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Studying Transitive of Types in topological transformation group with properties dynamics

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Abstract: This thesis sheds light on the transitivity property of points in topological spaces and topological transformation groups. The concept of transitivity has been further studied in the admissible set (syndetical set, semi-replete set- replete set, extended set). The thesis also focused on studying the types of transitivity in topological transformation groups. It also presented types of point transitivity and then established the relationship of these types to the dynamic properties (fixed point - periodic point - almost periodic point - almost periodic point invariant point - minimal point) as it is a basic axis in the work. This undoubtedly required providing detailed proofs for abbreviated proofs and proofs of problems not established in the reference, in addition to the introduction of new definitions and theorems.

Keywords— repellently regionally transitive ; transitive set ; syndetical set ; invariant set ; extensive set

1. INTRODUCTION

Dynamical systems are the study of the long-term behavior of evolving systems. The modern theory of dynamical systems originated at the end of the 19th century with fundamental questions concerning the stability and evolution of the solar system. Attempts to answer those questions led to the development of a rich and powerful field with applications to physics, biology, meteorology, astronomy, economics, and other areas. By analogy with celestial mechanics, the evolution of a particular state of a dynamical system is referred to as an orbit number of themes appear repeatedly in the study of dynamical systems: properties of individual orbits; periodic orbits; typical behavior of orbits; statistical properties of orbits; randomness vs. determinism; entropy; chaotic behavior; and stability under perturbation of individual orbits and patterns. [1-3] Interest in studying topological dynamics increased in the early fifties of the last century as a result of the development of human needs in quantity and quality, which necessitated the search for more flexible tools in simulating problem models [4] Classical Dynamics used Systems of differential equations provide a tool for modeling motion, and the kinetic behavior of the solutions of these systems is the behavior of the elements concerned in the model or example. However, these concepts have their limits that cannot be exceeded. A system of equations has conditions and is useless if it does not have a single, continuous solution [5]. The expert may choose a system for his model and may or may not find a solution for his system. The solution may or may not be continuous and

may or may not be unique, which prompted specialists to look from another window that makes the modeling more symbolic and more general, going beyond real spaces to topological spaces in general [6]. This was called Topological Transformation Groups. Topological groups have the algebraic structure of a group and the topological structure of a topological space and they are linked by the requirement that multiplication and inversion are continuous functions [7-8]. In a topological transformation group, a transitive set represents a subset whose influence, usually by multiplication or left-right transposition, can reach any other point in space from within that subset. This concept is closely related to topological transitivity, which describes how points move under the influence of a group.[9-10] In this work we introduced different forms of transitive such as transitive point and extensively transitive point , topological transformation group regionally transitive - topological transformation group universally transitive - topological transformation group repellently regionally transitive.

(1-1) Topological Transformation Group

In this section, we introduce the definition of the topological group and topological transformation group. Also highlight the periodic point Introduce the definition semi replete set in the topological transformation group and relationships between the admissible set (syndetical set - extensive set), providing many useful examples and notes for subsequent section.

Definition: (1-1-1) [11]
A topological group is a set G with tow structures :

1- G is a group

2- G is a topological space

such that the two structures are compatible i.e. the multiplication map $f: G \times G \to G$ and the inversion map $\upsilon: G \to G$ are both continuous.

REMARK (1-1-2):

Let (G ,*) be a topological transform group, K compact subset of G and $g \in G$ then :

1. $g^{-1}K$ compact subset of G .
2. Kg^{-1} compact subset of G.
3. gK compact subset of G

Definition (1-1-3) [12]

A right topological transformation group is defined to be an order triple ($\mathbb{Z}$, T, ϖ) consisting of a topological space $\mathbb{Z}$, a topological group T and mapping $\varpi: Z \times T \to \mathbb{Z}$ such that:

1) $\overline{\omega}(x.e) = x \ (x \in X)$ where is the identity element of T .
2) $\overline{\omega}(\overline{\omega}(x.t).g) = x\ (tg)\ (x \in \mathbb{Z}.t.g\ \in T\ )$
3) ϖ is continuous

Definition (1-1-4) [11]

Let ($\mathbb{Z}$, T, ϖ) be topological transformation group and A subset of $\mathbb{Z}$ is said to be invariant set if AT=A

Remark (1-1-5)

Let (ℤ, T, ϖ) be topological transformation group then:
1) ℤ and ∅ are invariant
2) If A is invariant subsets of ℤ and $A \subset B$ then B is invariant

Definition (1-1-6) [11]

Let ($\mathbb{Z}$, T, ϖ) be topological transformation group:

1) A subset of $\mathbb{Z}$ is said to be invariant set if AT=A
2) A subset of T is said to be extensive set if there exist semi replete set P such that $P \cap A \neq \emptyset$

Then A is said to be replete semi group if for each compact set K there exist $g \epsilon G$ such that $gK \subset$ A or Kg ⊂ A. group:

3) A subset of $\mathbb{Z}$ is said to be invariant set if AT=A
4) A subset of T is said to be extensive set if there exist semi replete set P such that $P \cap A \neq \emptyset$
Then A is said to be replete semi group if for each compact set K there exist $g \epsilon G$ such that $gK \subset$ A or Kg ⊂ A.
5) A subset of T is said to be a right syndetical and a left syndetical al if there is a exist compact subset K of G (K ⊂ T) such that ({T =AK}, T =KA)
6) $x \in Z$ is said to be periodic point under Z if there exist period P such that $xP = x$.

Theorem (1-1-7)

Let ($\mathbb{Z}$, T, ϖ) be a topological group and A extensive set of T then $g^{-1}A$ extensive subset of T

Corollary (1-1-8)

Let (Z, T, ϖ) be a topological transformation group and x periodic point under T then A semi replete set

Theorem (1-1-9)

Let ($\mathbb{Z}$, T, ϖ) be a topological transformation group. and T be an abelian Then the following statement are equivalent.

1) A extensive set of T

2) A syndetical set of T

Proof

Assume (1) we prove (2) Let A be a extensive subset of T there exist semi replete p of T such that$A \cap P \neq \emptyset$ Since intersects be abelian then P⊂ A form theorem (1-2-16) there exist a compact subset K such that P K =Tso T⊂ AK it follows that A syndetical set of G . ■

Assume (2) we prove (1) Let A be a syndetical subset of T there exist a compact subset K of T such that AK=T Since T abelian Then T = AK form hypothesis we obtain K^{-1} compact set ,for each $g \in T$there exist $a \in A\ .k \in K$ such that g = ak since T group there exist $K^{-1} \subset T$ Hence$gK^{-1} \subset A$ A semi replete p of T and $A \cap A \neq \emptyset$ therefore A extensive set of T

(2-1) Transitive Point

In this section, we will study the concept of transitivity in (point - topological transformation group). The section also presents different types of transitivity and finds the relationship between them through theorems and results.

Definition (2-1-1)

Let (Z, T, ϖ) be topological transformation group and $x \in Z$,the point x is said to be transitive under T provided that if U is a non-empty open set there exist $t \in T$ such that $xt \in U$

Definition (2-1-2)

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Let (Z, T, ϖ) be topological transformation group is said to be regionally transitive provided that if U and V non empty open subsets of X there exist $t \in T$ such that $Ut \cap V \neq \emptyset$

Definition (2-2-2)

Let (Z, T, ϖ) be topological transformation group ,the point x is said to be extensively transitive under T provided that if U non empty open set there exist extensive subset A of T such that $xA \subset U$

Definition (2-2-4)

Let (Z, T, ϖ)be topological transformation group is said to be universally transitive provided that if $x, y \in X$ then there exist $t \in T$ such that $xt = y$

Definition (2-2-5)

Let (Z, T, ϖ) be topological transformation group is said to be repellently regionally transitive if U and V non empty open subsets of X then there exist replete subset A of T such that $t \in A \; and \; Ut \cap V \neq \emptyset$

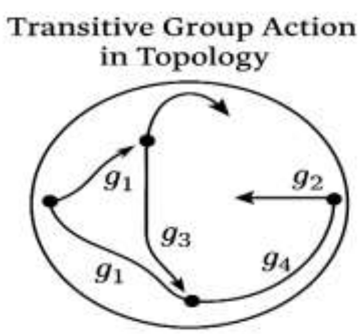

Figure (2-2): Transitive transformation group

Note (2-2-6)

1- In some transitive system the orbit forms periodic points f^n(x)=X
2- The existence of a dense orbit for at least one point
3-The point x has transitive orbit if $\overline{xT} = X$

Example (2-2-7)

Let $(Z.f)$ Discrete flow and $f: Z \to Z$ Continuous function define $f(x) = x^2 - 1$ so $0.-1 \in Z$ Such that $f(0) = -1. f(-1) = 0$ Since Z topological space therefore $f(0) \in Z \; and \; f(-1) \in Z$ Thus {0,-1} are transitive points

Remark (2-2-8)

Let (Z, T, ϖ) be a topological transformation group then:

1- If x fixed point then x transitive point under T

2-If (Z, T, ϖ) universally transitive then $xT = yT$

3-If (Z, T, ϖ) universally transitive then $\overline{xT} = \overline{yT}$

Proposition (2-2-9)

Let (Z, T, ϖ) be a topological transformation group then:

1) If x transitive point under T then U invariant set

2) If (Z, T, ϖ) repellently regionally transitive then V invariant set

3) xg extensively transitive under T then U invariant set

Proof:

Assume that xT is least invariant subset of Z which contains the point x since x transitive point under T and U non empty open set then there $t \in T$ such that $xt \in U$ so $xT \subset U$ since xT invariant set by remark (1-1- 5) U invariant set .

Theorem (2-2-10)

Let (Z, T, ϖ) be topological transformation group is repellently regionally transitive if and only if (Z, T, ϖ) is regionally transitive

Proof:

Assume that (Z, T, ϖ) is repellently regionally transitive and U and V non empty open subsets of X then there exist replete subset A of T such that $t \in A \; and \; Ut \cap V \neq \emptyset$, $UA \cap V \neq \emptyset$ Since A replete subset of T for each $g.g_1 \in T$ there exist compact subset K of T such that $gKg_1 \subset A$ $.UgKg_1 \subset UA \subset V$ $thus \; UgKg_1 \subset V$ since T topological group then there exist $g_1^{-1} \in T$ such that $UgK \subset Vg_1^{-1}$ since (Z, T, ϖ) repellently regionally transitive by proposition (2-2-9) $UTK \subset V$ since T syndetical set then $Ut \cap V \neq \emptyset$ for $t \in T$. Therefore (Z, T, ϖ) is regionally transitive

Conversely, direct proof

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Theorem (2-2-11)

Let (Z, T, ϖ) be topological transformation group then

1) x is extensively transitive under T

2) xg is extensively transitive under T

Proof:

($1 \to 2$) Assume that x extensively transitive under T then for each U non empty open set there exist extensive subset A of T such that $xA \subset U$ since T group then for each $g \in T$ there exist $g^{-1} \in T$ such that $xeA = xgg^{-1}A \subset U$, since A extensive subset of T by theorem(1-1-7) then $g^{-1}A$ extensive subset of T therefore xg extensively transitive under T.

($2 \to 1$) Assume that xg extensively transitive under T then for each U non empty open set, there exist extensive subset A of T such that $xg\, A \subset U$ for each $b \in U$ there exist $a \in$ A such that $xg\, a =$ b since T abelian group then there exist $g^{-1} \in T$ such that $xa = \mathrm{b}g^{-1}$ so $xA \subset \mathrm{U}g^{-1}$ since xg extensively transitive by proposition (3) in (2-2-9) U invariant set therefore $xA \subset$ U and x extensively transitive under T.

Theorem (2-2-12)

Let (Z, T, ϖ) be topological transformation group , x periodic point then x is extensively transitive under T

Proof:

Assume that $x \in U$ Since x periodic point under T and P period of point x then $xP = x$, by Corollary (1-1-8) P syndetical set since (Z,T) topological group then U open set hence $xP \subset U$ enough to prove P extensive , Since e a P syndetical subset of T there exist a compact subset K of T such that P K =T Since T abelian Then T = PK form hypothesis we obtain K^{-1} compact set ,for each $g \in T$there exist $k \in K^{-1}$, $a \in P$such that gk^{-1}= a and $gK^{-1} \subset$P Hence P semi replete of T and $P \cap P \neq \emptyset$ therefore P extensive set x is extensively transitive .

Corollary (2-2-13)

Let (Z, T, ϖ) be topological transformation group , x periodic point then xg is extensively transitive under T

Theorem (2-2-14)

Let (Z, T, ϖ) be topological transformation group then

1) x almost periodic point

2) x is extensively transitive under T

($1 \to 2$) Assume that x almost point then for each an neighborhood Ψ of x there exist syndetical al subset A of T such that $xA \subset \Psi$ since (Z,T) topological group then Ψ open set, Since e a A syndetical subset of T there exist a compact subset K of T such that P K =T Since T abelian Then T = AK form hypothesis we obtain K^{-1} compact set ,for each $g \in T$there exist $k \in K^{-1}$, $a \in A$ such that gk^{-1}= a and $gK^{-1} \subset$A Hence A semi replete of T and $P \cap P \neq \emptyset$ therefore P extensive set x is extensively transitive

($2 \to 1$) Assume that x extensively transitive under T then for each U non empty open set there exist extensive subset A of T such that $xA \subset U$ since A extensive subset of T by theorem(1-1-9) A syndetical subset of T by hypothesis U neighborhood of x.

Corollary (2-2-15)

Let (Z, T, ϖ) be topological transformation group then

1) x Invariant almost periodic point

2) x is extensively transitive under T

Theorem (2-2-16)

Let (Z, T, ϖ) be topological transformation group , x periodic point then repellently regionally transitive

Proof

Assume that U neighborhood of x Since x periodic point under T and P period of point x then $xP = x$, by Corollaries (1-1-8) P syndetical set since (Z,T) topological group then U open set hence $xP \subset U$ enough to prove P replete set , Since e a P syndetical subset of T by theorem (2-1-7) P replete set therefore for each $g.g_1 \in T$ there exist compact subset K of T such that $gKg_1 \subset P$ $.xgKg_1 \subset xP \subset U$ $thus\ UgKg_1 \subset V$ since T topological group then there exist $g_1^{-1} \in T$ such that $xgK \subset Ug_1^{-1} \subset UT$ by hypothesis U is invariant set and T syndetical therefore $xT \subset U$ so $UT \cap xT \subset UT \cap U$ since xT is least invariant subset of X which contains the point x then $UT \cap xT \neq \emptyset$ thus $UT \cap U \neq$

$\emptyset$ and $Ut \subset U$ for each $t \in T$, therefor (Z, T, ϖ) be repellently regionally transitive.

Theorem (2-1-17)

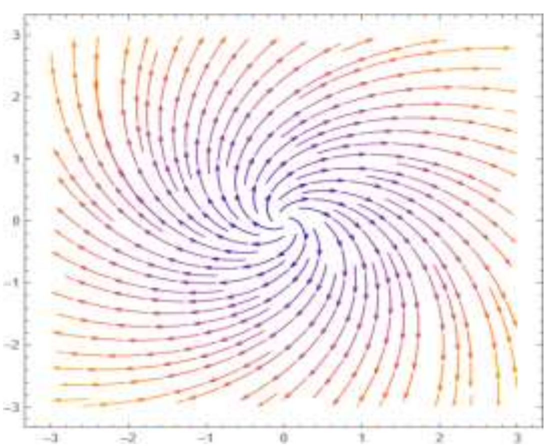
Let (Z, T, ϖ) be a transformation group repellently regionally transitive then x minimal point

Proof:

Assume that repellently regionally transitive if U and V non empty open subsets of X then there exist replete subset A of T such that $t \in A$ and $U\,t \cap V \neq \emptyset$, since A replete set then for each $g.g_1 \in T$ there exist compact K subset of T such that $gKg_1 \subset A$ and $gKg_1 \in gKT$ since K compact set by remark (1-1-7) gK compact set so T syndetical al this lead to $x\ gKT = xT$ so $xT \cap xA \neq \varphi$ by hypothesis $xT = xA$ for each $t \in T$ $threr\ exist$ $a \in A$ such that $xt = xa$, $xta^{-1} = x$ since T group then $TA^{-1} \in T$ hence $xT \subset \{x\}$ by hypothesis $xT = \{x\}$, xT invariant closed subset of X which contains the point x hence xT minimal set and x minimal point.

Proposition (2-1-18)

Let (Z, T, ϖ) be universally transitive, Then $\overline{\mathrm{xT}}$ minimal set



Proof

Assume that xT invariant subset of X which contains the point x since (Z, T, ϖ) be universally transitive then for each x ,$y \in X$ then there exist $t \in T$ such that $xt = y$ and $xT \subset yT$ By remark (2-2-9) $xT = yT$,$\overline{xT} \cap yT = \overline{\mathrm{xT}} \cap xT$ Since $\overline{\mathrm{xT}}$ invariant close subset of X which contains x the $\overline{xT} \cap yT \neq \emptyset$ so $\overline{\mathrm{xT}} \cap xT$ by hypothesis $xT \subset \overline{\mathrm{xT}}$ and $xT = \overline{\mathrm{xT}}$ then $\overline{\mathrm{xT}}$ minimal set.

Proposition (2-1-19)

Let (Z, T, ϖ) be a transformation group then

1- (Z, T, ϖ) universally transitive
2- x minimal point

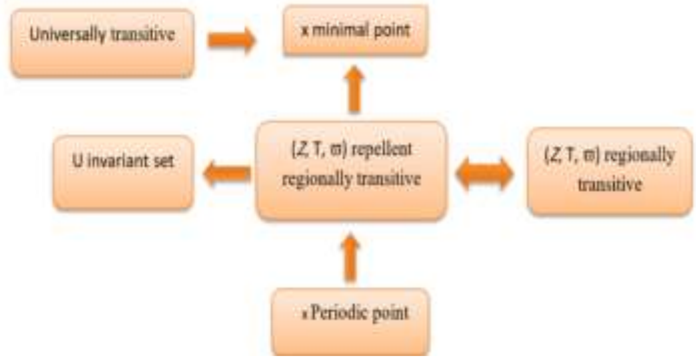


Figure (1) Properties repellent originally in topological group

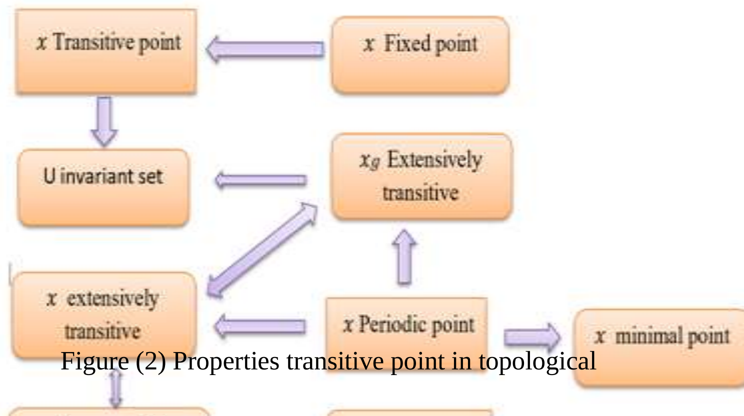


Figure (2) Properties transitive point in topological

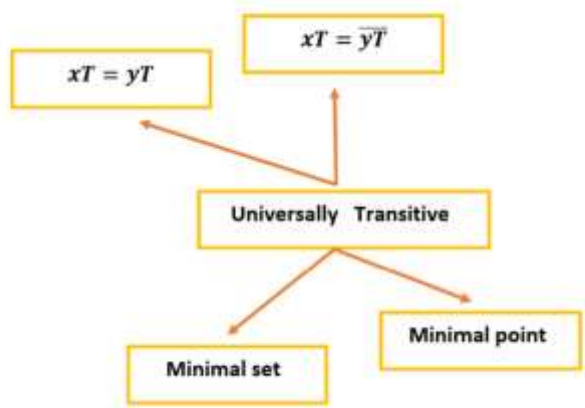


Figure (3) Properties Universally transitive in topological group

(a)

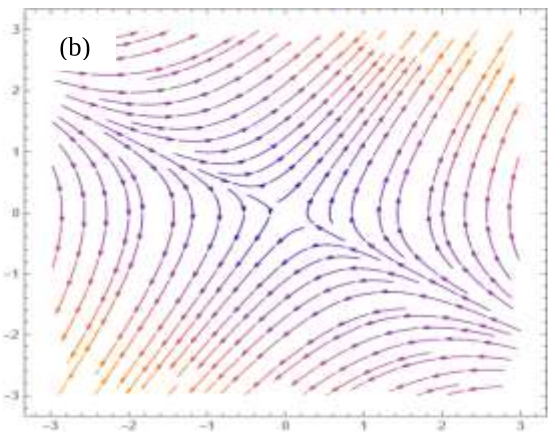


Figure (4) Show orbit point (a) periodic (b) transitive

3.1 Conclusion

After a detailed analysis of the algebraic and topological structure of the right topological transformation group, and highlighting the role of transitivity in these two structures, it becomes clear that the relationship between admissible groups and transitivity forms the cornerstone for understanding the properties and applications of topological groups. The relationship between the types of transitivity in topological transformation groups has added significantly to the study. The results have shown that . In this section, we will review the most important findings, based on the theories and mathematical treatments that supported these observations. We review the most important results we have reached in this thesis.

1. Topological group repellently regionally transitive be regionally transitive and universally transitive

2. Relationship types Topological group transitive such as (regionally- repellently- universally) with dynamical points (minimal point- periodic point

3. Closure Orbit of Topological group universally transitive be minimal set

4. Transitive at points is related to the orbit. Based on the relationship between transitive points and dynamic points, found that the orbit of transitive points includes points (periodic point- almost periodic point-invariant almost periodic point).

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