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A new Method for Solving the population growth equation using the Mohand transforms

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Abstract: *For mathematicians, engineers, and physicists, the transform is a crucial component of the mathematical foundation. Especially when it comes to solving integral equations, the integral transform methods offer simple and efficient solutions for a wide range of issues that arise in different scientific and engineering domains. Consequently, the precise solution was found with a minimal amount of computational effort and time. In addition to solving integrative equations using approximated methods and studying the integral transforms (Laplace, Mohand), the primary goal is to classify and study integral equations and integro-differential equations. Utilizing real-world examples, such as the population growth equation, to illustrate the efficacy and precision of these integral transforms is also a goal.*

Keywords: integro-differential equations, integral transforms (Laplace, Mohand), the population growth equation.

1. Introduction

Numerous scientific disciplines and applications, such as fluid dynamics, plasticity, elasticity, heat and mass transfer, filtration theory, oscillation theory, electrostatics, electrodynamics, biomechanics, game theory, control, queuing theory, electrical engineering, and economics., medicine, and more, involve the integral equation. Integral equations are defined mathematically as equations where the unknown functions appear under the integral sign, which is represented by:

$$\eta(x) = \mathrm{F}(x) + \lambda \int_{s(x)}^{r(x)} N(x,t)\, \eta(t)\, dt$$

where Since is a constant parameter, N(x,t)N(x,t) is a known function of two variables x and t, and s(x) and r(x) are the limits of integration. The kernel of the integral equation N(x,t)N(x,t) is also called the kernel of the integral equation. The unknown function (x)(x) is within the integral sign.that will be ascertained. In numerous additional instances, the unknown function η(x) can be found both inside and outside the integral sign. The functions N(x,t) and F(x) are provided beforehand. The correct comprehension of the qualitative aspects of numerous phenomena and processes in diverse fields of natural science depends on the precise solution of these equations. In many different applications, integral equations are derived from differential equations. For nearly 200 years, numerous problems in applied mathematics, mathematical physics, and engineering science have been effectively resolved through the use of integral transforms. The closed-form solution can be derived specifically using the integral transform technique. $L\{\mathrm{F}(x)\} = \int_0^\infty e^{-vx}\, \mathrm{F}(x)\, dx, \quad \text{for } x \geq 0$

Following that, in 2017, Mohand presented a brand-new integral transform that is based on the classical Fourier integral. This transform is known as the Mohand transform. For x≥0, the Mohand transform of a function f(x) is defined as [8]:.

$$Mo\{\mathrm{F}(x)\} = v^2 \int_0^\infty \mathrm{F}(x) e^{-vx}\, dx,$$

Mo is the Mohand transform operator, and v is a real number. This has been successfully applied to the solution of integral and linear differential equations [7].

2.Basic Concepts of Integral and Integro – Differential Equations

An equation with the unknown function η(x) (Solution) appearing under an integral sign is called an integral equation. [9]: is the form of a criterion integral equation in η(x).

$$\eta(x) = \mathrm{F}(x) + \lambda \int_{s(x)}^{r(x)} N(x,t)\, \eta(t)\, dt \qquad \dots (1)$$

This means that s(x) and r(x) represent the limits of integration, λ is a constant, and the function N(x,t) is a function of two variables, x and t, and is referred to as the integral equation's kernel function (known function). Additionally, the function F(x) is predetermined and is frequently referred to as the integral equation's driving term. Furthermore, η(x) makes appearances both inside and outside of the integral sign. The limits of integration, s(x) and r(x), can be variables, constants, or mixed, it is crucial to note

Integro - Differential Equations

When initial value or boundary value problems are transformed into integral equations, many scientific applications can be mathematically represented as integro-differential equations. The form [9]: represents a general Fredholm integro-differential equation since it includes both integral and derivatives of the unknown function η(x).

$$\eta^{(k)}(x) = \mathrm{F}(x) + \lambda \int_a^b N(x,t)\,\eta(t)\,dt \quad , \ \eta^{(k)} = \frac{d^k\eta}{dx^k} \qquad \dots (6)$$

4. integral transforms

In order to solve linear integral equations with constant coefficients, additional operational tools are the transforms. Three primary steps comprise the solution process: The given "hard" problem is converted into a "simple" equation. This straightforward equation can be resolved using only algebraic operations. Some well-known integral transforms, including the Laplace, Mohand, Mahgoub, Sadik, and Yang transforms, were illustrated in this chapter. The solution of the simple equation is transformation back in order to get the answer to the given problem [7]. In addition, then differences between these transforms will be explored along with a few examples. There are a few well-known integral transforms that will be discussed in this section along with examples of how well they work to solve integral equations. Eigenschaften of Integral Transform: 1) Cons We can readily derive any integral properties [9] from the previously mentioned definition of integral transforms.

Some Famous Integral Transforms

Several Well-Known Integral Transformations The efficiency and efficacy of various integral transformations for solving equations will be discussed in this part along with their presentation. Integral Transform Properties Any integral property can be readily derived from the previously described definition of integral transforms [9]:

Since Linearity property: T { F_1 (x)+ F_2 (x)}= T {F_1 (x)}+ T {F_2 (x)}, 1) Constant Multiple: T {a F(x)}=a T {F(x)},

Table 1: (L , M , Mo , …..) –Transforms of some elementary functions

$S.N$	$f(x)$	$L\{f(x)\}$	$Mo\{f(x)\}$
$S.N$	$f(x)$	$L\{f(x)\}$	$Mo\{f(x)\}$
1	1	$\frac{1}{v}$	v
2	x	$\frac{1}{v^2}$	1
3	x^2	$\frac{2!}{v^3}$	$\frac{2!}{v}$
4	$x^n, n \in N$	$\frac{n!}{v^{n+1}}$	$\frac{n!}{v^{n-1}}$

5	e^{ax}	$\frac{1}{v-a}$	$\frac{v^2}{v-a}$
6	$\sin ax$	$\frac{a}{v^2+a^2}$	$\frac{av^2}{v^2+a^2}$
7	$\cos ax$	$\frac{v}{v^2+a^2}$	$\frac{v^3}{v^2+a^2}$
8	$\sinh ax$	$\frac{a}{v^2-a^2}$	$\frac{av^2}{v^2-a^2}$
9	$\cosh ax$	$\frac{v}{v^2-a^2}$	$\frac{v^3}{v^2-a^2}$

2. Laplace Transform (L – Transform)

One effective technique for resolving differential and integral equations is then Laplace transform method. Differential equations and integral equations are transformed into easily solved polynomial equations by the Laplace transform, and the solution to the examined Utilizing the inverse Laplace transform yields the equation. The L-Transform of a function F(x), for x ≥ 0, can be defined as follows [9].

$$L\,\{\mathrm{F}(x)\} = \int_0^\infty e^{-vx}\,\mathrm{F}(x)\,dx,$$

Where L is referred to as the L-Transform operation and v is real. Both of the following conditions must be met for the L-Transform to exist for any function F(x). 1. . On the interval of integration 0 ≤ x >A, F(x) is piecewise continuous for any positive A. 2. . Since a is a real constant, F(x) is of exponential order e^ax as x→∞. Consequently, the L-Transform is present and needs to be satisfied

$$\lim_{v\to\infty} L\,\{\mathrm{F}(x)\} = 0\ .$$

The Convolution Theorem for L – Transform [9]

This is an important theorem that will be used for solving integral equations. Consider the integral equation:

$$\eta(x) = \mathrm{F}(x) + \lambda \int_{s(x)}^{r(x)} N(x,t)\,\eta(t)dt$$

If it depends on the difference between x and t, the kernel N(x,t) is called a difference kernel. It is possible to express the integral equation as follows by using the difference kernels as an example: *cos (x-t), sin h (x-t).*

Then we consider two function $\mathrm{F}_1(x)$ and $\mathrm{F}_2(x)$ that each have the prerequisites for the Laplace transform to exist. Let L {F_1 (x)} and L {F_2 (x)} be the L-Transforms for the two functions, F_1 (x) and F_2 (x).so the L- Transform convolution of these two functions can be defined as:

$$(\mathrm{F}_1 * \mathrm{F}_2)(x) = \int_0^x \mathrm{F}_1(x-t)\,\mathrm{F}_2(t)\,dt \ \text{ or } (\mathrm{F}_2 * \mathrm{F}_1)(x) = \int_0^x \mathrm{F}_2(x-t)\mathrm{F}_1(t)\,dt$$

Then $(\mathrm{F}_1 * \mathrm{F}_2)(x) = (\mathrm{F}_2 * \mathrm{F}_1)(x)$.

Thus the L - Transform of the convolution product $(\mathrm{F}_1 * \mathrm{F}_2)(x)$ is given by:

$L\,\{(\mathrm{F}_1 * \mathrm{F}_2)(x)\} = L\,\{\int_0^x \mathrm{F}_1(x-t)\mathrm{F}_2(t)\,dt\,\}$.

Inverse of L - Transform [9]

When the function F(x) has L {F(x)} as its L-Transform, the inverse of L-Transform of L {F(x)} is F(x), i. We write: $L^{-1}\{\ L\,\{\mathrm{F}(x)\}\,\} = \mathrm{F}(x)$.

Since the operator of the inverse of the L-transform is L^(-1) then the linearity property also applies to the inverse of the L-transform. That is, we write:

$$L^{-1}\{a\,L\,\{\mathrm{F}_1(x)\} + b\,L\{\mathrm{F}_2(x)\}\} = a\,L^{-1}\{L\,\{\mathrm{F}_1(x)\}\,\} + b\,L^{-1}\{L\,\{\mathrm{F}_2(x)\,\}\}$$

$$= a\,\mathrm{F}_1(x) + b\,\mathrm{F}_2(x)\ .$$

2. Mohand Transform (Mo – Transform)

An additional effective tool for solving integral equations is the Mo-Transform, which works similarly to the L-Transform in that it converts integral equations into easily solved polynomial equations

Definition 2: [2] The Mo-transform of a function $f(x)$, for $x \geq 0$ can be defined as :

$$Mo\{\,\mathrm{F}(x)\} = v^2 \int_0^\infty \mathrm{F}(x)e^{-vx}\,dx,$$

The Mo-Transform operator is denoted by Mo, and v is a real constant (number). Note that the Mo-transform only if F(x) is piecewise continuous and of exponential order does the Mo-transform of the function F(x) for x≥0 exist; these are only necessary conditions for the existence of the function.

The Convolution Theorem for Mo -Transform [2]

We should apply any transform to solve integral equations, as we have seen when solving integral equations using earlier transforms. Let's say that the Mo-Transform of F_1 (x) and F_2 (x) is Mo { F_1 (x) } and 〖Mo {F〗 _2 (x) }, respectively.

$$Mo\,\{\,\mathrm{F}_1(x) * \mathrm{F}_2(x)\,\} = \frac{1}{v^2}\cdot Mo\{\,\mathrm{F}_1(x)\}\,.\,Mo\{\,\mathrm{F}_2(x)\}\ .$$

Inverse of Mo -Transform [2]

If $Mo\,\{\mathrm{F}(x)\}$ is Mo – Transform of the function $\mathrm{F}(x)$, then $\mathrm{F}(x)$ is called the inverse of Mo -Transform and mathematically it can be define as:

$$\mathrm{F}(x) = Mo^{-1}\{\,Mo\,\{\mathrm{F}(x)\}\,\}\ .$$

Where the inverse of the Mo-Transform operator is the operator 〖Mo〗 ^(-1) and it unquestionably possesses the linearity property.

$$Mo^{-1}\{a\,Mo\,\{\mathrm{F}_1(x)\} + b\,Mo\,\{\mathrm{F}_2(x)\}\,\} = a\cdot Mo^{-1}\{Mo\,\{\mathrm{F}_1(x)\}\} + b\cdot Mo^{-1}\{Mo\,\{\mathrm{F}_2(x)\}\} = a\,\mathrm{F}_1(x) + b\mathrm{F}_2(x).$$

For a few basic functions, the following tables display the previously well-known transforms.

5.Dualities between Transforms

The dualities between L-Transform and some well-known transforms, including M-Transform, S-Transform, Mo-Transform, and γ-Transform, as well as between them, will be covered in this section. To illustrate the significance of these relationships for the integral transforms mentioned, we provide examples where the other transform result is represented by the symbol φ.

1- Laplace - Mohand Duality [1]

L - Transform of $f(x)$ for $x \geq 0$, denoted by $L\,\{f(x)\}$, is defined by the equation:

$$L\,\{f(x)\} = \int_0^\infty e^{-vx}\,f(x)\,dx$$

And Mo - Transform of $f(x)$ for $x \geq 0$, denoted by $Mo\{f(x)\}$, is defined by:

$$Mo\,\{f(x)\} = v^2 \int_0^\infty f(x)\, e^{-vx}\, dx = \varphi(v)$$

then $L\,\{f(x)\} = \int_0^\infty e^{-vx}\, f(x)\, dx = \frac{v^2}{v^2}\int_0^\infty e^{-vx}\, f(x)\, dx$

$$= \frac{1}{v^2}\,\{\, v^2 \int_0^\infty e^{-vx}\, f(x)\, dx\,\} = \frac{1}{v^2}\,\varphi(v)$$

Thus $L\{f(x)\} = \frac{1}{v^2}\,[Mo\{f(x)\}]$.

And similarly, we can find. $Mo\{f(x)\} = v^2 L\{f(x)\}$.

6.Illustrative Examples

The following examples are represented to solve integral equations using different transforms.

Example 1: Apply integral transforms to the following integral equation to solve it

$$u(x) = \frac{1}{6}\,x^3 + \int_0^x (x-t)\,u(t)\,dt$$

Solution:

A: ***L- Transform***

We obtain by applying the Laplace transform to both sides.

$$L\,\{u(x)\} = \frac{1}{6}\,L\,\{x^3\} + L\,\{\int_0^x (x-t)\,u(t)\,dt\}$$

$$L\,\{u(x)\} = \frac{1}{6}\,.\frac{3!}{v^4} + \frac{1}{v^2}\,.L\,\{u(x)\}$$

$$L\,\{u(x)\}\left(1 - \frac{1}{v^2}\right) = \frac{1}{v^4}$$

$$L\,\{u(x)\}\left(\frac{v^2-1}{v^2}\right) = \frac{1}{v^4}$$

$$L\,\{u(x)\} = \frac{1}{v^4}\,.\left(\frac{v^2}{v^2-1}\right)$$

$$L\,\{u(x)\} = \frac{1}{v^2}\,.\left(\frac{1}{v^2-1}\right)$$

$$\therefore u(x) = L^{-1}\left[\frac{1}{v^2}\,.\left(\frac{1}{v^2-1}\right)\right] = x\, sinhx.$$

B: ***Mo- Transform***

By taking *s*- Transform on both sides we have:

$$Mo\{u(x)\} = Mo\left\{\frac{1}{6}x^3\right\} + Mo\left\{\int_0^x (x-t)u(t)dt\right\}$$

$$Mo\{u(x)\} = \frac{1}{6}\cdot\frac{3!}{v^2} + 1\cdot\frac{1}{v^2}\cdot Mo\{u(x)\}$$

$$Mo\{u(x)\}\left(1 - \frac{1}{v^2}\right) = \frac{1}{v^2}$$

$$Mo\{u(x)\}\left(\frac{v^2-1}{v^2}\right) = \frac{1}{v^2}$$

$$Mo\{u(x)\} = \frac{1}{v^2}\cdot\frac{v^2}{v^2-1}$$

$$Mo\{u(x)\} = 1\cdot\frac{1}{v^2}\cdot\frac{v^2}{v^2-1}$$

$u(x) = Mo^{-1}\left\{1 \cdot \frac{1}{v^2} \cdot \frac{v^2}{v^2-1}\right\}$

$u(x) = x \cdot \sinh x$.

Example2: Use integral transforms to solve the integral equation that followes

$$u(x) = cosx + sinx + \int_0^x u(t)\, dt$$

Solution:

A: L – Transform

By taking L – Transform on both side, we have

$L\{u(x)\} = L\{\cos x\} + L\{\sin x\} - L\{\int_0^x u(t)dt\}$

$L\{u(x)\} = \frac{v}{v^2+1} + \frac{1}{v^2+1} - \frac{1}{v} \cdot L\{u(x)\}$

$L\{u(x)\}\left(1 + \frac{1}{v}\right) = \frac{v}{v^2+1} + \frac{1}{v^2+1}$

$L\{u(x)\}\left(\frac{v+1}{v}\right) = \frac{v}{v^2+1} + \frac{1}{v^2+1}$

$L\{u(x)\} = \frac{v+1}{v^2+1} \cdot \frac{v}{v+1} = \frac{v}{v^2+1}$

$u(x) = L^{-1}\left\{\frac{v}{v^2+1}\right\}$

$\therefore u(x) = \cos x$.

since $Mo\,\{u(x)\} = v^2\,.L\,\{u(x)\}$

$Mo\,\{u(x)\} = v^2 \cdot\ \frac{v}{v^2+1} = \frac{v^3}{v^2+1}$

$Mo\{u(x)\} = \frac{v^3}{v^2+1}$

$\therefore u(x) = Mo^{-1}\left\{\frac{v^3}{v^2+1}\right\} = \cos x$.

Since $M\,\{u(x)\} = \frac{1}{v} \cdot\ \ Mo\,\{u(x)$

$M\{u(x)\} = \frac{1}{v}\ \cdot\ \frac{v^3}{v^2+1} = \frac{v^2}{v^2+1}$

$\therefore u(x) = M^{-1}\left(\frac{v^2}{v^2+1}\right) = \cos x$

Fig. 1: $\mathrm{u(x)} = \cos x$

6. Realistic Examples

Population growth is examined in this section as a significant practical use of integral equations and integral transforms.

Population growth equation:

The rate at which a city's population increases is proportionate to the number of people currently residing there, or it covers the prediction of any future increases in the birthrate, which is crucial for global planning. Furthermore, the development of a plant, bacteria, cell, organ, or species are additional examples. For instance, the growth of the human population can be represented as an integral equation. Since the survival function is x and then number of the children born at the time *x=0* is b (where b is an arbitrary constant), we obtain [3]:

$$f(x) = bx + \int_0^x N(x-y)f(y)dy \qquad \text{.......... (8)}$$

Since, we will solve that equation with some transformations as shown below

A: ***L- Transform***

$f(x) = bx + \int_0^x N(x-y) \, . \, f(y)dy.$

$L\{f(x)\} = L\{bx\} + L\{\int_0^x N(x-y) \, . \, f(y)dy\}$

$L\{f(x)\} = \frac{b}{v^2} + \frac{1!}{v^2} \, . L\{f(x)\}$

$L\{f(x)\}\left(1 - \frac{1}{v^2}\right) = \frac{b}{v^2} \rightarrow L\{f(x)\}\left(\frac{v^2-1}{v^2}\right) = \frac{b}{v^2}$

$\therefore L\{f(x)\} = \frac{b}{v^2} \, . \frac{v^2}{v^2-1} = \frac{b}{(v^2-1)}$

$\therefore f(x) = L^{-1}\{\frac{b}{v^2-1}\} = b\, L^{-1}\left(\frac{1}{v^2-1}\right) = b.sinhx \, .$

B: ***Mo – Transform***

$Mo\{f(x)\} = Mo\{bx\} + Mo\{\int_0^x N(x-y)\, f(y)\, dy\}.$

$Mo\{f(x)\} = b + \frac{1}{v^2} \cdot Mo\{f(x)\}$

$Mo\{f(x)\}\left(1 - \frac{1}{v^2}\right) = b$

$Mo\{f(x)\}\left(\frac{v^2-1}{v^2}\right) = b$

$Mo\{f(x)\} = b \cdot \frac{v^2}{v^2-1}$

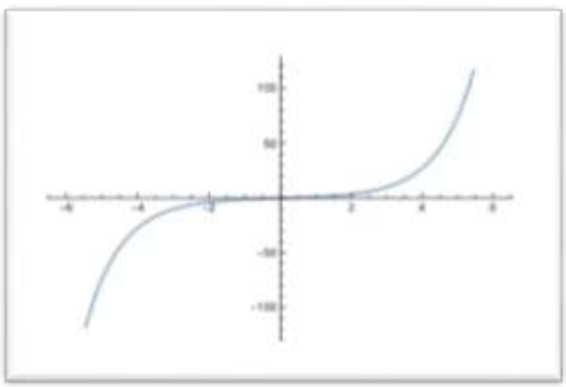

Fig.2: $f(x) = b \, .\sinh x$

Conclusion and Future Work :

In that peaper, we focus on studying and classification integral equation , especially, linear integral equation, and using famous integral transformations, such as Laplace,…… to solve this kind of integral equation due to more accuracy and less error obtain from these integral transformations. As application for integral transformations, we discuss the population growth equation,

For future work, we can recommend the following :

- Solve non-linear integral equations using these famous integral transformations with their relationships .

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