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Numerical Method RS(RSS) to evaluate Triple Integrals with for Continuous Integrands

Eman Jasim Fadhil Khudhair[1] and Safaa M. Aljassas[2]

1emanj.alabodi@student.uokufa.edu.iq, 2safaam.musa@uokufa.edu.iq

1,2Department of Mathematics, College of Education for Girls, University of Kufa, Najaf, Iraq

Abstract : The main objective of this research is to calculate the triple integrals with continuous integrators numerically using Romberg's method with Simpson's rule RS on the outer dimension Z and Romberg's acceleration of the component rule RSS on both the inner dimensions X and the middle Y. We called the method $RS(RSS)$,as we obtained high accuracy in the results with relatively few partial periods and in a short time.

Keywords: Triple Integrals, Continuous Functions ,Simpson's rule and Romberg Accelerating.

1-Introduction

Numerical integration refers to the process of approximating the value of a particular integral. Triple integrals serve as a significant tool in mathematics, enabling the calculation of integrals within three-dimensional space. These integrals are utilized across various fields, including physics and engineering, where they assist in determining volumes and areas in three dimensions.

Triple integrals are important in finding volumes , intermediate centres and moments of inertia of volumes , for example Calculate the triple integral of ($\mathfrak{f}(\mathfrak{p},\mathfrak{t},\mathfrak{s}) = \mathfrak{p}^2$) over the region $\mathcal{R}$ between the parabola $\mathfrak{p}^2 = 9 - \mathfrak{s}$ and the plane $\mathfrak{s} = 0$. Calculate the volume cut out from the cone $Q = \frac{1}{4}\pi$ by the sphere $\mathfrak{p} = 2\mathfrak{a}\cos Q$. calculate the center of the volume under $\mathcal{M}^2 = \mathcal{X}\mathcal{Y}$ and above the triangle $\mathcal{X} = \mathcal{Y}, \mathcal{Y} = 0, \mathcal{X} = 4$ in the plan $\mathcal{M} = 0$, Calculate the volume specified by coordinate planes and level $6\mathcal{X} + \mathcal{X}4\mathcal{Y} + 3\mathcal{Z} = 12$.[1]

There are many researchers who have worked in the field of triple integrals including:

And in 2010, Eghaar presented a numerical formula for calculating the values of triple integrals using the Mid- point rule on the three dimensions when number of subintervals of interval of interior integral equal to the number of subintervals o of interval of middle integral and equal to the number of subintervals of exterior integral and improved the results by Romberg's acceleration on the values resulting from the application of the formula derived and symbolized by$RMMM$ and obtained good results.[2]

Additionally, in 2018 Safaa et al. introduced two numerical methods $R(MSM)$ and $R(SMM)$ to evaluate the value of triple integral with continuous integrands, these methods obtained from Romberg acceleration with two rules from Newton–Cotes formulas (Midpoint and Simpson) and they got good results in terms of accuracy and the access of approximate values to the real values was fast in a relatively few sub intervals.[3]

Mohammed et al. [4] presented in 2013 numerical method to evaluate the value of triple integrals with continuous integrands by $RSSS$ method that obtained from Romberg acceleration with Simpson's rule on three dimensions X,Y and Z as the same approach of Eghaar [2].

In this research, we presented a new method for calculating triple integrals with continuous integrals numerically using the Romberg method with Simpson's rule (RS) on the outer dimension and the Romberg acceleration with Simpson's rule on both the inner and middle dimensions (RSS). We symbolized this method with the symbol ($RS(RSS)$). We obtained good results in terms of accuracy and speed in approaching the true value of the integrals with the least number of partial periods used and in a short time.

2-Numerical method for calculating double integrals with continuous integrals using Simpson's rule:

by applying the Romberg acceleration method to the values resulting from using the following rules: Rule Simpson's on both dimensions (internal x and exterior y) and symbolize this base with the SS symbol where S is the (Simpson's rule) [5]

$$ss = \frac{\varepsilon^2}{9}\Big[g(\sigma,\mathfrak{c}) + g(\sigma,\mathfrak{d}) + g(\tau,\mathfrak{c}) + g(\tau,\mathfrak{d}) + 4\sum_{i=1}^{n}\big(g(\mathfrak{x}_{2i-1},\mathfrak{c}) + g(\mathfrak{x}_{2i-1},\mathfrak{d})\big) + 2\sum_{i=1}^{n-1}\big(g(\mathfrak{x}_{2i},\mathfrak{c}) + g(\mathfrak{x}_{2i},\mathfrak{d})\big) + 4\sum_{j=1}^{n}\Big(g\big(\sigma,y_{2j-1}\big) + g\big(\tau,y_{2j-1}\big) + 4\sum_{i=1}^{n} g\big(\mathfrak{x}_{2i-1},y_{2j-1}\big) + 2\sum_{i=1}^{n-1} g\big(\mathfrak{x}_{2i},y_{2j-1}\big)\Big)\, 2\sum_{j=1}^{n-1}\Big(g\big(\sigma,y_{2j}\big) + g\big(\tau,y_{2j}\big) + 4\sum_{i=1}^{n} g\big(\mathfrak{x}_{2i-1},y_{2j}\big) + 2\sum_{i=1}^{n-1} g\big(\mathfrak{x}_{2i},y_{2j}\big)\Big)\Big] \quad (1)$$

Whereas $i = 1,2, \dots, n$, $x_{2i-1} = \sigma + (2i-1)\varepsilon$ $i = 1,2, \dots, n-1$, $x_{2i} = \sigma + 2i$ $j = 1,2, \dots, 2n$, $y_j = c + \frac{2j-1}{2}\varepsilon$

The formula for the correction limits for it if the integral function is continuous is

$$\mathfrak{U} - SS(h) = \mathcal{A}_{SS}\varepsilon^4 + \mathcal{B}_{SS}\varepsilon^6 + \mathcal{C}_{SS}\varepsilon^8 + \cdots \quad (2)$$

Simpson's rule and its correction limits, the general form of which is

$$S(\epsilon) = \frac{\varepsilon}{3}\left[g(\sigma) + g(\tau) + 2\sum_{s=1}^{\frac{l}{2}-1} g(\tau + 2s\varepsilon) + 4\sum_{s=1}^{\frac{l}{2}} g(\sigma + (2s-1)\varepsilon)\right] \quad (3)$$

$$\mu_S = \vartheta_S\varepsilon^4 + £_S\varepsilon^6 + ¥_S\varepsilon^8 + \cdots \quad (4)$$

$\vartheta, £, ¥$ are constants.

3- Romberg Accelerating

Suppose that we have the following integral

$$\mathcal{W} = \int_t^\ell f(x)\,dx \quad (5)$$

If we calculate the integral $\mathcal{W}$ for different values of $\mathcal{L}$ ($\mathcal{L}_1$ and $\mathcal{L}_2$) and let first value is $\mathcal{T}_1$ and another one is $\mathcal{T}_2$ with respect to $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, then $\mathcal{W}$ will be

$$\mathcal{W} = \int_t^\ell f(x)\,dx = \mathcal{T}_1 + \sum_{j=\delta}^{\infty} \mathcal{A}_j \mathcal{L}_1^j \quad (6)$$

$$\mathcal{W} = \int_t^\ell f(x)\,dx = \mathcal{T}_2 + \sum_{j=\delta}^{\infty} \mathcal{A}_j \mathcal{L}_2^j \quad (7)$$

$\mathcal{L}$ Where $\mathcal{T}_1$ and $\mathcal{T}_2$ are approximately values for the integral by Trapezoidal rule, $\mathcal{W}$ is exactly integral value and $\sum_{j=\delta}^{\infty} \mathcal{A}_j \mathcal{L}_1^j$ is corrections bounds. By solve the equations (6) and (7) we get

$$\mathcal{W} = \int_t^\ell f(x)\,dx = \frac{\mathcal{L}_1^\delta \mathcal{T}_2 - \mathcal{T}_1}{\mathcal{L}_1^\delta - \mathcal{L}_2^\delta} + \sum_{j=\delta+1}^{\infty} \mathcal{A}_j \frac{\mathcal{L}_1^\delta \mathcal{T}_2^j - \mathcal{L}_2^\delta \mathcal{T}_1^j}{\mathcal{L}_1^\delta - \mathcal{L}_1^\delta} \quad (8)$$

If suppose that $\mathcal{L}_2 = 2\mathcal{L}_1$ in equation (8), then we get

$$\mathcal{W} = \frac{2^\delta \mathcal{T}_2 - \mathcal{T}_1}{2^\delta - 1} + \sum_{j=\delta}^{\infty} \mathcal{A}_j \mathcal{L}^j \frac{2^\delta - 2^j}{2^\delta - 1} \quad (9)$$

Where $\mathcal{L} = \frac{\ell - t}{u}$, u is the number of partial intervals of $[t, \ell]$. Ralston [6]

And $\mathcal{A}_j \frac{2^\delta - 2^j}{2^\delta - 1}$ are constants do not depend on $\mathcal{L}$ and δ is the power of corrections terms, Fox [7] and Fox [8].

So $\mathcal{W} \cong \frac{2^\delta \mathcal{T}_2 - \mathcal{T}_1}{2^\delta - 1}$ (10)

The above form is Romberg accelerating to improve integral values numerically.

4- Numerical Method RS(RSS) to evaluate Triple Integrals with for Continuous Integrands

We now review how to calculate approximate values for the triple integral using the RS (RSS) method.

We assume that the integral Ψ is defined in the form:

$$\Psi = \int_e^f \int_c^d \int_\sigma^\tau f(x, y, z)\,dx\,dy\,dz \quad (11)$$

To find its approximate value, we write it as:

$$\int_e^f \mathcal{K}(z)\,dz \quad (12)$$

Where

$$\mathcal{K}(z) = \int_c^d \int_\sigma^\tau f(x, y, z)\,dx\,dy \quad (13)$$

The approximate value of the integral (12) over the outer dimension z using the midpoint rule is:

$$\Psi = \varepsilon \sum_{\ell=1}^{m} \mathcal{K}(z_\ell) + \mathcal{N}(\varepsilon) \quad (14)$$

If $z_\ell = e + \frac{2k-1}{2}\varepsilon$, $\ell = 1,2, \dots, m$, m It is the number of sub-periods into which the period $[e, g]$ is divided and $\varepsilon = \frac{f-e}{m}$ and that $\mathcal{N}(\varepsilon)$ It is the correction limit on the outer dimension z.

To calculate the values of $\mathcal{K}(z_\ell)$ approximately, we substitute the values of (z_ℓ) in (13) and we get:

$$\mathcal{K}(z_\ell) = \int_c^d \int_\sigma^\tau f(x, \mathcal{Y}, z_\ell)\,dx\,d\mathcal{Y} \quad (15)$$

When applying the (SS) rule to integration (15) Rule Simpson's on both dimensions (internal x and exterior y), we get the formula:

$$\mathcal{K}(z_\ell) = \frac{\bar{\varepsilon}}{9} \sum_j^{n_1} \sum_i^{n_2} f(x_i, y_j, z_\ell) + \mathcal{N}(\varepsilon) \quad (16)$$

Whereas : $i = 1,2, \dots, n$, $x_{2i-1} = a + (2i-1)h$ $i = 1,2, \dots, n-1$, $x_{2i} = a + 2ih$
$j = 1,2, \dots, 2n$, $y_j = c + \frac{2j-1}{2}h$

We can improve the values$\mathcal{K}(z_\ell)$of the Romberg acceleration method using the values calculated from eq. (16) and resulting from using the rule SS eq.(1).The resulting $\mathcal{K}(z_\ell)$values are only approximate values of the integral eq. (13) after applying the Romberg acceleration to them. After substituting $\mathcal{K}(z_\ell)$each$\ell = 1,2,\dots,m$ in eq. (14), we can apply the Romberg acceleration method to these values after knowing the formula $\mathcal{N}(\varepsilon)$, and thus we obtain the values of the integral eq.(11) by the Romberg acceleration method with the Simpson rule eq.(3), which leads to knowing the value of the integral eq.(12) approximately. The error formula$\overline{\mathcal{N}(\varepsilon)}$ is similar to eq.(11) and the error formula $\mathcal{N}(\varepsilon)$ is similar to eq. (17).

When dividing the periods$[\sigma,\tau]$, $[\mathfrak{c},\mathfrak{d}]$ and $[\mathfrak{e},\mathfrak{f}]$ into $n1n2$ $(n1 = n2)$ and m respectively ,of the partial periods when applying the aforementioned method, they used the following values: $m = 1,2,4,8,\dots, n1 = n2 = 1,2,4,8,\dots$

When we put $m = 1$, we calculate the value of the integral eq. (12) by the Romberg acceleration method on the values of the Simpson eq.(3), then we put $m = 2$ and calculate the value of the integral eq. (12) in the same way above, and so on until we get a value in which the absolute error is less than or equal to a certain value called $Eps1$ (on the outer dimension z). In order to find the value of the integral eq.(12), we are required to find $\mathcal{K}(z_\ell)$by the Romberg acceleration method with the SS rule eq.(1) on the integral eq. (15). For example, when we put $m = 1$ in eq. (14), we have to calculate $\mathcal{K}(z_1)$from eq. (16) when $n1 = n2 = 1$ then $n1 = n2 = 2$ and so on until we get a value in which the absolute error is Eps (on the inner and middle dimensions). Suppose that we obtained this value when $n1 = n2 = 8$, we will prove in the tables the approximate value of the integral as the resulting value when $m = 1$ and $n1 = n2 = 8$. However, if it is $m = 2$, we must calculate $\mathcal{K}(z_1)$ and $\mathcal{K}(z_2)$ for eq. (14) by applying eq. (16) when $n1 = n2 = 1$ then $n1 = n2 = 2$ and so on. Assuming that the absolute error in the value of $\mathcal{K}(z_1)$ is less than or equal to Eps when $n1 = n2 = 16$ and the absolute error in the value of $\mathcal{K}(z_2)$ is less than or equal to Eps when $n1 = n2 = 64$, in this case we will prove in our tables the approximate value of the integral defined by eq.(11) as the result obtained when $m = 2$ and $n1 = n2 = 64$. we prove the larger value of $n1 = n2$ and so on when $m > 2$.

5- Examples

Example 1/

The integrant of $\int_0^1\int_0^1\int_0^1 xe^{(-x-y-z)}\,dx\,dy\,dz$

that defined for $(x,y,z) \in [0,1]\times[0,1]\times[0,1]$ But the integration is unknown the analytical value. (approximates to Twelve decimal) applying the t method $RS(RSS)$ with$\overline{Eps} = 10^{-12}$for the outer dimension z and $\overline{\overline{Eps}} = 10^{-14}$for the internal x and medial y dimensions and applying the RS (RSS) method, we obtained the values of the code in Table (1), which indicates that we can obtain a correct value to thirteen decimal places when $m = 32$ and $n_1 = n_2 = 32$

Table (1)

m	RS(RSS)	n1=n2
2	0.017837996079	32
4	0.017831992897	32
8	0.017831983981	32
16	0.017831983977	32
32	0.017831983977	32

Example 2/

The integrant of $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin\frac{(x+y+z)}{2}\,dx\,dy\,dz$

that defined for $(x,y,z) \in \left[0,\frac{\pi}{2}\right]\times\left[0,\frac{\pi}{2}\right]\times\left[0,\frac{\pi}{2}\right]$ which analytic value is 3.313708498985 (approximates to Twelve decimal) applying the method $RS(RSS)$ with$\overline{Eps} = 10^{-12}$for the outer dimension z and $\overline{\overline{Eps}} = 10^{-14}$for the internal x and medial y dimensions

Appling $RS(RSS)$ with $m = 32, n1 = n2 = 32$

Table (2)

m	RS(RSS)	n1=n2
2	3.314154474358	32

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4	3.313708088838	32
8	3.313708499084	32
16	3.313708498985	32
32	3.313708498985	32

Example 3/

The integrant of $\int_1^2 \int_1^2 \int_1^2 \log(x+y+z)\, dx\, dy\, dz$

that defined for $(x, y, z) \in [1,2] \times [1,2] \times [1,2]$ which analytic value is 1.497802288575

(approximates to Twelve decimal) applying the method $RS(RSS)$ with $\overline{Eps} = 10^{-12}$ for the outer dimension z and $\overline{\overline{Eps}} = 10^{-14}$ for the internal x and medial y dimensions

Appling $RS(RSS)$ with $m = 32, n1 = n2 = 32$

Table(3)

m	RS(RSS)	n1=n2
2	1.497796686735	32
4	1.497802279459	32
8	1.497802288567	32
16	1.497802288575	32
32	1.497802288575	32

Example4/

The integrant of $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\, e^{(y+z)}\, dx\, dy\, dz$

that defined for $(x, y, z) \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \times \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \times \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ which analytic value is 2.0062370851190

(approximates to Twelve decimal) applying the method $RS(RSS)$ with $\overline{Eps} = 10^{-12}$ for the outer dimension z and $\overline{\overline{Eps}} = 10^{-14}$ for the internal x and medial y dimensions

Appling $RS(RSS)$ with $m = 64, n1 = n2 = 32$

Table (4)

m	RS(RSS)	n1=n2
2	2.006497359605	32
4	2.006237323584	32
8	2.006237085176	32
16	2.006237085119	32
32	2.006237085119	32
64	2.006237085119	32

Example 5/

The integrant of $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x \sin y + \cos z}{\sqrt{x^4+y^4+z^4}}\, dx\, dy\, dz$

that defined for $(x, y, z) \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \times \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \times \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ But the integration is unknown the analytical value. (approximates to Twelve decimal) applying the t method $RS(RSS)$ with $\overline{Eps} = 10^{-12}$ for the outer dimension z and $\overline{\overline{Eps}} = 10^{-14}$ for the internal x and medial y dimensions and applying the RS (RSS) method, we obtained the values of the code in Table (1), which indicates that we can obtain a correct value to thirteen decimal places when $m = 64$ and $n_1 = n_2 = 64$

Table (5)

m	RS(RSS)	n1=n2
2	0.380531501916	64
4	0.380494507508	64
8	0.380493149977	64
16	0.380493169903	64
32	0.380493169857	64
64	0.380493169857	64

6-Conclusions:

The results in this research show that the calculation of the approximate values for the triple integrals with continuous functions using method $RS(RSS)$ as it gives good results in terms of accuracy and the speed of approaching the approximate values to the real integral values and the time used.

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