omega$ are anti-fuzzy ψ-subalgebra of X.

Conversely, let $\mu^-{}_\Omega$ and $\mu^+{}_\Omega$ are fuzzy ψ-subalgebras of , $v^-{}_\Omega$ and $v^+{}_\Omega$ are anti-fuzzy ψ-subalgebra of X and $x, y \in X$, then

$\mu^-{}_\Omega(x+y) \geq \min\{\mu^-{}_\Omega(x), \mu^-{}_\Omega(y)\}$,

$\mu^+{}_\Omega(x+y) \geq \min\{\mu^+{}_\Omega(x), \mu^+{}_\Omega(y)\}$ and

$v^-{}_\Omega(x+y) \leq \max\{v^-{}_\Omega(x), v^-{}_\Omega(y)\}$,

$v^+{}_\Omega(x+y) \leq \max\{v^+{}_\Omega(x), v^+{}_\Omega(y)\}$.

Now, $\tilde{\mu}_\Omega(x+y) = [\mu^-{}_\Omega(x+y), \mu^+{}_\Omega(x+y)]$

$\succcurlyeq [\min\{\mu^-{}_\Omega(x), \mu^-{}_\Omega(y)\}, \min\{\mu^+{}_\Omega(x), \mu^+{}_\Omega(y)\}]$

$= \text{rmin}\{[\mu^-{}_\Omega(x), \mu^+{}_\Omega(x)], [\mu^-{}_\Omega(y), \mu^+{}_\Omega(y)]\}$

$= \text{rmin}\{\tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(y)\}$, therefore $\tilde{\mu}_\Omega(x+y) \succcurlyeq \text{rmin}\{\tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(y)\}$ and

$\tilde{v}_\Omega(x+y) = [v^-{}_\Omega(x+y), v^+{}_\Omega(x+y)]$

$\preccurlyeq [\max\{v^-{}_\Omega(x), v^-{}_\Omega(y)\}, \max\{v^+{}_\Omega(x), v^+{}_\Omega(y)\}]$

$= \text{rmax}\{[v^-{}_\Omega(x), v^+{}_\Omega(x)], [v^-{}_\Omega(y), v^+{}_\Omega(y)]\}$

$= \text{rmax}\{\tilde{v}_\Omega(x), \tilde{v}_\Omega(y)\}$, therefore $\tilde{v}_\Omega(x+y) \preccurlyeq \text{rmax}\{\tilde{v}_\Omega(x), \tilde{v}_\Omega(y)\}$.

Similarly, $\tilde{\mu}_\Omega(x+y^-) \succcurlyeq \text{rmin}\{\tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(y)\}$ and $\tilde{v}_\Omega(x-y) \preccurlyeq \text{rmax}\{\tilde{v}_\Omega(x), \tilde{v}_\Omega(y)\}$.

Hence Ω is an interval-valued bifuzzy ψ-subalgebra of X. □

Theorem 3.9.

If an interval-valued bifuzzy subset $\Omega = <\tilde{\mu}_\Omega , \tilde{\nu}_\Omega>$ is an interval-valued bifuzzy ψ-subalgebra of X, then the upper $[t_1,t_2]$-Level and Lower $[s_1,s_2]$-Level of Ω are ψ-subalgebras of X.

Proof:

Let $x, y \in U(\tilde{\mu}_\Omega |[t_1,t_2])$, then $\tilde{\mu}_\Omega (x) \succcurlyeq [t_1,t_2]$ and $\tilde{\mu}_\Omega (y) \succcurlyeq [t_1,t_2]$. It follows that

$\tilde{\mu}_\Omega (x+y) \succcurlyeq \text{rmin}\{\tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(y)\} \succcurlyeq [t_1,t_2]$, so that

$x+y \in U(\tilde{\mu}_\Omega |[t_1,t_2])$.

Similarly, $x + y^- \in U(\tilde{\mu}_\Omega |\tilde{t})$.

Hence $U(\tilde{\mu}_\Omega |\tilde{t})$ is ψ-subalgebra of X.

Let $x, y \in L(\tilde{\nu}_\Omega|[s_1,s_2])$, then $\tilde{\nu}_\Omega(x) \preccurlyeq [s_1,s_2]$ and $\tilde{\nu}_\Omega(y) \preccurlyeq [s_1,s_2]$. It

follows that $\tilde{\nu}_\Omega(x+y) \succcurlyeq \text{rmax}\{\tilde{\nu}_\Omega(x), \tilde{\nu}_\Omega(y)\} \preccurlyeq \tilde{s}$,

so that $x + y \in L(\tilde{\nu}_\Omega|\tilde{s})$.

Similarly, $x + y^- \in L(\tilde{\nu}_\Omega|\tilde{s})$

Hence $L(\tilde{\nu}_\Omega|\tilde{s})$ is ψ-subalgebra of X. ⌂

4. A homomorphic image and pre-image of an interval-valued bifuzzy ψ-subalgebra

In this section, we will introduce a new notion called a homomorphic image and pre-image of an interval-valued bifuzzy ψ-subalgebra of ψ-algebra and study several properties of it.

Definition 4.1.

Let $f: (X; +, -, 0) \to (Y; +', -', 0')$ be a mapping from the set X to a set Y.

If $\Omega = <\tilde{\mu}_\Omega , \tilde{\nu}_\Omega >$ is an interval-valued bifuzzy subset of X, then the interval-valued bifuzzy subset $\beta = <\tilde{\mu}_\beta , \tilde{\nu}_\beta>$ of Y defined by:

$$f(\tilde{\mu}_\Omega)(y) = \tilde{\mu}_\beta(y) = \begin{cases} \underset{x \in f^{-1}(y)}{rsup}\ \tilde{\mu}_\Omega(x) & if \quad f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ 0 & otherwise \end{cases}$$

$$f(\tilde{\nu}_\Omega)(y) = \tilde{\nu}_\beta(y) = \begin{cases} \underset{x \in f^{-1}(y)}{inf}\ \tilde{\nu}_\Omega(x) & if \quad f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ 1 & otherwise \end{cases}$$

is said to be **the image of Ω under** f **.**

Similarly if $\beta = <\tilde{\mu}_\beta , \tilde{\nu}_\beta>$ is an interval-valued bifuzzy subset of Y, then the interval-valued bifuzzy subset $\Omega = (\beta \circ f)$ in X (i.e the interval-valued bifuzzy subset defined by $\tilde{\mu}_\Omega (x) = \tilde{\mu}_\beta (f(x))$, $\tilde{\nu}_\Omega (x) = \tilde{\nu}_\beta (f(x))$ for all $x \in X$) is called **the pre-image of β under** f).

Theorem 4.2.

A homomorphic pre-image of an interval-valued bifuzzy ψ-subalgebra is also an interval-valued bifuzzy ψ-subalgebra.

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 9 Issue 7 July - 2025, Pages: 46-55

Proof:

Let $f:(X;+,-,0) \to (Y;+',-',0')$ be homomorphism from an

ψ-algebra X into an ψ-algebra Y. If $\beta = <\tilde{\mu}_\beta , \tilde{v}_\beta>$ is an interval-valued

bifuzzy ψ-subalgebra of Y and $\Omega = <\tilde{\mu}_\Omega , \tilde{v}_\Omega>$ the pre-image of β under f,

then $\tilde{\mu}_{f^{-1}(\beta)}(x) = \tilde{\mu}_\Omega(f(x))$, $\tilde{v}_{f^{-1}(\beta)}(x) = \tilde{v}_\Omega(f(x))$, for all $x \in X$.

Now, let $x, y \in X$, then

$$\begin{aligned}\tilde{\mu}_{f^{-1}(\beta)}(x+y) &= \tilde{\mu}_\Omega(f(x+y)) \\ &= \tilde{\mu}_\Omega(f(x) +' f(y)) \\ &\succcurlyeq \text{rmin}\,\{\tilde{\mu}_\Omega(f(x), \tilde{\mu}_\Omega(f(y))\} \\ &= \text{rmin}\,\{\tilde{\mu}_{f^{-1}(\beta)}(x), \tilde{\mu}_{f^{-1}(\beta)}(y)\} \text{ and}\end{aligned}$$

$$\begin{aligned}\tilde{v}_{f^{-1}(\beta)}(x+y) &= \tilde{v}_\Omega(f(x+y)) \\ &= \tilde{v}_\Omega(f(x)+' f(y)) \\ &\preccurlyeq \text{rmax}\,\{\tilde{v}_\Omega(f(x), \tilde{v}_\Omega(f(y))\} \\ &= \text{rmax}\,\{\tilde{v}_{f^{-1}(\beta)}(x), \tilde{v}_{f^{-1}(\beta)}(y)\}.\end{aligned}$$

Similarly, $\tilde{\mu}_{f^{-1}(\beta)}(x-y) \succcurlyeq \text{rmin}\{\tilde{\mu}_{f^{-1}(\beta)}(x), \tilde{\mu}_{f^{-1}(\beta)}(y)\}$ and $\tilde{v}_{f^{-1}(\beta)}(x+y^-) \preccurlyeq \text{rmax}\,\{\tilde{v}_{f^{-1}(\beta)}(x), \tilde{v}_{f^{-1}(\beta)}(y)\}$. □

Theorem 4.3.

Let $f:(X;+,-,0) \to (Y;+',-',0')$ be an epimorphism from an ψ-algebra X into an ψ-algebra Y. For every interval-valued bifuzzy ψ-subalgebra $\Omega = <\tilde{\mu}_\Omega , \tilde{v}_\Omega>$ of X with sup and inf properties, then $f(\Omega)$ is an interval-valued bifuzzy ψ-subalgebra of Y.

Proof:

By definition $f(\tilde{\mu}_\Omega)(y') = \underset{x \in f^{-1}(y')}{\text{rsup}} \tilde{\mu}_\Omega(x)$ and

$f(\tilde{v}_\Omega)(y') = \underset{x \in f^{-1}(y')}{\text{rinf}} \tilde{v}_\Omega(x)$ for all $y' \in Y$ and

$\text{rsup}(\emptyset) = [0,0]$ and $\text{rinf}(\emptyset) = [1,1]$. We have prove that

$f(\tilde{\mu}_\Omega)(x'+'y') \succcurlyeq \text{rmin}\,\{f(\tilde{\mu}_\Omega)(x'), f(\tilde{\mu}_\Omega)(y')\}$, and

$f(\tilde{v}_\Omega)(x'+'y') \preccurlyeq \text{rmax}\{f(\tilde{v}_\Omega)(x'), f(\tilde{v}_\Omega)(y')\}$, for all $x', y' \in Y$.

$$\begin{aligned}f(\tilde{\mu}_\Omega)(x'+'y') &= \underset{t \in f^{-1}(x'+'y')}{\text{rsup}} \tilde{\mu}_\Omega(t) = \tilde{\mu}_\Omega(x+y) \\ &\succcurlyeq \text{rmin}\,\{\tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(y)\}, \\ &= \text{rmin}\,\{\underset{t \in f^{-1}(x')}{\text{rsup}} \tilde{\mu}_\Omega(t), \underset{t \in f^{-1}(y')}{\text{rsup}} \tilde{\mu}_\Omega(t)\}\end{aligned}$$

$= \text{rmin}\ \{\ f(\tilde{\mu}_{\Omega})(x'), f(\tilde{\mu}_{\Omega})(y')\ \}$ and

$$f(\tilde{v}_{\Omega})(x'+'y') \ = \inf_{t\in f^{-1}(x'+'y')} \tilde{v}_{\Omega}(t) = \tilde{v}_{\Omega}(x+y)$$

$$\preccurlyeq \ \text{rmax}\ \{\tilde{v}_{\Omega}(x), \tilde{v}_{\Omega}(y)\}$$

$$= \text{rmax}\ \{\inf_{t\in f^{-1}(x')} \tilde{v}_{\Omega}(t), \inf_{t\in f^{-1}(y')} \tilde{v}_{\Omega}(t)\}$$

$$= \text{rmax}\ \{\ f(\tilde{v}_{\Omega})(x'), f(\tilde{v}_{\Omega})(y')\}.$$

Similarly, $f(\tilde{\mu}_{\Omega})(x'-y') \succcurlyeq$ rmin$\{\ f(\tilde{\mu}_{\Omega})(\ x'), f(\tilde{\mu}_{\Omega})\ (y')\}$ and $f(\tilde{v}_{\Omega})\ (x'-y') \preccurlyeq$ rmax $\{f(\tilde{v}_{\Omega})\ (x'), f(\tilde{v}_{\Omega})\ (y')\}$.

Hence, $f(\Omega)$ is an interval-valued bifuzzy ψ-subalgebra of . □

References

[1] A.T. Hameed and B.H. Hadi, **Anti-Fuzzy AT-Ideals on AT-algebras,** Journal Al-Qadisyah for Computer Science and Mathematics, vol.10, no.3(2018), 63-74.

[2] A.T. Hameed and B.H. Hadi, **Interval-valued bifuzzy AT-subalgebras and Fuzzy AT-Ideals on AT-algebra,** World Wide Journal of Multidisciplinary Research and Development, vol.4, no.4(2018), 34-44.

[3] A.T. Hameed and E.K. Kadhim, Interval-valued IFAT-ideals of AT-algebra, Journal of Physics: Conference Series (IOP Publishing), 2020, pp:1-5.

[4] A.T. Hameed and N.H. Malik, (2021), (β, α)-Fuzzy Magnified Translations of AT-algebra, Journal of Physics: Conference Series (IOP Publishing), 2021, pp:1-13.

[5] A.T. Hameed and N.H. Malik, (2021), Magnified translation of intuitionistic fuzzy AT-ideals on AT-algebra, Journal of Discrete Mathematical Sciences and Cryptography, (2021), pp:1-7.

[6] A.T. Hameed and N.J. Raheem, (2020), Hyper SA-algebra, International Journal of Engineering and Information Systems (IJEAIS), vol.4, Issue 8, pp.127-136.

[7] A.T. Hameed and N.J. Raheem, (2021), **Interval-valued Fuzzy SA-ideals with Degree (λ,κ) of SA-algebra**, Journal of Physics: Conference Series (IOP Publishing), 2021, pp:1-13.

[8] A.T. Hameed, N.J. Raheem and A.H. Abed, (2021), **Anti-fuzzy SA-ideals with Degree (λ,κ) of SA-algebra**, Journal of Physics: Conference Series (IOP Publishing), 2021, pp:1-16.

[9] A.T. Hameed, S.H. Ali and , R.A. Flayyih, **The Bipolar-valued of Fuzzy Ideals on AT-algebra**, Journal of Physics: Conference Series (IOP Publishing), 2021, pp:1-9.

[10] A.T. Hameed, H.A. Mohammed and A.H. Abed, Anti-fuzzy ideals of BZ-algebras, (2023).

[11] A.T. Hameed, S.M. Mostafa and A.H. Abed, Cubic KUS-ideals of KUS-algebras , Asian Journal of Mathematical Sciences,vol.8, no.2, pp:36 - 43, (2017).

[12] K. Is´eki and S. Yanaka, An Introduction to Theory of BCK-algebras, Math. Japonica, vol. 23 (1979), pp:1-20."

[13] Jabir N.H. and Hameed A.T. (2023), On The ψ-subalgebras of ψ-algebra, International Journal of Academic and Applied Reserch, ISSN: 2643-9603,Vol.7, Issue 4, Pages:8-11.

[14] Jabir N.H. and Hameed A.T. (2023), On the ψ-algebra, Journal of Interdisciplinary Mathematics, ISSN:0972-0502 (Pirnt), ISSN: 2169-012X(On line).

[15]