International Journal of Academic and Applied Research (IJAAR)
ISSN: 2643-9603
Vol. 9 Issue 8 August - 2025, Pages: 1-7

ρ - Uniform Exponential Stability of Random Dynamical Systems by Lypanouv Function

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Abstract. *In this research, the necessary and sufficient conditions of ρ- uniform exponential stability are given for* ***Random Dynamical Systems*** *in Metric space. Furthermore, generalizations of some results of uniform stability are obtained.*
Keywords: Random Dynamical Systems(RDSs), ρ- uniform exponential stability, Lypanouv Function, Uniformly Exponentially Stable.

1 .Introduction

"In recent years, an important progress has been made in the study of the asymptotic behaviour of evolution equations in infinite-dimensional Banach spaces. Significant progress has been made in this direction pointing out that an impressive list of classical problems can be treated using the theory of linear LRDSs(see, for example, Sacker and Sell [17], Chow and Leiva [5]-[8], Chicone and Latushkin [4] and Latushkin, Montgomery - Smith and Randolph [13]). There have been obtained results concerning dichotomy of linear skew-product flows over locally compact Banach spaces (see Latushkin, Montgomery-Smith and Randolph [13]) and dichotomy of linear LRDSs over compact Hausdorff spaces, respectively (see Chow and Leiva [6], [7] and [8]). The asymptotic behaviour of the linear skew-product flow has been also characterized in terms of spectral properties of the evolution semigroup associated to the skew-product flow (see Latushkin, Montgomery-Smith and Randolph [13]).
Skew-product semiflows which is an extension of the classical concept of exponential stability for time-dependent linear differential equations in Banach spaces (see, for example, Datko [10] and Daleckii and Krein [11]). We give necessary and sufficient conditions for ρ **-**uniform exponential stability of linear LRDSs using a Banach function spaces technique. We not only answer questions concerning stability of linear LRDSs but also obtain generalizations of some well-known results due to Datko ([10]), Zabczyk ([18]), Neerven ([15]) and Rolewicz ([16]). The theory developed here is applicable for a large class of systems described in Chow and Leiva ([5]-[8]). M. Megan A. L. Sasu B. Sasu **[14]** give necessary and sufficient conditions for uniform exponential stability of evolution equations in Banach spaces. K. Horbacz , J. Myiak and T. Szarek, [12] They consider a stochastic process generated by random dynamical systems on Banach spaces and they show that Under the suitable assumptions this process is weakly convergent to some limit .
A. Barbata, M. Zasadzinski , R. Chatbouri and H. Souley Ali [3], they study the uniform asymptotic stability in probability when a nonlinear stochastic differential equation does not have a trivial solution.

2. Notations and Preliminaries

In this section we shall present some definitions, notations and results about RDSs and Matric function spaces.
We begin with the notion of RDS on the trivial Matric bundle $\mathcal{E} = \Omega \times X$, where X is a fixed Matric space - the state space - and $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space. The set $B(X)$ denoted to the all bounded (continuous) operators from X into itself.

Definition 2.1 [1,9]. A metric dynamical system (MDS) is the tuple $(\mathbb{R}, \Omega, \mathcal{A}, \mathbb{P}, \theta)$, if it has the following properties:
(f1) $\theta(0, \omega) = \omega$; for all $\omega \in \Omega$;
(f2)$\theta(s + t, \omega) = \theta(s, \theta(t, \omega))$; for all $(s, t, \omega) \in \mathbb{R}^2 \times \Omega$;
(f3) $\theta : \mathbb{R} \times \Omega \longrightarrow \Omega$ is measurable, and
(f4) $\mathbb{P}(\theta_t A) = \mathbb{P}(A)$, for all $t \in \mathbb{R}$ and $A \in \mathcal{A}$.

Definition 2.2 [1,9] A random variable $r: \Omega \longrightarrow \mathbb{R}^+$ is called **tempered random variable** (TRV), if

$$\sup_{t \in \mathbb{T}} \left\{ e^{-\lambda |t|} \, |r(\theta_t \omega)| \right\} < \infty , \text{ for every } \lambda > 0 \text{ and } \omega \in \Omega.$$

or equivalently if

$$\lim_{t \longrightarrow +\infty} \frac{1}{|t|} \log |r(\theta_t \omega)| = 0, \text{for } \omega \in \Omega .$$

In the following we will make simple modification on a definition of RDS given in [1,9].
Definition 2.3[1]: A pair (θ, Φ) is said to be a random dynamical system (RDS) on $\mathcal{E} = \Omega \times X$ if θ is a MDS on Ω and $\Phi : \mathbb{R} \times \Omega \longrightarrow B(X)$ admit the following properties:
(s1) $\Phi(0, \omega) = I$, the identity;
(s2) $\Phi(t + s, \omega) = \Phi(s, \theta_t \omega) \Phi(t, \omega)$; for all $\omega \in \Omega$, $s, t \in \mathbb{R}$;

(s3) $\mathbb{P}\left\{\omega: \lim_{t\longrightarrow 0^+} \Phi(0,\omega)x = x\right\} = 1$ uniformly in the following sense: for every $x \in X$ and every TRV ε there is $\delta = \delta(x;\varepsilon) > 0$ such that

$$\mathbb{P}\{\omega: \|\Phi(t,\omega)x - x\| < \varepsilon(\theta_t\omega)\} = 1; \text{ for all } 0 \le t \le \delta.$$

Example 2.4.[9]: Let θ MDS on the compact Hausdorff space Ω (which can be considered as a probability space) and let $S = \{S(t)\}_{t\geq 0}$ be a C^0 −semigroup on the Banach space X. For every strongly continuous mapping $D: \Omega \longrightarrow B(X)$ there is a linear RDS $(\theta, \Phi D)$ on $\mathcal{E} = \Omega \times X$ such that

$$\Phi D(t,\omega)x = S(t)x + \int_0^t S(t-s)D(\theta_s\omega)\Phi D(s,\omega)x\, ds$$

for all $(t,\omega,x) \in \mathbb{R} \times \Omega \times X$.
The RDS (θ,Φ) is called the RDS generated by the triplet $(S;\ D;\ \sigma)$.

Remark 2.5. If (θ,Φ) is a RDS on $\mathcal{E} = \Omega \times X$, then

$$\Phi(nt,\omega) = \Phi\left(t, \theta_{(n-1)t}\omega\right) \circ \ldots \circ \Phi(t,\theta_{2t}\omega) \circ \Phi(t,\theta_t\omega) \circ \Phi(t,\omega)$$

for all $(n,t,\omega) \in \mathbb{N} \times \mathbb{R} \times \Omega$.

3. Uniformly Exponentially Stable

In this section we will study the Uniformly Exponentially Stable in Random Dynamical Systems (RDS).
First we generalize an important result given in [6].

Proposition 3.1. If (θ,Φ) is a RDS on $\mathcal{E} = \Omega \times X$, then there exist constant $\alpha > 0$ and a TRV $M: \Omega \longrightarrow [1,\infty)$ and such that

$$d(\Phi(t,\omega)x, 0) \le M(\theta_t\omega)e^{\alpha t},\ (t,\omega) \in \mathbb{R} \times \Omega$$

Proof We claim that there is $k > 0$ such that

$$M(\omega) = \sup\{d(\Phi(t,\omega),0): \omega \in \Omega, 0 \le t \le k\} < \infty.$$

Suppose that there are sequences $\omega_n \in \Omega$, $t_n \in \mathbb{R}^+$ such that $t_n \longrightarrow 0^+$ and $d(\Phi(t_n,\omega_n),0) > n$, implies that $x \in X$ such that

$$\{d(\Phi(t_n,\omega_n)x, 0): n \in \mathbb{N}\}$$

is unbounded. This contradicts the fact that

$$\mathbb{P}\left\{\omega: \lim_{t\longrightarrow 0^+} \Phi(0,\omega)x = x\right\} = 1.$$

Therefore $M(\omega) < \infty$.
Since $\Phi(0,\omega) = I$, then $M(\omega) \ge 1$.
Now fix $t \in \mathbb{R}^+$. Let m be an integer satisfying $m \le \frac{t}{k} \le m+1$, i.e., $km \le t \le km + k$. For every $\omega \in \Omega$ we admit

$$\begin{aligned} d(\Phi(t,\omega),0) &= d(\Phi(t-km+km,\omega),0) \\ &= d(\Phi(t-km+km-k+k,\omega),0) \\ &= d(\Phi(t-km+k(m-1)+K,\omega).0) \\ &= d(\Phi(t-km+k(m-1),\theta_k\omega) \circ \Phi(k,\omega),0). \end{aligned}$$

Now putting

$$\omega_0 = \omega, \omega_1 = \theta_k\omega_0, \omega_2 = \theta_k\omega_1, \ldots, \omega_m = \theta_k\omega_{m-1}\ ;$$

we get the following

$$\begin{aligned} d(\Phi(t,\omega),0) &= d(\Phi(t-km,\omega_m)\Phi(k,\omega_{m-1}) \ldots \Phi(k,\omega_1)\Phi(k,\omega_0),0) \\ &\le M^{m+1}(\omega) \le M.M^{t/k}(\omega) \end{aligned}$$

If we put $\alpha := \left(\frac{1}{k}\right) \ln M\,(\omega)$, then

$$d(\Phi(t,\omega),0) \le M(\omega)e^{\alpha t}.$$

Definition 3.2. Let (θ,φ) be RDS on $\mathcal{E} = \Omega \times X$, where X is a metric space with metric d and $0 \in X$ is an equilibrium point. The system is said to be **:**
uniformly global exponentially stable (UGES) if for every $(t,\omega,x) \in \mathbb{R}^+ \times \mathcal{E}$ there is a TRV $x: \Omega \longrightarrow (0,\infty)$ and a constant $\lambda > 0$ such that

$$d(\varphi(t,\omega)x, 0) \le M(\theta_t\omega)e^{-\lambda t}d(x,0) \qquad 3.1$$

Proposition 3.3. The RDS (θ,Φ) is uniformly exponentially stable if there are $t_0 > 0$ and $c \in (0,1)$ such that

$$d(\Phi(t,\omega),0) \le c,\ \omega \in \Omega,\ d(\Phi(t,\omega),0) \in B$$

For every bounded subset $B \in X$.
Proof: Suppose that $M: \Omega \longrightarrow [1,\infty)$ be a TRV and $\alpha > 0$ as in Proposition 3.1. Let $\nu > 0$ such that $c = e^{-\nu t_0}$. Let $\omega \in \Omega$ be fixed. If $t \in \mathbb{R}^+$ there are $n \in \mathbb{N}$ and $\in [0,t_0)$, $t = nt_0 + r$. Then

$$d(\Phi(t,\omega),0) \le d\left(\Phi\left(r,\theta_{nt_0}\omega\right),0\right). d(\Phi(nt_0,\omega),0)$$

International Journal of Academic and Applied Research (IJAAR)
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$$\leq Me^{\alpha t_0} d(\Phi(t_0, \theta_{(n-1)t_0}\omega), 0) \ldots d(\Phi(t_0, \theta_{t_0}\omega), 0) d(\Phi(t_0, \omega), 0)$$
$$\leq M(\theta_t\omega) e^{\alpha t_0} e^{-n\nu t_0} \leq N(\theta_t\omega) e^{-\nu t},$$

Where $N(\theta_t\omega) = M(\theta_t\omega) e^{(\alpha+\nu)t_0}$.

So, (θ, Φ) is uniformly exponentially stable.

Theorem 3.4. The RDS (θ, Φ) is UES if and only if there are Matric sequence space $B \in \mathcal{B}(\mathbb{N})$ and $\{t_n\} \subset \mathbb{R}^+$ such that

i) $\sup_n |t_{n+1} - t_n| < \infty$;

ii) the sequence

$$\varphi_{\omega,x}: \mathbb{N} \longrightarrow \mathbb{R}^+; \ \varphi_{\omega,x}(n) := d(\Phi(t_n, \omega)x, 0)$$

be in B for every $(\omega; x) \in \mathcal{E}$;

iii) there is a function $K: X \longrightarrow (0,1)$ satisfying

$$\left|\varphi_{\omega,x}\right|_B \leq K(x), \ (\omega; x) \in \mathcal{E}. \qquad (3.2)$$

Proof: The " if" It is direct by choose $B = \ell^1$ and $t_n = n$.

The "only if" part: There are two possible cases.

Case 1. $T = \sup_n t_n < \infty$, so

$$d(\Phi(T, \omega)x, 0) \leq d(\Phi(T - t_n, \theta_{t_n}\omega)x, 0) \,.\, d(\Phi(t_n, \omega)x, 0)$$
$$\leq Me^{\alpha T} \|\Phi(t_n, \omega)x\| = \varphi_{\omega,\tilde{x}}(n), n \in \mathbb{N}, (\omega; x) \in \mathcal{E}$$

where $\tilde{x} = Me^{\alpha T} x$ and $M(\omega) \geq 1$; $\alpha > 0$ are in Proposition 3.1. So

$$d(\Phi(T, \omega)x, 0)_{\chi_{\{0,\ldots,n-1\}}} \leq \varphi_{\omega,\tilde{x}}, \ n \in \mathbb{N}^* = \mathbb{N} - \{0\}.$$

The inequality (3.2) implies that

$$F_B(n) d(\Phi(T, \omega)x, 0) \leq \left|\varphi_{\omega,\tilde{x}}\right|_B \leq K(\tilde{x}), \ n \in \mathbb{N}^*$$

Because $\in \mathcal{B}(\mathbb{N})$, then

$$\Phi(T, \omega)x = 0, (\omega; x) \in \mathcal{E}$$

and so (θ, Φ) is UES.

Case 2. Assume that $\{t_n\}$ is unbounded sequence . Since $B \in \mathcal{B}(\mathbb{N})$, it follows that there is $c > 0$ with

$$\left|\chi_{\{n\}}\right|_B \geq c, n \in \mathbb{N}.$$

From

$$\varphi_{\omega,x}(n)\chi_{\{n\}} \leq \varphi_{\omega,x}, n \in \mathbb{N}, (\omega, x) \in \mathcal{E}.$$

So

$$c \,.\, d(\Phi(t_n, \omega)x, 0) \leq \left|\varphi_{\omega,x}\right|_B \leq K(x), n \in \mathbb{N}, \ (\omega; x) \in \mathcal{E}.$$

By the uniform boundedness principle there is $N > 0$ satisfy

$$d(\Phi(t_n, \omega), 0) \leq N, n \in \mathbb{N}, \omega \in \Omega.$$

If$\omega \in \Omega$ and $\geq t_0$, then the unboundedness of $\{t_n\}$ and (i) implies that there is $n(s) \in \mathbb{N}$ such that

$$t_{n(s)} \leq s \leq t_{n(s)} + k$$

where $k = \sup_n |t_{n+1} - t_n|$. Then

$$d(\Phi(s, \omega), 0) \leq d(\Phi(s - t_{n(s)} + t_{n(s)}, \omega), 0)$$
$$= d(\Phi\left(s - t_{n(s)}, \theta_{t_{n(s)}}\omega\right) \Phi(t_{n(s)}, \omega), 0)$$
$$= d\left(\Phi\left(s - t_{n(s)}, \theta_{t_{n(s)}}\omega\right), 0\right) . d(\Phi(t_{n(s)}, \omega), 0)$$
$$\leq MNe^{k\alpha}, s \geq t_0, \omega \in \Omega.$$

It follows that

$$d(\Phi(s, \omega), 0) \leq L := \max \{Me^{kt_0}, MNe^{k\alpha}\}, s \in \mathbb{R}^+, \omega \in \Omega.$$

Define a sequence (k_n) by $k_0 = 0$; $k_n = \min\{j: t_j \geq t_{k_n}\}$. So $k_n \longrightarrow \infty$ and

$$t_j \leq t_{k_n}, \ j \in \{0, \ldots, k_n\}, n \in \mathbb{N}.$$

From

$$d(\Phi(t_{k_n}, \omega)x, 0) \leq d\left(\Phi\left(t_{k_n} - t_j, \theta_{t_j}\omega\right), 0\right) . d(\Phi(t_j, \omega)x, 0)$$
$$\leq L. d(\Phi(t_j, \omega)x, 0), \ j \in \{0, \ldots, k_n\}, n \in \mathbb{N}.$$

it results

$$d(\Phi(t_{k_n}, \omega)x, 0)_{\chi_{\{0,\ldots,k_n\}}} \leq L\varphi_{\omega,x}, n \in \mathbb{N}, \quad (\omega, x) \in \mathcal{E}$$

and hence

$$d(\Phi(t_{k_n}, \omega)x, 0) \, F_B(k_n + 1) \leq LK(x), n \in \mathbb{N}, \quad (\omega, x) \in \mathcal{E}$$

It follows that from there is $K \geq 1$ with

$$d\big(\Phi\big(t_{k_n},\omega\big),0\big)\,F_B(k_n+1)\le K,\, n\in\mathbb{N},\quad \omega\in\Omega.$$

(The boundedness). Since $B\in\mathcal{B}(\mathbb{N})$ then $\exists\ m\in\mathbb{N}$ with

$$d(\Phi\big(t_{k_m},\omega\big),0)\le \tfrac{1}{2},\ \omega\in\Omega.$$

By Proposition 3.2 we get the result.

4. Lyapunov Characterization $\rho-$ Uniform Exponential Stability

Here, we will generalize the concept of Uniform Exponential Stability for LRDS, which was introduced in Section 3, and characterize this concept using Lyapunov.

we consider the function $\rho\colon\Omega\times X\longrightarrow\mathbb{R}^+$ with the following properties:

(P1) $\rho(\cdot,x)\colon\Omega\longrightarrow\mathbb{R}^+$, is measurable for every $x\in X$;

(P2) $\rho(\omega,\cdot)\colon \mathrm{X}\longrightarrow\mathbb{R}^+$, is continuous for every $\omega\in\Omega$;

Definition 4.1 A RDS (θ,Φ) is called $\rho-$ uniformly global exponentially stable ($\rho-$UGES) with respect to $\rho\colon\Omega\times X\longrightarrow\mathbb{R}^+$ if there is a TRV $M\colon\Omega\longrightarrow(0,\infty)$ and a $\lambda>0$ such that

$$d(\Phi(t,\omega)x,0)\le\ \rho(\omega,d(x,0))e^{-\lambda t},\ \forall(t,\omega,x)\in\mathbb{R}^+\times\Omega\times \mathrm{X}.\quad (4.1)$$

In the following, we present a simple modification of the Lyapunov function given in [2].

Definition 4.2 A function $L:\mathbb{R}^+\times\Omega\times\mathbb{R}^n\longrightarrow\mathbb{R}$ is said to be Lyapunov-Like function for (θ,Φ) if

(1) $L(\cdot,\omega,\cdot)\colon\mathbb{R}^+\times\mathbb{R}^n\longrightarrow\mathbb{R}$ is a continuously differentiable in $(t,x)\in\mathbb{R}^+\times\mathbb{R}^n$ for every $\omega\in\Omega$,

(2) $L(t,\cdot,x)\colon\Omega\longrightarrow\mathbb{R}$ is measurable for every $(t,x)\in\mathbb{R}^+\times\mathbb{R}^n$ and

(3) there exist positive numbers $\lambda_1,\lambda_2,\lambda_3,p,q,r,\delta$ and a TRV k, such that for every $(t,\omega,x)\in\mathbb{R}^+\times\Omega\times\mathbb{R}^n$:

$$\lambda_1 d(\Phi(t,\omega)x.0)^{1/p}\le L\,(t,\omega,\Phi(t,\omega)x)\le\lambda_2 d(\Phi(t,\omega)x.0)^{1/q},\qquad \text{(i)}$$

$$\frac{\partial}{\partial t}L(t,\theta_t\omega,\Phi(t,\omega)x)\le-\lambda_3\,d(\Phi(t,\omega)x.0)^{\frac{1}{r}}-k(\theta_t\omega)e^{-\delta t},\qquad \text{(ii)}$$

Theorem 4.3: The RDS (θ,Φ) is $\rho-$UGES if it admits a Lyapunov Like function and satisfy the following for every $(t,\omega,x)\in\mathbb{R}^+\times\Omega\times\mathbb{R}^n$:

(a) there exists $\gamma>0$ with

$$L\,(t,\theta_t\omega,\Phi(t,\omega)x)-[L\,(t,\theta_t\omega,\Phi(t,\omega)x)]^{\frac{q}{r}}\le\gamma e^{-\delta t},\ \text{where } \delta>\lambda_3/\lambda_2^{q/r}$$

(b) $m\gamma\ge k(\omega)$, for every $\omega\in\Omega$ where $m=\lambda_3/\lambda_2^{q/r}$

Proof Consider $Q(t,\theta_t\omega,\Phi(t,\omega)x)\coloneqq L\,(t,\theta_t\omega,\Phi(t,\omega)x)e^{mt}$, where $L\,(t,\omega,x)$ is a Lyapunov-Linke function.

$$\frac{\partial}{\partial t}Q(t,\theta_t\omega,\Phi(t,\omega)x)\coloneqq e^{mt}\frac{\partial}{\partial t}L\,(t,\theta_t\omega,\Phi(t,\omega)x)+mL\,(t,\theta_t\omega,\Phi(t,\omega)x)e^{mt}$$

From (ii) for all $(t,\omega,x)\in\mathbb{R}^+\times\Omega\times\mathbb{R}^n$ we have

$$\frac{\partial}{\partial t}Q(t,\theta_t\omega,\Phi(t,\omega)x)\le\left[-\lambda_3 d(\Phi(t,\omega)x.0)^{\frac{1}{r}}-k(\omega)e^{-\delta t}\right]e^{mt}$$
$$+mL\,(t,\theta_t\omega,\Phi(t,\omega)x)e^{mt}.$$

From the right hand of (i) we have

$$-d(\Phi(t,\omega)x.0)^{\frac{1}{r}}\le-\left[\frac{L\,(t,\omega\theta_t,\Phi(t,\omega)x)}{\lambda_2}\right]^{\frac{q}{r}}.\,d(\Phi(t,\omega)x.0)^{1/q}\ge\frac{L\,(t,\theta_t\omega,\Phi(t,\omega)x)}{\lambda_2}$$

which implies that

$$\frac{\partial}{\partial t}Q(t,\theta_t\omega,\Phi(t,\omega)x)\le\left[-\frac{\lambda_3}{\lambda_2^{r/q}}[L\,(t,\theta_t\omega,\Phi(t,\omega)x)]^{\frac{q}{r}}-k(\omega)e^{-\delta t}\right]e^{m(t-t_0)}$$
$$+mL\,(t,\theta_t\omega,\Phi(t,\omega)x)e^{m(t-t_0)}\ .$$

From the condition (a) we obtain

$$\frac{\partial}{\partial t}Q(t,\theta_t\omega,\Phi(t,\omega)x)\le m\gamma e^{-\delta t}e^{mt}-k(\omega)e^{-\delta t}e^{mt}$$
$$=(m\gamma-k(\omega))e^{(m-\delta)t}.$$

Hence

$$Q(t,\theta_t\omega,\Phi(t,\omega)x)-Q(0,\omega,x)\le\int_0^t(m\gamma-k(\theta_t\omega))e^{(m-\delta)s}\,ds$$

$$Q(t,\theta_t\omega,\Phi(t,\omega)x)-Q(0,\omega,x)\le\int_0^t(m\gamma-k(\theta_t\omega))e^{(m-\delta)s}\,ds$$

$$=\int_0^t m\gamma e^{(m-\delta)s}\,ds-\int_0^t k(\theta_t\omega)e^{(m-\delta)s}\,ds$$
$$=\frac{m\gamma}{(m-\delta)}e^{(m-\delta)t}-\frac{m\gamma}{(m-\delta)}-\int_0^t k(\theta_t\omega)e^{(m-\delta)s}\,ds$$

Then

$$Q(t,\omega,\Phi(t,\omega)x) \le Q(0,\omega,x) + \frac{m\gamma}{(m-\delta)}e^{(m-\delta)t} - \frac{m\gamma}{(m-\delta)} - \int_0^t k(\theta_t\omega)e^{(m-\delta)s}\,ds$$

$$= Q(0,\omega,x) - \frac{m\gamma}{(\delta-m)}e^{(m-\delta)t} + \frac{m\gamma}{(\delta-m)} - \int_0^t k(\theta_t\omega)e^{(m-\delta)s}\,ds$$

$$\le Q(0,\omega,x) + \frac{m\gamma}{(\delta-m)} - \int_0^t k(\theta_t\omega)e^{(m-\delta)s}\,ds$$

Since $\frac{m\gamma}{(\delta-m)}e^{(m-\delta)t} \ge 0$ and $(0,\omega,x) = L\,(0,\omega,x)$, it follows that

$$Q(t,\theta_t\omega,\Phi(t,\omega)x) \le L\,(0,\omega,x) + \frac{m\gamma}{(\delta-m)} - \int_0^t k(\theta_s\omega)e^{(m-\delta)s}\,ds\ ,$$

From the right hand of (i) we obtain

$$\lambda_1 d(\Phi(t,\omega)x.0)^{1/p} \le L\,(t,\theta_t\omega,\Phi(t,\omega)x) \le \lambda_2 d(\Phi(t,\omega)x.0)^{1/q}$$

Hence $\quad Q(t,\theta_t\omega,\Phi(t,\omega)x) \le \lambda_2 d(x,0)^{\frac{1}{q}} + + \frac{m\gamma}{(\delta-m)} - \int_0^t k(\theta_s\omega)e^{(m-\delta)s}\,ds.$

From the left han of (i)

$d(\Phi(t,\omega)x,0) \le [L\,(t,\omega,\Phi(t,\omega)x)/\lambda_1]^p$, then

$$d(\Phi(t,\omega)x,0) \le \left[\frac{Q(t,\theta_t\omega\omega,\Phi(t,\omega)x)}{\lambda_1 e^{mt}}\right]^p = \left[\frac{Q(t,\theta_t\omega\omega,\Phi(t,\omega)x)}{\lambda_1}\right]^p e^{-mpt}$$

$$\le \left[\frac{\lambda_2 d(x,0)^{\frac{1}{q}} + \frac{m\gamma}{(\delta-m)} - \int_0^t k(\theta_s\omega)e^{(m-\delta)s}ds}{\lambda_1}\right]^p e^{-mpt}$$

Define

$$\rho(\omega,x) = \left[\frac{\lambda_2 d(x,0)^{\frac{1}{q}} + \frac{m\gamma}{(\delta-m)} - \int_0^t k(\theta_s\omega)e^{(m-\delta)s}\,ds}{\lambda_1}\right]^p$$

Then

$$d(\Phi(t,\omega)x,0) \le \ \rho(\omega,x)e^{-mpt}$$

Hence (θ,Φ) is $\rho-$UGES**.**

Proposition 4.4 The $\rho-$UGES implies UES .

Proof. Let $M\colon\Omega \longrightarrow \mathbb{R}^+$ be TRV and $d\colon X \longrightarrow \mathbb{R}^+$ be a matric function on X.

The Cartesian product is $M \times d : \Omega \times X \longrightarrow \mathbb{R}^+ \times \mathbb{R}^+$, then the function

$\rho \coloneqq \mu \circ (M \times d)\colon \Omega \times X \longrightarrow \mathbb{R}^+$, (where $\mu\colon \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a usual multiplication) satisfy (P1) and (P2).

Theorem 4.5. The RDS (θ,Φ) with above Lyapunov-Like function is $\rho-$UGES if there exists TRV γ such that

$$L\,(t,\theta_t\omega,\Phi(t,\omega)x) - [L\,(t,\theta_t\omega,\Phi(t,\omega)x)]^{\frac{r}{q}} \le \gamma(\theta_t\omega)e^{-\delta t},$$

where $\delta > m = \lambda_3/\lambda_2^{r/q}$.

Proof. Let $Q(t,\theta_t\omega,\Phi(t,\omega)x) = L\,(t,\theta_t\omega,\Phi(t,\omega)x)e^{mt}$, where $L\,(t,\theta_t\omega,\Phi(t,\omega)x)$ is the above Lyapunov-Like function

$$\frac{\partial}{\partial t}Q(t,\theta_t\omega,\Phi(t,\omega)x) \coloneqq e^{mt}\frac{\partial}{\partial t}L\,(t,\theta_t\omega,\Phi(t,\omega)x) + mL\,(t,\theta_t\omega,\Phi(t,\omega)x)e^{mt}$$

$$\le \left[-\lambda_3\|x\|^r - ke^{\delta t}e^{-mt}\right]e^{mt} + mL\,(t,\theta_t\omega,\Phi(t,\omega)x)e^{mt}$$

$$\le -\lambda_3\|x\|^r e^{mt} - ke^{\delta t} + mL\,(t,\theta_t\omega,\Phi(t,\omega)x)e^{mt}.$$

Since

$$d(x,0)^r \le -\left[\frac{L\,(t,\theta_t\omega,\Phi(t,\omega)x)}{\lambda_2}\right]^{\frac{r}{q}} d(x,0) \ge \left[\frac{L\,(t,\theta_t\omega,\Phi(t,\omega)x)}{\lambda_2}\right]^{\frac{1}{q}}$$

Then

$$\frac{\partial}{\partial t}Q(t,\theta_t\omega,\Phi(t,\omega)x) \le -[L\,(t,\theta_t\omega,\Phi(t,\omega)x)]^{\frac{r}{q}}\left[\lambda_3/\lambda_2^{r/q}\right]e^{mt} - ke^{\delta t}$$

$$+mL\,(t,\theta_t\omega,\Phi(t,\omega)x)e^{mt}$$

$$\le m\left\{L\,(t,\theta_t\omega,\Phi(t,\omega)x) - [L\,(t,\theta_t\omega,\Phi(t,\omega)x)]^{\frac{r}{q}}\right\}e^{mt} - ke^{\delta t}$$

From the condition of theorem we obtain:

$$\frac{\partial}{\partial t}Q(t,\theta_t\omega,\Phi(t,\omega)x) \le m\gamma e^{(m-\delta)t} - ke^{\delta t}$$

$$Q(t,\theta_t\omega,\Phi(t,\omega)x) - Q(0,\omega,x) \le \int_0^t m\gamma e^{(m-\delta)s}\,ds - k\int_0^t e^{\delta s}\,ds$$

$$\le \frac{m\gamma}{(m-\delta)}\left[e^{(m-\delta)t} - 1\right] - \frac{k}{\delta}\left[e^{\delta t} - 1\right]$$

$$\leq \frac{m\gamma}{(\delta-m)} - \frac{m\gamma}{(\delta-m)} e^{(m-\delta)t} + \frac{k}{\delta} - \frac{k}{\delta} e^{\delta t}$$

$$\leq \frac{m\gamma}{(\delta-m)} + \frac{k}{\delta}$$

$$Q(t, \theta_t\omega, \Phi(t,\omega)x) \leq Q(0,\omega,x) + \frac{m\gamma}{(\delta-m)} + \frac{k}{\delta}$$

$$\leq L\,(0,\omega,x) + \frac{m\gamma}{(\delta-m)} + \frac{k}{\delta}$$

If $k > 0$ then

$$Q(t, \theta_t\omega, \Phi(t,\omega)x) \leq \lambda_2 \|x\|^q + \frac{m\gamma}{(\delta-m)} + \frac{k}{\delta}.$$

Define $\rho: \Omega \times X \longrightarrow \mathbb{R}^+$ by $\rho(\omega, x) := \lambda_2 d(x,0)^q + \frac{m\gamma}{(\delta-m)} + \frac{k}{\delta}.$

Since $\lambda_1 d(\Phi(t,\omega)x, 0)^p \leq L\,(t, \theta_t\omega, \Phi(t,\omega)x)$, then

$$d(\Phi(t,\omega)x, 0) \leq [L\,(t, \theta_t\omega, \Phi(t,\omega)x)/\lambda_1]^{1/p}$$

$$\leq [Q(t, \theta_t\omega, \Phi(t,\omega)x)/\lambda_1 e^{mt}]^{1/p}$$

$$\leq [\ \rho(\omega,x)/\lambda_1]^{1/p} e^{-\frac{m}{p}t}$$

Hence the system (θ, Φ) is ρ −UGES.

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