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On The Bifuzzy ψ-subalgebra of ψ-algebra

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Abstract: *The concept tripolar fuzzy subset is a generalization of fuzzy subset. In this paper, the concept ψ-algebra, ψ-subalgebras, fuzzy ψ-subalgebra of ψ-algebras are introduced and several properties are investigated. Also, we introduce the notion bifuzzy ψ-subalgebra of ψ-algebras and we explain the relation between bifuzzy ψ-subalgebra and ψ-subalgebra.*

Keywords: ψ-algebra, ψ- subalgebra, fuzzy ψ-subalgebra, bifuzzy ψ-subalgebra.

1. INTRODUCTION

In 1965, L.A. Zadeh introduced the notion of fuzzy subset, [1]. In 1976, K. Is´eki and S. Tanaka studied the notion of BCK-algebra, [2]. In 1991, O.G. Xi studied the notion of fuzzy BCK-algebra, [3]. In 2006, A.B. Saoid introduced fuzzy QS-algebra with interval-valued membership function, [4]. Also, T. Priya and T. Ramachandran introduced anti-fuzzy ideals of CI-algebra and its lower level cuts, [5]. Jun[6,7] studied the notion of cubic set as generalization of fuzzy set and interval-valued fuzzy set. In 2015, A.T. Hameed introduced the idea of SA-algebras. She stated some concepts related to it such as SA-subalgebra, SA-ideal, fuzzy SA-subalgebra and fuzzy SA-ideal of SA-algebra. She introduced the concept of homomorphisms on SA-algebra and fuzzy homomorphisms on SA-algebra, [9]. In 2023, A.T. Hameed and N.H. Jaber introduced the notion of ψ -subalgebra, ψ -ideal, bifuzzy ψ -subalgebra, bifuzzy ψ-ideal and they introduced the concept of homomorphisms on ψ -algebra and fuzzy homomorphisms on ψ -algebra.

2. Preliminaries

In this section, we give some basic definitions and preliminaries proprieties of ψ-subalgebras and fuzzy ψ -sebalgebra of ψ -algebra such that we include some elementary aspects that are necessary for this paper.

Definition 2.1.([14]). Let $(X;+,-,0)$ be an algebra with two operations $(+)$ and $(-)$ and constant (0). X is called an ***ψ*-algebra** if it satisfies the following properties: for all $x,y,z \in X$,

(ψ_1) $x-x=0,$

(ψ_2) $(0-x)+x=0,$

(ψ_3) $(x-y)-z=x-(z+y),$

(ψ_4) $(y+x)-(x-z)=y+z.$

In , we can define a binary relation ($\leq$) by : $x \leq y$ if and only if $x+y=0$ and $x-y=0$, $x,y \in X$.

Definition 2.2. [13].

Let $(X;+,-,0)$ be a ψ-algebra and let S be a nonempty set of X. S is called **a ψ- subalgebra of X** if $x+y \in S$ *and* $x-y \in S$, whenever $x,y \in S$.

Definition 2.3.[4].

Let X be a nonempty set, a fuzzy subset μ of X is a mapping $\mu: X \rightarrow [0,1]$.

Definition 2.4.[14].

For any $t \in[0,1]$ and a fuzzy subset μ in a nonempty set X, the set

$U(\mu,t) = \{x \in X \mid \mu(x) \geq t\}$ is called **an upper t-level cut of μ,** and the set $L(\mu,t) = \{x \in X \mid \mu(x) \leq t\}$ is called **a lower t-level cut of μ**.

Definition 2.5.[13].

Let $(X;+,-,0)$ be a ψ-algebra, a fuzzy subset μ of X is called **a fuzzy ψ-subalgebra of X** if for all $x, y \in X$,

1- $\mu(x+y) \geq min\{\mu(x),\mu(y)\}$ and
2- $\mu(x-y) \geq min\{\mu(x),\mu(y)\}$.

Definition 2.6. [14].

Let $(X;+,-,0)$ be an ψ-algebra, a fuzzy subset μ of X is called **an anti-fuzzy ψ-subalgebra of X** if for all $x, y \in X$,

AFψS$_1$) $\mu(x+y) \leq max\{\mu(x),\mu(y)\}$,

AFψS$_2$) $\mu(x-y) \leq max\{\mu(x),\mu(y)\}$.

Proposition 2.7. [4].

Let μ be an anti-fuzzy subset of an ψ-algebra $(X;+,-,0)$.

1- If μ is an anti-fuzzy ψ-subalgebra of , then it satisfies for any t∈[0, 1], $L(\mu,t) \neq \emptyset$ implies $L(\mu,t)$ is a ψ-subalgebra of X.

2- If $L(\mu,t)$ is a ψ-subalgebra of X, for all t∈[0, 1], $L(\mu,t) \neq \emptyset$, then μ is an anti-fuzzy ψ-subalgebra of X .

Proposition 2.8. [4].

Let μ be an anti-fuzzy subset of an ψ-algebra $(X;+,-,0)$.

1- If μ is an anti-fuzzy ψ-ideal of , then it satisfies for any t∈[0, 1], $L(\mu,t) \neq \emptyset$ implies $L(\mu,t)$ is an ψ- ideal of X.

2- If $L(\mu,t)$ is an ψ- ideal of X, for all t∈[0, 1], $L(\mu,t) \neq \emptyset$, then μ is an anti-fuzzy ψ-ideal of X .

3. Bifuzzy ψ-subalgebra of ψ-algebra

In this section, we will introduce a new notion called bifuzzy ψ -subalgebra of ψ -algebra and study several properties of it.

Definition 3.1. A bifuzzy subset A of an ψ-algebra $(X;+,-,0)$ is an object having the form $A = \{(x,\mu_A(x),\nu_A(x)) \mid x \in X\}$ where the functions $\mu_A: X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and $\nu_A: X \to [0,1]$ denote the degree of non membership (namely $\nu_A(x)$) which is called anti-fuzzy function and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$.

Definition 3.2.

Let $(X;+,-,0)$ be an ψ-algebra, a fuzzy subset ν of X is called **an anti-fuzzy ψ-subalgebra of X** if for all $x, y \in X$,
$\nu(x+y) \leq \max\{\nu(x),\nu(y)\}$ and $\nu(x+y^-) \leq \max\{\nu(x),\nu(y)\}$.

Definition 3.3.

Let $A = \{(x,\mu_A(x),\nu_A(x)) \mid x \in X\}$ be a bifuzzy subset of an ψ-algebra X. A is said to be **a bifuzzy ψ-subalgebra of X** if:
for all $x, y \in X$,
(IFS$_1$) $\mu_A(x+y) \geq \min\{\mu_A(x),\mu_A(y)\}$ and
$\mu_A(x+y^-) \geq \min\{\mu_A(x),\mu_A(y)\}$.

(IFS2) $\nu_A(x+y) \leq \max\{\nu_A(x), \nu_A(y)\}$ and
$\nu_A(x+y^-) \leq \max\{\nu_A(x), \nu_A(y)\}$.

i.e., μ_A is a fuzzy ψ-subalgebra of an ψ-algebra and ν_A is an anti-fuzzy ψ-subalgebra of an ψ-algebra.

Example 3.4.

Let $X = \{0,1,2,3\}$ in which $(+, -)$ be defined by the following table

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

-	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X; +, -, 0)$ is an ψ-algebra. It is easy to show that $S_1 = \{0,1\}$, $S_2 = \{0,2\}$ and $S_3 = \{0,3\}$ are ψ-subalgebras of X. Define a fuzzy subset
$\mu_A: X \to [0,1]$ such that $\mu_A(0) = 0.7$, $\mu_A(1) = \mu_A(2) = 0.6$, $\mu_A(3) = 0.4$, $\nu_A: X \to [0,1]$ such that
$\nu_A(0) = 0.3$, $\nu_A(1) = \nu_A(2) = 0.4$, $\nu_A(3) = 0.6$.

Routine calculation gives that μ_A is a fuzzy ψ-subalgebra of X and that ν_A is anti-fuzzy ψ-subalgebra of X.

Proposition 3.5.

Every bifuzzy ψ-subalgebra $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ of an ψ-algebra $(X; +, -, 0)$ satisfies the inequalities

$\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$, for all $x \in X$.

Proof:

$\mu_A(0) = \mu_A(x + x^-) \geq \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x)$ and

$\nu_A(0) = \nu_A(x + x^-) \leq \max\{\nu_A(x), \nu_A(x)\} = \nu_A(x)$. □

Definition 3.6.

For fuzzy subsets μ_A and ν_A of an ψ-algebra $(X; +, -, 0)$ and $t \in Im(\mu_A)$, $U(\mu_A, t) = \{x \in X \mid \mu_A(x) \geq t\}$ and $s \in Im(\nu_A)$,

$L(\nu_A, s) = \{x \in X \mid \nu_A(x) \leq s\}$.

Remark 3.7.

1- If μ_A is a fuzzy ψ-subalgebra of ψ-algebra $(X; +, -, 0)$, then it is that $U(\mu_A, t)$ is a ψ-subalgebra of X, for any $t \in Im(\mu)$.

Let $, y \in U(\mu_A, t)$, then $\mu_A(x) \geq t$, and $\mu_A(y) \geq t$, then

$\min\{\mu_A(x), \mu_A(y)\} \geq t$, since μ_A is a fuzzy ψ-subalgebra, then $\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y)\} \geq t$, therefore $x + y \in U(\mu_A, t)$ and

$\mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y)\} \geq t$, therefore $x - y \in U(\mu_A, t)$.

2- If ν_A is anti-fuzzy ψ-subalgebra of X, then it is that $L(\nu_A, s)$ is a ψ-subalgebra of X, for any $s \in Im(\nu)$.

Let $x, y \in L(\nu_A, s)$, then $\nu(x) \leq s$ and $\nu(y) \leq s$, then

$\max\{\nu_A(x), \nu_A(y)\} \leq s$, since ν_A is anti-fuzzy ψ-subalgebra, then

$\nu_A\ (x+y) \leq \max\{\ \nu_A(x)\ , \nu_A(y)\} \leq$ s, therefore $x+y \in\ L(\nu_A, s)$ and $\nu_A\ (x+y^-) \leq \max\{\ \nu_A(x)\ , \nu_A(y)\} \leq$ s, therefore $+y^- \in\ L(\nu_A, s)$.

3- But if we do not give a condition that μ_A is a fuzzy ψ-subalgebra of X, then $U\ (\mu_A, t)$ is not a ψ-subalgebra of X or ν_A is anti-fuzzy ψ-subalgebra of X, then $L(\nu_A, s)$is not a ψ-subalgebra of X as seen in the following example.

Example 3.8.

Let $X = \{0,1,2, 3\}$ is an ψ-algebra which is given in Example (4.1.4).

Define a fuzzy subset μ_A of X:

X	0	1	2	3
μ_A	0.7	0.6	0.5	0.3

Then μ_A is not a fuzzy ψ-subalgebra of X.

Since $\mu_A(1+2) = 0.3 \ngeq 0.5 = \min\{\ \mu_A\ (1), \mu_A\ (2)\}$.

For t = 0.5, we obtain $U\ (\mu_A, t)\ = \{0, 1, 2\}$ which is not an ψ-subalgebra of X since 1+2 = 3 $\notin\ U\ (\mu_A, t)$.

Proposition 3.9.

Let $A = \{(x, \mu_A(x), \nu_A\ (x)) \mid x \in X\}$ be a bifuzzy subset of an ψ-algebra ($X; +, -, 0$). If A is a bifuzzy ψ-subalgebra of X, then for any t, s ∈ [0, 1], $U(\mu_A, t) \neq \emptyset$ implies $U(\mu_A, t)$ is a ψ-subalgebra of X and $L(\nu_A, s) \neq \emptyset$ implies $L(\nu_A, s)$ is a ψ-subalgebra of X.

Proof:

Assume that μ_A is a fuzzy ψ-subalgebra of X, let t∈ [0,1] be such that $U(\mu_A, t) \neq \emptyset$, and let $x, y \in X$ be such that $x, y \in U(\mu_A, t)$, then

$\mu_A(x) \geq t$ and $\mu_A(y) \geq t$, so $\mu_A(x+y)\ \geq\ \min\{\mu_A\ (x), \mu_A\ (y)\} \geq\ t$, so that $(x+y) \in U(\mu_A, t)$. Similarly, $(x+y^-) \in U(\mu_A, t)$.

Hence $U(\mu_A, t)$ is a ψ-subalgebra of X.

Assume that ν_A is an anti-fuzzy ψ-subalgebra of X, let s∈ [0,1] be such that $L(\nu_A, s) \neq \emptyset$, and let $x, y \in X$ be such that $x, y \in L(\nu_A, s)$, then $\nu_A(x) \leq s$ and $\nu_A(y) \leq s$, so $\nu_A\ (x+y) \leq \max\{\nu_A\ (x), \nu_A\ (y)\} \leq s$,

so that $(x+y) \in L(\nu_A, s)$.

Similarly, $(x+y^-) \in U(\nu_A, s)$.

Hence $L(\nu_A, s)$ is a ψ-subalgebra of X. □

Proposition 3.10.

Let $A = \{(x, \mu_A(x), \nu_A\ (x)) \mid x\ \in X\}$ be a bifuzzy subset of an ψ-algebra ($X; +, -, 0$). If $U(\mu_A, t)$ and $L(\nu_A, s)$ are ψ-subalgebras of X, for all t, s ∈ [0, 1], $U(\mu_A, t) \neq \emptyset \neq L(\nu_A, s)$, then A is a bifuzzy ψ-subalgebra of X.

Proof:

Suppose that A is not bifuzzy ψ-subalgebra of X, satisfies $U(\mu_A, t)$ is a ψ-subalgebra of X. Now, assume $\mu_A(x+y) < \min\{\mu_A(x), \mu_A(y)\}$, taking $t_0 = {(\mu_A\,(x+y) + \min\{\mu_A\,(x), \mu_A(y)\})}/_{2}$, we have $t_0 \in [0,1]$

and $\min\{\mu_A(x), \mu_A(y)\} > t_0 > \mu_A\,(x+y)$, it follows that $x, y \in U(\mu_A, t_0)$

and $x + y \notin U(\mu_A, t_0)$, this is a contradiction since $U(\mu_A, t_0)$ is a ψ-subalgebra of X. Similarity, $(x-y) \in U(\mu_A, t_0)$.

Since $L(\nu_A, s)$ is a subalgebra of X, assume $\nu_A\,(x+y) > \max\{\nu_A(x), \nu_A\,(y)\}$,

taking $s_0 = {(\nu_A\,(x+y) + \max\{\nu_A\,(x), \nu_A\,(y)\})}/_{2}$, we have $s_0 \in [0, 1]$ and $\max\{\nu_A(x), \nu_A\,(y)\} < s_0 < \nu_A\,(x+y)$, it follows that $x, y \in L(\nu_A, s_0)$ and $x + y \notin L(\nu_A, s_0)$,

this is contradiction since $L(\nu_A, s_0)$is a ψ-subalgebra of X.

Similarly, $(x + y^-) \in L(\nu_A, s_0)$.

Therefore A is a bifuzzy ψ-subalgebra of X. □

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