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On the Cubic fuzzy Bd-ideal of Bd-algebra

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Abstract— In this paper, we define the notion of cubic fuzzy Bd-subalgebras and cubic fuzzy Bd-ideals of Bd-algebra and prove their generalizations. Further, we display the relation between them their level cuts, several theorems, properties are studied and proved.

Keywords— Bd-subalgebra, Bd-ideal, Cubic fuzzy Bd-subalgebra, Cubic fuzzy Bd-ideal.

1. INTRODUCTION

The idea of fuzzy sets begins to be introduced by Za... in 1965 as a broader generalization of classical set theory. [8]. In 1966 Imai and Iseki established that abstract algebras fall into two dalles Bck-algebras and BCI-algebras [4] and [5]. In 1999, J. Meggers and H.S. Kim introduced d-algebras [2], then they introduced and investigated B-algebras in 2002 [3] , In 2010, Jun and etc introduced the notion of cubic, as a generalization of fuzzy set and interval-valued fuzzy set [6]. In 2011, Jun and Jung and Kim applied the notion of cubic set of group and introduced the notion of cubic[9], In 2022, Thanatpora Bantaaja and authors introduced Bd-algebras [7]

2. PRELIMINARIES

In this section, we give some elementary aspects in Bd-algebra such that we deem it necessary for these papers

Definition 2.1[3]

B-algebra is anon-empty set with a constant 0 and binary operation $*$ satisfying the following axioms:

$(B_1)\ x * x = 0$

$(B_2)\ x * 0 = x,$

$(B_3)\ (x * y) * z = x * \big(z * (0 * y)\big)$, for all $x, y \in X$.

Definition 2.2[2]

A nonempty set X with a constant 0 and a binary operation $*$ is called a d-algebra, if for all $x, y \in X$. it satisfies the following axioms:

$(d_1)\ x * x = 0$

$(d_2)\ 0 * x = 0$

$(d_3)\ x * y = 0$ if and only if $x * y = 0$

Definition 2.3[7]

An algebra $(X; *, 0)$ is called a Bd-algebra if it satisfies the following axioms. for every $x, y \in X$.

$(Bd_1) X * 0 = x$

(Bd_2) if $x * y = 0$ and $y * x = 0$ then $x = y$

Definition 2.4 [7]

A non-empty subset S of X is called a Bd-subalgebra of X if

1. $0 \in S$
2. If $\forall x, y \in X$, then $(x * y) \in S$

Definition 2.5[7]

A non-empty subset I of X is called an Bd-ideal of X if it satisfies

1.0∈I

2. (∀x,y∈X) (x+y)∈I and y∈I, and y∈I, then y∈I

3. (∀x∈I and y∈X, then (x+y)∈I

Definition 2.6 [8]

Let X be a nonempty set. A fuzzy subset μ of x. X is mapping $\mu: x \to [0,1]$

Definition 2.7

Let $(X; +,0)$ be an Bd-algebra, a fuzzy subset μ of X is called a Fuzzy

Bd-subalgebra of X if for all $x, y \in X$, $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$

Definition 2.8

Let $(X; +,0)$ be an Bd-algebra, fuzzy subset μ of X is called a Fuzzy Bd-ideal of X if it satisfies: for all $x, y \in X$,

$(\mathrm{F}I_1)$ $\mu(0) \geq \mu(x)$

$(\mathrm{F}I_2)$ $\mu(x) \geq \min\{\mu(x + y), \mu(y)\}$,

$(\mathrm{F}I_3)$ $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$.

Definition 2.9 [1]

Let $(\mathrm{X}; *, 0)$ and $(\mathrm{Y}; *', 0')$ be B-algebra. A function ψ From X to Y denoted by $\psi: \mathrm{X} \to \mathrm{Y}$ is called B-homomorphism if for every $\mathrm{x}, \mathrm{y} \in \mathrm{X}$ applies

$\psi(\mathrm{x} * \mathrm{y}) = \psi(\mathrm{x}) *' \psi(\mathrm{y})$, where $\psi(\mathrm{x}), \psi(\mathrm{y}) \in \mathrm{Y}$

Definition 2.10 [8]

Let $(X; *, 0)$ be a nonempty set. By a cubic subset of a BCK-algebra $(X; *, 0)$, we mean a structure $\Omega = \{< x, \tilde{\mu}_{\mathrm{Ш}}(x), \lambda(x) > | x \in X\}$ in which $\tilde{\mu}_{\mathrm{Ш}} = (x)$ is an interval valued fuzzy set on X and λ is a fuzzy set in X. A cubic set $\mathrm{Ш} = \{x, \tilde{\mu}_{\mathrm{Ш}}(x), \lambda(x) | x \in X\}$ is simply denoted by Ш

$=< \tilde{\mu}_{\mathrm{Ш}}(x), \lambda(x) >$. Denote by C^X the collection of all cubic sets in X.

Definition 2.11 [9]

For a family $\mathrm{Ш}_i = \{< x, \tilde{\mu}_{\mathrm{Ш}_i}(x), > | x \in X\}$ on fuzzy subsets of a BCK-algebra $(X;*,0)$, where $i \in \Lambda$ and Λ is index set, we define the join ($\vee$) and meet ($\wedge$) operations as follows:

$$\vee_{i\in\Lambda}\, \mathrm{Ш}_i = \left(\vee_{i\in\Lambda}\, \tilde{\mu}_{\mathrm{Ш}_i}\right)(x) = \sup\{\tilde{\mu}_{\mathrm{Ш}_i}(x) | i \in \Lambda\}$$

$$\wedge_{i\in\Lambda}\, \mathrm{Ш}_i = \left(\wedge_{i\in\Lambda}\, \tilde{\mu}_{\mathrm{Ш}_i}\right)(x) = \inf\{\tilde{\mu}_{\mathrm{Ш}_i}(x) | i \in \Lambda\}$$

Definition 2.12

Let $(X;+,0)$ be an Bd-algebra. A cubic subset $\mathrm{Ш} =< \tilde{\mu}_{\mathrm{Ш}}(x), \lambda_{\mathrm{Ш}}(x) >$ of X is called cubic fuzzy Bd-subalgebra of X if, for all $x, y \in X$:

1. $\tilde{\mu}_{\mathrm{Ш}}(x + y) \geqslant r \min\{\tilde{\mu}_{\mathrm{Ш}}(x), \tilde{\mu}_{\mathrm{Ш}}(y)\}$, and
2. $\lambda_{\mathrm{Ш}}(x + y) \leq \max\{\lambda_{\mathrm{Ш}}(x), \lambda_{\mathrm{Ш}}(y)\}$.

Example 2.13

Let $X = \{0, a, b, c\}$ be Bd-algebra in Example (3.1.2). It is easy to show that

$I = \{0, a\}$ is an Bd-subalgebra of X Define a cubic subset $\mathrm{Ш} =< \tilde{\mu}_{\mathrm{Ш}}, \lambda_{\mathrm{Ш}} >$ of X of fuzzy subset $\mu: X \to [0,1]$ by:

$$= \begin{cases} [0.3,0.8] & if\ x = \{0, a\} \\ [0.2,0.5] & otherwise \end{cases} \text{ and } \lambda_{\mathrm{Ш}} = \begin{cases} 0.2 & if\ x = \{0, a\} \\ 0.3 & otherwise \end{cases}$$

The cubic set $\mathrm{Ш} =< \mu_{\tilde{\mathrm{Ш}}}, \lambda_{\mathrm{Ш}} >$ is a cubic fuzzy Bd-subalgebra of X.

Proposition 2.14

Let $\mathrm{Ш} =< \tilde{\mu}_{\mathrm{Ш}}, \lambda_{\mathrm{Ш}} >$ be a cubic fuzzy Bd-subalgebra of Bd-algebra (X;+,0), then $\tilde{\mu}_{\mathrm{Ш}}(\tilde{0}) \geqslant \tilde{\mu}_{\mathrm{Ш}}(x)$ and $\lambda_{\mathrm{Ш}}(0) \leq \lambda_{\mathrm{Ш}}(x)$, for all $x \in X$.

Proof:

For an $x \in X$, we have

$$\tilde{\mu}_{\mathrm{Ш}}(\tilde{0}) = \tilde{\mu}_{\mathrm{Ш}}(0 + x)$$

$$\geqslant r \min\{\tilde{\mu}_{\mathrm{Ш}}((0 + x) + 0), \tilde{\mu}_{\mathrm{Ш}}(x)\}$$

$$= r \min\{[\tilde{\mu}_B((0 + x) + 0), \mu_B^-(x)], [\mu_B^+((0 + x) + 0), \mu_B^+(x)]\}$$

$$= r \min\{[\mu_B^-(0), \mu_B^-(x)], [\mu_B^+(0), \mu_B^+(x)]\}$$

$$= r \min\{[\mu_B^-(0), \mu_B^-(x)], [\mu_B^+(0), \mu_B^+(x)]\}$$

$$= [\mu_B^-(x), \mu_B^+(x)] = \tilde{\mu}_{\mathrm{Ш}}(\tilde{x}).$$

Similarly, we can show that $\lambda_{\mathrm{Ш}}(0) \leq \max\{[\lambda_{\mathrm{Ш}}(0), \lambda_{\mathrm{Ш}}(x)]\} = \lambda_{\mathrm{Ш}}(x)$. ■

Definition 2.15

Let $\mathrm{Ш}_i =< \tilde{\mu}_{\mathrm{Ш}_i}, \lambda_{\mathrm{Ш}_i} >$ be a cubic subset of Bd-algebra $(X;+,0)$, where $i \in \Lambda$, then

1. The R-intersection of any set of cubic subset of X is

   $(\cap \tilde{\mu}_{\mathrm{w}_i})(x) = r \inf \tilde{\mu}_{\mathrm{w}_i}(x)$ and $(\vee \lambda_{\mathrm{w}_i})(x) = \sup \lambda_{\mathrm{w}_i}(x)$.

2. The p-intersection of any set of cubic subset of X is

   $(\cap \tilde{\mu}_{\mathrm{w}_i})(x) = r\, inf\, \tilde{\mu}_{\mathrm{w}_i}(x)$ and $(\wedge \lambda_{\mathrm{w}_i})(x) = \inf \lambda_{\mathrm{w}_i}(x)$.

3. The P-union of any set of cubic subset of X is $(\cup \tilde{\mu}_{\mathrm{w}_i})(x) = r\, sup\, \tilde{\mu}_{\mathrm{w}_i}(x)$ and $(\vee \lambda_{\mathrm{w}_i})(x) = \sup \lambda_{\mathrm{w}_i}(x)$.

4. The R-union of any set of cubic subset of X is

   $(\cup \tilde{\mu}_{\mathrm{w}_i})(x) = r\, sup\, \tilde{\mu}_{\mathrm{w}_i}(x)$ and $(\wedge \lambda_{\mathrm{w}_i})(x) = \inf \lambda_{\mathrm{w}_i}(x)$.

Proposition 2.16

The R-intersection of any set of cubic fuzzy Bd-subalgebra of X is also cubic fuzzy Bd-subalgebra of X.

Proof:

Let $\mathrm{W}_i =< \tilde{\mu}_{\mathrm{w}_i}, \lambda_{\mathrm{w}_i} >$ where $i \in \Lambda$, be a set of cubic fuzzy Bd-subalgebras of X and $x, y \in X$, them

$$(\cap \tilde{\mu}_{\mathrm{w}_i})(x+y) = \inf(\tilde{\mu}_{\mathrm{w}_i})(x+y)$$

$$\geqslant \inf\{r \min\{(\tilde{\mu}_{\mathrm{w}_i})(x), (\tilde{\mu}_{\mathrm{w}_i})(y)\}\}$$

$$= r \min\left\{\inf\left(\tilde{\mu}_{\mathrm{w}_i}(x)\right), \inf\left(\tilde{\mu}_{\mathrm{w}_i}(y)\right)\right\}$$

$$= r \min\{(\cap \tilde{\mu}_{\mathrm{w}_i})(x), (\cap \tilde{\mu}_{\mathrm{w}_i})(y)\}$$

Hence $(\cap \tilde{\mu}_{\mathrm{w}_i})$ is a fuzzy Bd-subalgebra of X.

$$(\vee \lambda_{\mathrm{w}_i})(x+y) = \sup(\lambda_{\mathrm{w}_i})(x+y)$$

$$\leq \sup\{\max\{(\lambda_{\mathrm{w}_i})(x), (\lambda_{\mathrm{w}_i})(y)\}\}$$

$$= \max\left\{\sup\left(\lambda_{\mathrm{w}_i}(x)\right), \sup\left(\lambda_{\mathrm{w}_i}(y)\right)\right\}$$

$$= \max\{(\vee \lambda_{\mathrm{w}_i})(x), (\vee \lambda_{\mathrm{w}_i})(y)\}$$

Hence $(\vee \lambda_{\mathrm{w}_i})$ is anti-fuzzy Bd-subalgebra of X.

Hence, R-intersection of $\mathrm{W}_i =< \tilde{\mu}_{\mathrm{w}_i}, \lambda_{\mathrm{w}_i} >$ is a cubic fuzzy of X. ■

Remark 2.17

The P-intresection of any set of cubic fuzzy Bd-subalgebras need not be a cubic fuzzy Bd-subalgebra, for Example:

Example 2.18

let $X = \{0, a, b, c, d, e\}$ be a set with the following table

+	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	0	b	d	e	c
b	b	a	0	e	c	d
c	c	d	e	0	b	a
d	d	e	c	a	0	b
e	e	c	d	b	a	0

Then $(X; +,0)$ is an Bd-algebra. It is easy to show $S_1 = \{0, a, b\}$ and $S_2 = \{0, e\}$ are Bd-subalgebras of X we deftined two cubic Sets $\mathbb{W}_1 = \{X; \mu_{\mathbb{W}_1}(x), \lambda_{\mathbb{W}_2}(x) | x \in X\}$ and $\mathbb{W}_2 = \{X, \tilde{\mu}_{\mathbb{W}_2}(x), \lambda_{\mathbb{W}_2}(x) | x \in X\}$ of X by:

$$\tilde{\mu}_{\mathbb{W}_1}(x) = \begin{cases} [0.5,0.8] & ,if\ x \in S_1 \\ [0.4,0.7] & ,if\ x \in \{c, e\}, \\ [0.3,0.4] & ,otherwise \end{cases} \lambda_{\mathbb{W}_1}(x) = \begin{cases} 0.2 & ,if\ x \in S_1 \\ 0.4 & ,if\ x \in \{c, e\} \\ 0.6 & ,otherwise \end{cases}$$

$$\tilde{\mu}_{\mathbb{W}_2}(x) = \begin{cases} [0.4,0.9] & ,if\ x \in S_2 \\ [0.3,0.7] & ,otherwise \end{cases}, \lambda_{\mathbb{W}_2}(x) = \begin{cases} 0.1 & ,if\ x \in S_2 \\ 0.3 & ,otherwise \end{cases}$$

Thin $\mathbb{W}_1$ and $\mathbb{W}_2$ are cubic fuzzy Bd-subalgebra of X, but P-intersection of

$\mathbb{W}_1$ and $\mathbb{W}_2$ are not cubic fuzzy Bd-subalgebra of X, Since

$(\tilde{\mu}_{\mathbb{W}_1} \cap \tilde{\mu}_{\mathbb{W}_2})(c + a) = \min[0.3,0.4], [0.3,0.7] = [0.3,0.4] \not\geq$

$\min\{(\tilde{\mu}_{\mathbb{W}_1} \cap \tilde{\mu}_{\mathbb{W}_2})(c), (\tilde{\mu}_{\mathbb{W}_1} \cap \tilde{\mu}_{\mathbb{W}_2})(a)\} = \min\{\min\{[0.4,0.7], [0.3,0.7], [0.5,0.8], [0.3,0.7]\}\}$

$= \min\{[0.3,0.7], [0.3,0.7]\}$

$= [0.3,0.7]$

And $(\lambda_{\mathbb{W}_1} \vee \lambda_{\mathbb{W}_2})(c + a) = \max[0.6,0.3] = 0.6 \not\leq$

$= \max\{(\lambda_{\mathbb{W}_1} \vee \lambda_{\mathbb{W}_2})(c), (\lambda_{\mathbb{W}_1} \vee \lambda_{\mathbb{W}_2})(a)\}$

$= \max\{\max\{0.4,0.3\}, \max\{0.2,0.3\} = 0.4$

Proposition 2.19

Let $\mathbb{W}_i =< \tilde{\mu}_{\mathbb{W}_i}, \lambda_{\mathbb{W}_i} >$ be a cubic fuzzy Bd-subalgebra of Bd-algebra $(X; +,0)$, where $i \in \Lambda$,such that $\inf\{\max\{\lambda_{\mathbb{W}_i}(x), \lambda_{w_i}(y)\}\} = \max\{\inf \lambda_{\mathbb{W}_i}(x), \inf \lambda_{\mathbb{W}_i}(y)\}$, for all $x \in X$, then the P-intersection of $\mathbb{W}_i$ is also a cubic fuzzy Bd-subalgebras of X.

Proof:

Let $\mathbb{W}_i =< \tilde{\mu}_{\mathbb{W}_i}, \lambda_{\mathbb{W}_i} >$ where $i \in \Lambda$, be a set of cubic fuzzy

Bd-subalgebra of X , and $x, y \in X$, then

$(\cap \tilde{\mu}_{\mathrm{Ш}_i})(x+y) = r\inf \tilde{\mu}_{\mathrm{Ш}_i}(x+y)$

$\geqslant r\inf\{r\min\{\tilde{\mu}_{\mathrm{Ш}_i}(x), \tilde{\mu}_{\mathrm{Ш}_i}(y)\}\}.$

$= r\min\{r\inf \mu_{\mathrm{Ш}_i}(x), r\inf \mu_{\mathrm{Ш}_i}(y)\}$

$\geqslant r\min\{(\cap \tilde{\mu_{\mathrm{Ш}_i}})(x), (\cap \tilde{\mu_{\mathrm{Ш}_i}})(y)\}.$

Hence $(\cap \tilde{\mu}_{\mathrm{Ш}_i})$ is a fuzzy Bd-subalgebra of X.

and

$(\wedge \lambda_{\mathrm{Ш}_i})(x+y) = \inf \lambda_{\mathrm{Ш}_i}(x+y)$

$\leq \inf\{\max\{\lambda_{\mathrm{Ш}_i}(x), \lambda_{\mathrm{Ш}_i}(y)\}\}.$

$= \max\{\inf \lambda_{\mathrm{Ш}_i}(x), \inf \lambda_{\mathrm{Ш}_i}(y)\}$

$= \max\{(\wedge \lambda_{\mathrm{Ш}_i})(x), (\wedge \lambda_{\mathrm{Ш}_i})(y)\}$

Hence $(\wedge \lambda_{\mathrm{Ш}_i})$ is a anti-fuzzy Bd-subalgebra of X.

Hence, P-intersection of $\mathrm{Ш}_i$ is a cubic fuzzy Bd-subalgebra of X. ■

Proposition 2.20

Let $\mathrm{Ш}_i = < \tilde{\mu}_{\mathrm{Ш}_i}, \lambda_{\mathrm{Ш}_i}(x) >$ be a cubic fuzzy Bd-subalgebra of Bd-algebra $(X; +, 0)$, where $i \in \Lambda$ $r\sup\{r\min\{\tilde{\mu}_{\mathrm{Ш}_i}(x), \tilde{\mu}_{\mathrm{Ш}_i}(y)\}\} = r\min\{r\sup \tilde{\mu}_{\mathrm{Ш}_i}(x), r\sup \tilde{\mu}_{\mathrm{Ш}_i}(y)\}$ for all $x \in X$ thin the P-union of $\mathrm{Ш}_i$ is also a cubic fuzzy Bd-subalgebra of X.

Proof:

Let $\mathrm{Ш}_i = < \tilde{\mu}_{\mathrm{Ш}_i}, \lambda_{\mathrm{Ш}_i} >$ where $i \in \Lambda$, be a set of cubic fuzzy

Bd-subalgebra of X and x,y $\in X$

$(\cup \tilde{\mu}_{\mathrm{Ш}_i})(x+y) = r\sup \tilde{\mu}_{\mathrm{Ш}_i}(x+y)$

$\geqslant r\sup\{r\min\{\tilde{\mu}_{\mathrm{Ш}_i}(x), \tilde{\mu}_{\mathrm{Ш}_i}(y)\}\}.$

$= r\min\{\sup(\mu_{\mathrm{Ш}_i}(x)), \sup(\mu_{\mathrm{Ш}_i}(y))\}$

$= r\min\{(\cup \tilde{\mu}_{\mathrm{Ш}_i})(x), (\cup \tilde{\mu}_{\mathrm{Ш}_i})(y)\}$

Hence $(\cup \tilde{\mu}_{\mathrm{Ш}_i})$ is a fuzzy Bd-subalgebra of X.

and $(\vee \lambda_{\mathrm{Ш}_i})(x+y) = \sup \lambda_{\mathrm{Ш}_i}(x+y)$

$\leq \sup\{\max\{\lambda_{\mathrm{Ш}_i}(x), \lambda_{\mathrm{Ш}_i}(y)\}\}.$

$$= \max\{\sup \lambda_{\mathrm{Ш}_i}(x), \sup \lambda_{\mathrm{Ш}_i}(y)\}$$

$$= \max\{(\vee \lambda_{\mathrm{Ш}_i})(x), (\vee \lambda_{\mathrm{Ш}_i})(y)\}$$

Hence $(\vee \lambda_{\mathrm{Ш}_i})$ is a anti-fuzzy Bd-subalgebra of X.

Hence, P-union of $\mathrm{Ш}_i$ is a cubic fuzzy Bd-subalgebras of X.■

Remark 2.21

The R-union of any sets of cubic fuzzy Bd-subalgebra need not be a cubic

Bd-subalgebra, for Example :

Example 2.22

let $X = \{0, a, b, c, d, e\}$ be a set with the following table

+	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	0	b	d	e	c
b	b	a	0	e	c	d
c	c	d	e	0	b	a
d	d	e	c	a	0	b
e	e	c	d	b	a	0

Then $(X; +, 0)$ is an Bd-algebra. It is easy to show $S_1 = \{0, a, b\}$ and $S_2 = \{0, e\}$ are Bd-subalgebras of X we deftined two cubic Set $\mathrm{Ш}_1 = < \mu_{\mathrm{Ш}_1}, \lambda_{\mathrm{Ш}_1} >$ and

$\mathrm{Ш}_2 = < \mu_{\mathrm{Ш}_2}, \lambda_{\mathrm{Ш}_2} >$ of X by:

$$\tilde{\mu}_{\mathrm{Ш}_1}(x) = \begin{cases} [0.5,0.8] & , if\ x \in S_1 \\ [0.4,0.7] & , if\ x \in \{c, e\}, \\ [0.3,0.4] & , otherwise \end{cases} \lambda_{\mathrm{Ш}_1}(x) = \begin{cases} 0.2 & , if\ x \in S_1 \\ 0.4 & , if\ x \in \{c, e\} \\ 0.6 & , otherwise \end{cases}$$

$$\tilde{\mu}_{\mathrm{Ш}_2}(x) = \begin{cases} [0.4,0.9] & , if\ x \in S_2 \\ [0.3,0.7] & , otherwise \end{cases}, \lambda_{\mathrm{Ш}_2}(x) = \begin{cases} 0.1 & , if\ x \in S_2 \\ 0.3 & , otherwise \end{cases}$$

Thin $\mathrm{Ш}_1$ and $\mathrm{Ш}_2$ are cubic fuzzy Bd-subalgebras of X, but P-union of

$\mathrm{Ш}_1 \cup \mathrm{Ш}_2$ are not cubic fuzzy Bd-subalgebra of X, Since

$(\tilde{\mu}_{\mathrm{Ш}_1} \cup \tilde{\mu}_{\mathrm{Ш}_2})(c + a) = \max[0.3,0.4], [0.3,0.7] = [0.3,0.7] \not\geq$

$\max\{(\tilde{\mu}_{\mathrm{Ш}_1} \cup \tilde{\mu}_{\mathrm{Ш}_2})(c), (\tilde{\mu}_{\mathrm{Ш}_1} \cup \tilde{\mu}_{\mathrm{Ш}_2})(a)\} = \max\{\max\{[0.4,0.7], [0.3,0.7],$

$[0.5,0.8], [0.3,0.7]\}\}$

$= \max\{[0.4,0.7], [0.5,0.8]\}$

$= [0.5,0.8]$

And

$(\lambda_{\mathrm{Ш}_1} \wedge \lambda_{\mathrm{Ш}_2})(c+a) = \min[0.6,0.3] = 0.3 \not\leq$

$= \min\{(\lambda_{\mathrm{Ш}_1} \wedge \lambda_{\mathrm{Ш}_2})(c), (\lambda_{\mathrm{Ш}_1} \wedge \lambda_{\mathrm{Ш}_2})(a)\}$

$= \min\{\min\{0.4,0.3\}, \min\{0.2,0.3\}$

$= 0.2$

Proposition 2.23

Let $\mathrm{Ш}_i = < \tilde{\mu}_{\mathrm{Ш}_i}, \lambda_{\mathrm{Ш}_i} >$ be a cubic fuzzy Bd-subalgebras of Bd-algebra $(X; +,0)$ where $i \in \Lambda$ $r\sup\{r\min\{\tilde{\mu}_{\mathrm{Ш}_i}(x), \tilde{\mu}_{\mathrm{Ш}_i}(y)\}\} = r\min\{r\sup\tilde{\mu}_{\mathrm{Ш}_i}(x), r\sup\tilde{\mu}_{\mathrm{Ш}_i}(y)\}$ and $\inf\{\max\{\lambda_{\mathrm{Ш}_i}(x), \lambda_{\mathrm{Ш}_i}(y)\}\} = \max\{\inf\lambda_{\mathrm{Ш}_i}(x), \inf\lambda_{\mathrm{Ш}_i}(y)\}$ for all $x \in X$ then the R-union of $\mathrm{Ш}_i$ is also a cubic fuzzy Bd-subalgebra of X.

Proof:

Let $\mathrm{Ш}_i = \{< x, \tilde{\mu}_{\mathrm{Ш}_i}(x), \lambda_{\mathrm{Ш}_i}(x) > | x \in X\}$ where $i \in \Lambda$, be a $\mathrm{Ш}_i$ set of cubic fuzzy Bd-subalgebra of X such that

$r\sup\{r\min\{\tilde{\mu}_{\mathrm{Ш}_i}(x), \tilde{\mu}_{\mathrm{Ш}_i}(y)\}\} = r\min\{r\sup\tilde{\mu}_{\mathrm{Ш}_i}(x), r\sup\tilde{\mu}_{\mathrm{Ш}_i}(y)\}$ and $\inf\{\max\{\lambda_{\mathrm{Ш}_i}(x), \lambda_{\mathrm{Ш}_i}(y)\}\} = \max\{\inf\lambda_{\mathrm{Ш}_i}(x), \inf\lambda_{\mathrm{Ш}_i}(y)\}$, for all $x \in X$ then

$x, y \in X$,

$(\cup\, \tilde{\mu}_{\mathrm{Ш}_i})(x+y) = r\sup\tilde{\mu}_{\mathrm{Ш}_i}(x+y)$

$\geqslant r\sup\{r\min\{\tilde{\mu}_{\mathrm{Ш}_i}(x), \tilde{\mu}_{\mathrm{Ш}_i}(y)\}\}.$

$= r\min\{r\sup\mu_{\mathrm{Ш}_i}(x), r\sup\mu_{\mathrm{Ш}_i}(y)\}$

$= r\min\{(\cup\, \tilde{\mu}_{\mathrm{Ш}_i})(x), (\cup\, \tilde{\mu}_{\mathrm{Ш}_i})(y)\}$

Hence $(\cup\, \tilde{\mu}_{\mathrm{Ш}_i})$ is a fuzzy Bd-subalgebra of X.

and $(\wedge\, \lambda_{\mathrm{Ш}_i})(x+y) = \inf\lambda_{\mathrm{Ш}_i}(x+y)$

$\leq \inf\{\max\{\lambda_{\mathrm{Ш}_i}(x), \lambda_{\mathrm{Ш}_i}(y)\}\}.$

$= \max\{\inf\lambda_{\mathrm{Ш}_i}(x), \inf\lambda_{\mathrm{Ш}_i}(y)\}$

$= \max\{(\wedge\, \lambda_{\mathrm{Ш}_i})(x), (\wedge\, \lambda_{\mathrm{Ш}_i})(y)\}$

$\leq \max\{(\wedge\, \lambda_{\mathrm{Ш}_i})(x), (\wedge\, \lambda_{\mathrm{Ш}_i})(y)\}$

Hence $(\wedge\, \lambda_{\mathrm{Ш}_i})$ is a anti- fuzzy Bd-subalgebra of X.

Hence, R-union of $\mathrm{Ш}_i$ is a cubic fuzzy Bd-subalgebra of X.■

Definition 2.24

Let $(X;+,0)$, be an Bd-algebra. A cubic fuzzy subset $\mathbb{W} =< \tilde{\mu}_{\mathbb{W}}, \lambda_{\mathbb{W}} >$ of X for all $\tilde{t} \in D[0,1]$ and $\gamma \in [0,1]$, the set $\tilde{U}(\mathbb{W};\tilde{t},\gamma) = \{x \in X | \tilde{\mu}_{\mathbb{W}}(x) \succcurlyeq \tilde{t}, \lambda_{\mathbb{W}}(x) \leq \gamma\}$ is an level set of X.

Proposition 2.25

Let $(X;+,0)$ be an Bd-algebra. A cubic subset $\mathbb{W} =< \tilde{\mu}_{\mathbb{W}}, \lambda_{\mathbb{W}} >$ of X. If $\mathbb{W}$ is a cubic Bd-subalgebra of X, then. for any $\tilde{t} \in D[0,1]$ and $\gamma \in [0,1]$ th set $\tilde{U}(\mathbb{W};\tilde{t},\gamma)$ is an Bd-subalgebra of X.

Proof:

Assume that $\mathbb{W} =< \tilde{\mu}_{\mathbb{W}}, \lambda_{\mathbb{W}} >$ is a cubic fuzzy Bd-subalgebra of X and let

$\tilde{t} \in D[0,1]$ and $\gamma \in [0,1]$, be such that $\tilde{U}(\mathbb{W};\tilde{t},\gamma) \neq \emptyset$, and let $x, y \in X$ Such that

$x, y \in \tilde{U}(\mathbb{W};\tilde{t},\gamma)$, then $\tilde{\mu}_{\mathbb{W}}(x) \succcurlyeq \tilde{t}$, $\tilde{\mu}_{\mathbb{W}}(y) \succcurlyeq \tilde{t}$ and $\lambda_{\mathbb{W}}(x) \leq \gamma, \lambda_{\mathbb{W}}(y) \leq \gamma$ since $\mathbb{W}$ is a cubic fuzzy Bd-subalgebra of X, we get

$\tilde{\mu}_{\mathbb{W}}(x+y) \succcurlyeq r \min\{\tilde{\mu}_{\mathbb{W}}(x), \tilde{\mu}_{\mathbb{W}}(y)\} \geq \tilde{t}$ and

$\lambda_{\mathbb{W}}(x+y) \leq \max\{\lambda_{\mathbb{W}}(x), \lambda_{\mathbb{W}}(y)\} \leq \gamma$.

Hence the set $\tilde{U}(\mathbb{W};\tilde{t},\gamma)$ is an Bd-subalgebra of X. ■

Proposition 2.26

Let $(X;+,0)$ be an Bd-algebra. A cubic subset $\mathbb{W} =< \tilde{\mu}_{\mathbb{W}}, \lambda_{\mathbb{W}} >$ of X. If for all

$\tilde{t} \in D[0,1]$ and $\gamma \in [0,1]$, the set $\tilde{U}(\mathbb{W};\tilde{t},\gamma)$ is an Bd- subalgebra of X, then w is a cubic fuzzy Bd-subalgebra of X.

Proof:

Suppose that $\tilde{U}(\mathbb{W};\tilde{t},s)$ is an Bd-subalgebra of X and let $x, y \in X$ be such that $\tilde{\mu}_{\mathbb{W}}(x+y) \prec r \min\{\tilde{\mu}_{\mathbb{W}}(x), \tilde{\mu}_{\mathbb{W}}(y)\}$, and

$\lambda_{\mathbb{W}}(x+y) > \max\{\lambda_{\mathbb{W}}(x), \lambda_{\mathbb{W}}(y)\}$.

Consider $\tilde{\beta} = 1/2 \{\tilde{\mu}_{\mathbb{W}}(x+y) + r \min\{\tilde{\mu}_{\mathbb{W}}(x), \tilde{\mu}_{\mathbb{W}}(y)\}\}$ and

$$\tilde{\beta} = 1/2 \{\lambda_{\mathbb{W}}(x+y) + \max\{\lambda_{\mathbb{W}}(x), \lambda_{\mathbb{W}}(y)\}\}$$

We have $\tilde{\beta} \in D[0,1]$ and $\tilde{\beta} \in [0,1]$ and

$\tilde{\mu}_{\mathbb{W}}(x+y) < \tilde{\beta} < r \min\{\tilde{\mu}_{\mathbb{W}}(x), \tilde{\mu}_{\mathbb{W}}(y)\}$, and

$\lambda_{\mathbb{W}}(x+y) > \beta > \max\{\lambda_{\mathbb{W}}(x), \lambda_{\mathbb{W}}(y)\}$

It follows that $x, y \in \tilde{U}(\mathbb{W};\tilde{t},s)$, and $(x+y) \notin \tilde{U}(\mathbb{W};\tilde{t},s)$. This is a contradiction

Hence $\tilde{\mu}_{\mathbb{W}}(x+y) \succcurlyeq r \min\{\tilde{\mu}_{\mathbb{W}}(x), \tilde{\mu}_{\mathbb{W}}(y)\} \succcurlyeq \tilde{t}$, and

$$\lambda_{\mathbb{W}}(x+y) \leq \max\{\lambda_{\mathbb{W}}(x), \lambda_{\mathbb{W}}(y)\} \leq \gamma$$

and therefore $\mathbb{W} =< \tilde{\mu}_{\mathbb{W}}, \lambda_{\mathbb{W}} >$ is a cubic fuzzy Bd-ideal of X.

Proposition 2.27

Cubic set $\mathbb{W} =< \tilde{\mu}_{\mathbb{W}}, \lambda_{\mathbb{W}} >$ is a cubic fuzzy Bd-subalgebra of Bd-algebra X if and only if, $\mu_{\mathbb{W}}^{-}$ and $\mu_{\mathbb{W}}^{+}$ are fuzzy Bd-subalgebras of X and $\lambda_{\mathbb{W}}$ are anti fuzzy

Bd- subalgebra of X.

Proof

Assume that $\mathbb{W}$ is a cubic fuzzy Bd-subalgebra of X, for any $x, y \in X$.

$[\mu_{\mathbb{W}}^{-}(x+y), \mu_{\mathbb{W}}^{+}(x+y)] = \tilde{\mu}_{\mathbb{W}}(x+y)$

$\geqslant r \min\{\tilde{\mu}_{\mathbb{W}}(x), \tilde{\mu}_{\mathbb{W}}(y)\}$

$= r \min\{[\mu_{\mathbb{W}}^{-}(x), \mu_{\mathbb{W}}^{+}(x)], [\mu_{\mathbb{W}}^{-}(y), \mu_{\mathbb{W}}^{+}(y)]\}$

$= [\min\{\mu_{\mathbb{W}}^{-}(x), \mu_{\mathbb{W}}^{-}(x)\}, \min\{\mu_{\mathbb{W}}^{+}(y), \mu_{\mathbb{W}}^{+}(y)\}]$

Thus $\mu_{\mathbb{W}}^{-}(x+y) \geq \min\{\mu_{\mathbb{W}}^{-}(x), \mu_{\mathbb{W}}^{-}(x)\}, \quad \mu_{\mathbb{W}}^{+}(x+y) \geq \min\{\mu_{\mathbb{W}}^{+}(x), \mu_{\mathbb{W}}^{+}(x)\}$

and $\lambda_{\mathbb{W}}(x+y) \leq \max\{\lambda_{\mathbb{W}}(x), \lambda_{\mathbb{W}}(y)\}$,

Conversely, let $\mu_{\mathbb{W}}^{-}$ and $\mu_{\mathbb{W}}^{+}$ are fuzzy Bd-subalgebra of X and $\lambda_{\mathbb{W}}$ are anti-fuzzy

Bd-subalgebra of X and $x, y \in X$, then

$\mu_{\mathbb{W}}^{-}(x+y) \geq \min\{\mu_{\mathbb{W}}^{-}(x), \mu_{\mathbb{W}}^{-}(y)\}, \mu_{\mathbb{W}}^{+}(x+y) \geq \min\{\mu_{\mathbb{W}}^{+}(x), \mu_{\mathbb{W}}^{+}(y)\}$

and $\lambda_{\mathbb{W}}(x+y) \leq \max\{\lambda_{\mathbb{W}}(x), \lambda_{\mathbb{W}}(y)\}$.

Now, $\tilde{\mu}_{\mathbb{W}}(x+y) = [\mu_{\mathbb{W}}^{-}(x+y), \mu_{\mathbb{W}}^{+}(x+y)]$

$\geqslant [\min\{\mu_{\mathbb{W}}^{-}(x), \mu_{\mathbb{W}}^{-}(y), \mu_{\mathbb{W}}^{+}(x), \mu_{\mathbb{W}}^{+}(y)\}]$

=rmin{[$\mu_{\mathbb{W}}^{-}$(x), $\mu_{\mathbb{W}}^{+}$(x)], $\mu_{\mathbb{W}}^{-}$(y), $\mu_{\mathbb{W}}^{+}$(y)]

=rmin{, $\tilde{\mu}_{\mathbb{W}}(x)$, , $\tilde{\mu}_{\mathbb{W}}(y)$}, therefore

Hence $\mathbb{W}$ is a cubic fuzzy Bd-subalgebra of X. ■

Theorem 2.28

if a cubic set $\mathbb{W} =< \tilde{\mu}_{\mathbb{W}}, \lambda_{\mathbb{W}} >$ is a cubic subalgebra of X, then the upper $[\gamma_1, \gamma_2]$. Level and Lower t-Level of $\mathbb{W}$ are subalgebras of X.

Proof:

Let $x, y \in U(\tilde{\mu}_{\mathbb{W}} | [\gamma_1, \gamma_2])$, then $\tilde{\mu}_{\mathbb{W}}(x) \geqslant [\gamma_1, \gamma_2]$ and $\tilde{\mu}_{\mathbb{W}}(y) \geqslant [\gamma_1, \gamma_2]$. It follows that

$\tilde{\mu}_{\mathbb{W}}(x+y) \geqslant r \min\{\tilde{\mu}_{\mathbb{W}}(x), \tilde{\mu}_{\mathbb{W}}(y)\} \geqslant [\gamma_1, \gamma_2]$, So that

$x + y \in U(\tilde{\mu}_{\mathbb{W}} | [\gamma_1, \gamma_2])$

Hence $U(\tilde{\mu}_{\mathrm{W}}|[\gamma_1,\gamma_2])$ is Bd-subalgebra of X.

Let $x, y \in L(\lambda_{\mathrm{W}}|t)$, then $\lambda_{\mathrm{W}}(x) \leq t$ and $\lambda_{\mathrm{W}}(y) \leq t$. It follows that $\lambda_{\mathrm{W}}(x+y) \leq \max\{\lambda_{\mathrm{W}}(x), \lambda_{\mathrm{W}}(y)\} \leq t$, so that x+$y \in L(\lambda_{\mathrm{W}}|t)$.

Hence $L(\lambda_{\mathrm{W}}|t)$ is Bd-subalgebra of X. ■

Corollary 2.29

let $\mathrm{W} =< \tilde{\mu}_{\mathrm{W}}, \lambda_{\mathrm{W}} >$ be a cubic fuzzy Bd-subalgebra of Y, then

$\mathrm{W}([\gamma_1,\gamma_2]; t) = U(\tilde{\mu}_{\mathrm{W}}|[\gamma_1,\gamma_2]) \cap L(\lambda_{\mathrm{W}}|t) = \{x \in X|\tilde{\mu}_{\mathrm{W}}(x) \succcurlyeq [S_1, S_2], \lambda_{\mathrm{W}}(x) \leq t\}$ is a cubic fuzzy Bd-subalgebra of X.

The following example shows that the converse of corollary ( ) is not valid

Example 2.30

let $X = \{0, a, b, c, d, e\}$ be Bd-algebra in example ( ) and cubic set $\mathrm{W} = (\tilde{\mu}_{\mathrm{W}}, \lambda_{\mathrm{W}})$

of X by

$$\tilde{\mu}_{\mathrm{W}}(x) = \begin{cases} [0.6,0.8] & ,if\ x = 0 \\ [0.5,0.6] & ,if\ x \in \{c,d,e\}, \\ [0.3,0.4] & ,if\ x \in \{a,b\} \end{cases} \lambda_{\mathrm{W}_1}(x) = \begin{cases} 0.1 & ,if\ x = 0 \\ 0.3 & ,if\ x \in \{a,,b,c,d\} \\ 0.8 & ,if\ x \in \{a,e\} \end{cases}$$

$\mathrm{W}([\gamma_1,\gamma_2]; t) = U(\tilde{\mu}_{\mathrm{W}}|[\gamma_1,\gamma_2]) \cap L(\lambda_{\mathrm{W}}|t)$

$= \{x \in X|\tilde{\mu}_{\mathrm{W}}(x) \succcurlyeq [\gamma_1,\gamma_2], \lambda_{\mathrm{W}}(x) \leq t\}$

$= \{0,c,d,e\} \cap \{0,,b,c,d,\} = \{0,c,d\}$

Bd-subalgebra of X, but $\mathrm{W} =< \tilde{\mu}_{\mathrm{W}}, \lambda_{\mathrm{W}} >$ is not a cubic fuzzy Bd-subalgebra since $\tilde{\mu}_{\mathrm{W}}(d+c) \not\succcurlyeq r \min\{\tilde{\mu}_{\mathrm{W}}(d), \tilde{\mu}_{\mathrm{W}}(c)\}$ and $\lambda_{\mathrm{W}}(d+c) \not\leq \max\{\lambda_{\mathrm{W}}(d), \lambda_{\mathrm{W}}(c)\}$

Theorem 2.31

let $\mathrm{W} =< \tilde{\mu}_{\mathrm{W}}, \lambda_{\mathrm{W}} >$ be a cubic fuzzy subset of X Such that the sets $(\tilde{\mu}_{\mathrm{W}}|[\gamma_1,\gamma_2])$ and Bd-subalgebras of X, for every $[\gamma_1,\gamma_2]$ and $L(\lambda_{\mathrm{W}}|t)$,$t \in [0,1]$, then

$\mathrm{W} =< \tilde{\mu}_{\mathrm{W}}, \lambda_{\mathrm{W}} >$ is a cubic fuzzy Bd-subalgebra of X.

Proof:

Let $U(\tilde{\mu}_{\mathrm{W}}|[\gamma_1,\gamma_2])$ and $L(\lambda_{\mathrm{W}}|t)$ are Bd-subalgebras of X, for every

$[\gamma_1,\gamma_2] \in D[0,1]$ and $t \in [0,1]$ on the contrary. let $x_0, y_0 \in X$ be such that

$\tilde{\mu}_{\mathrm{W}}(x_0 + y_0) < r \min\{\tilde{\mu}_{\mathrm{W}}(x_0), \tilde{\mu}_{\mathrm{W}}(y_0)\}$.

Let $\tilde{\mu}_{\mathrm{W}}(x_0) = [\eta_1,\eta_2]$ and $\tilde{\mu}_{\mathrm{W}}(y_0) = [\eta_1,\eta_2]$ and $\tilde{\mu}_{\mathrm{W}}(x_0 + y_0) = [\gamma_1,\gamma_2]$. Than

$[\gamma_1,\gamma_2] < r \min\{[\eta_1,\eta_2],[\eta_3,\eta_4]\} = \{\min[\eta_1,\eta_3], \min[\eta_2,\eta_3]\}$.

So $\gamma_1 < \min\{[\eta_1,\eta_2] \cdot [\eta_3,\eta_4]\} = \{\min\{\eta_1,\eta_3\}$ and $\gamma_2 < \min\{\eta_2,\eta_4\}$. Let us considers,

$[\gamma_1, \gamma_2] = \frac{1}{2}\ [\tilde{\mu}_{\text{Ш}}(x_0 + y_0) + r \min\{\tilde{\mu}_{\text{Ш}}(x_0), \tilde{\mu}_{\text{Ш}}(y_0)\}]$

$= \frac{1}{2}\big[[\gamma_1, \gamma_2] + [\min\{\eta_1, \eta_3\}, \min\{\eta_2, \eta_4\}]\big]$

$= \frac{1}{2}(\gamma_1 + \min\{\eta_1, \eta_3\}), \frac{1}{2}(\gamma_2 + \min\{\eta_2, \eta_4\})$

Therefore $\min\{\eta_1, \eta_3\} > \rho_1 = \frac{1}{2}(\gamma_1 + \min\{\eta_1, \eta_3\}) > \gamma_1$

and $\min\{\eta_2, \eta_4\} > \rho_2 = \frac{1}{2}(\gamma_2 + \min\{\eta_2, \eta_4\}) > \gamma_2$

Hence $[\min\{\eta_1, \eta_3\}, \min\{\eta_2, \eta_4\}] > [\rho_1, \rho_2] > [\gamma_1, \gamma_2]$, so that

$(x_0 + y_0) \notin U(\tilde{\mu}_{\text{Ш}} | [\gamma_1, \gamma_2])$ which is a contradiction, since $\tilde{\mu}_{\text{Ш}}(x_0) = [\eta_1, \eta_2] > \min\{\eta_1, \eta_3\}, \min\{\eta_2, \eta_4\} > [\rho_1, \rho_2]$ this implies

$(x_0 + y_0) \in U(\tilde{\mu}_{\text{Ш}} | [\gamma_1, \gamma_2])$. Thus

$\tilde{\mu}_{\text{Ш}}(x + y) \succcurlyeq r \min\{\tilde{\mu}_{\text{Ш}(x)}, \tilde{\mu}_{\text{Ш}}(y)\}$ for all $x, y \in X$.

and, Let $x_0, y_0 \in X$ such that

$\lambda_{\text{Ш}}(x_0 + y_0) > \max\{\lambda_{\text{Ш}}(x_0, \lambda_{\text{Ш}}(y_0)\}$

Let $\lambda_{\text{Ш}}(x_0) = \nu_1, \lambda_{\text{Ш}}(y_0) = \nu_2$ and $\lambda_{\text{Ш}}(x_0 + y_0) = t$

Then $t > \max\{\nu_1, \nu_2\}$. let us consider

$t_1 = \frac{1}{2}[\lambda_{\text{Ш}}(x_0 + y_0) + \max\{\lambda_{\text{Ш}}(x_0), \lambda_{\text{Ш}}(y_0)\}]$

we get that $t_1 = \frac{1}{2}(t + \max\{\nu_1, \nu_2\})$, therefore $\nu_1 < t_1 = \frac{1}{2}(t + \max\{\nu_1, \nu_2\}) < t$ and $\nu_2 < t_2 = \frac{1}{2}(t + \max\{\nu_1, \nu_2\}) < t$, hence,

$\max\{\nu_1, \nu_2\} < t_1 = \lambda_{\text{Ш}}(x_0 + y_0)$

3. CUBIC FUZZY BD-IDEALS OF BD-ALGEBRA

In this section, the researcher introduces a new notion called cubic fuzzy Bd-ideal of Bd-algebras and studies several properties.

Definition 3.1

Let $(X; +, 0)$ be an Bd-algebra. A cubic set $\text{Ш} = < \tilde{\mu}_{\text{Ш}}(x), \lambda_{\text{Ш}}(x) >$ of X is called cubic fuzzy Bd-ideal of X if, for all $x, y \in X$.

(1) $\tilde{\mu}_{\text{Ш}}(0) \succcurlyeq \tilde{\mu}_{\text{Ш}}(x)$, and $\lambda_{\text{Ш}}(0) \leq \lambda_{\text{Ш}}(x)$,

(2) $\tilde{\mu}_{\text{Ш}}(x) \succcurlyeq r \min\{\tilde{\mu}_{\text{Ш}}(x + y), \tilde{\mu}_{\text{Ш}}(y)\}$, and

$\lambda(x) \leq r \max\{\lambda_{\text{Ш}}(x + y), \lambda_{\text{Ш}}(y)\}$

(3) $\tilde{\mu}_{\mathrm{Ш}}(x+y) \geqslant r\min\{\tilde{\mu}_{\mathrm{Ш}}(x), \tilde{\mu}_{\mathrm{Ш}}(y)\}$, and

$\lambda_{\mathrm{Ш}}(x+y) \leq \max\{\lambda_{\mathrm{Ш}}(x), \lambda_{\mathrm{Ш}}(y)\}$.

Example 3.2

Let $X = \{0, a, b, c\}$ be Bd-algebra in Example (3.2.2), It is easy to show that

$I = \{0, c\}$ is Bd-ideal of X. Define a cubic set $\mathrm{Ш} = < \tilde{\mu}_{\mathrm{Ш}}, \lambda_{\mathrm{Ш}} >$ of X of such that $: X \to [0,1]$ by:

$\tilde{\mu}_{\mathrm{Ш}}(x) = \begin{cases} [0.3,0.9] & if\ x \in [0,c] \\ [0.2,0.7] & otherwise \end{cases}$ and $\lambda_{\mathrm{Ш}} = \begin{cases} 0.3 & if\ x \in [0,c] \\ 0.4 & otherwise \end{cases}$. The cubic set. $\mathrm{Ш} = < \tilde{\mu}_{\mathrm{Ш}}, \lambda_{\mathrm{Ш}} >$ is a cubic fuzzy Bd-ideal of X.

Proposition 3.3

The R-intersection of any set of cubic fuzzy Bd- ideal of X is also cubic fuzzy

Bd-ideal of X.

Proof:

Let $\{\mathrm{Ш}_i | i \in \Lambda\}$ be a family of cubic fuzzy Bd-ideals of X, then for any $x, y \in X$.

$(\cap \tilde{\mu}_{\mathrm{Ш}_i})(0) = r\inf\left(\tilde{\mu}_{\mathrm{Ш}_i}(0)\right) \geqslant r\inf\left(\tilde{\mu}_{\mathrm{Ш}_i}(x)\right) = (\cap \tilde{\mu}_{\mathrm{Ш}_i})(x)$ and

$(\vee \lambda_{\mathrm{Ш}_i})(0) = \sup \lambda_{\mathrm{Ш}_i}(0) \leq \sup \lambda_{\mathrm{Ш}_i}(y) = (\vee \lambda_{\mathrm{Ш}_i})(y)$.

$$(\cap \tilde{\mu}_{\mathrm{Ш}_i})(x) = r\inf\left(\tilde{\mu}_{\mathrm{Ш}_i}(x)\right) \geqslant r\inf\left\{r\min\left(\tilde{\mu}_{\mathrm{Ш}_i}(x+y), \tilde{\mu}_{\mathrm{Ш}_i}(y)\right)\right\}$$

$$= r\min\left\{r\inf\left(\tilde{\mu}_{\mathrm{Ш}_i}(x+y)\right), r\inf\left(\tilde{\mu}_{\mathrm{Ш}_i}(y)\right)\right\}$$

$$= r\min\{(\cap \tilde{\mu}_{\mathrm{Ш}_i})(x+y), (\cap \tilde{\mu}_{\mathrm{Ш}_i})(y)\}$$

$$(\vee \lambda_{\mathrm{Ш}_i})(x) = \sup\left(\lambda_{\mathrm{Ш}_i}(x)\right)$$

$$\leq \sup\{\max\{\lambda_{\mathrm{Ш}_i}(x+y), \lambda_{\mathrm{Ш}_i}(y)\}\}$$

$$= \max\left\{\sup\left(\lambda_{\mathrm{Ш}_i}(x+y)\right), \sup\left(\lambda_{\mathrm{Ш}_i}(y)\right)\right\}$$

$$= \max\{(\vee \lambda_{\mathrm{Ш}_i})(x+y), \sup(\lambda_{\mathrm{Ш}_i})(y)\}$$

Hence, R-intersection of $\mathrm{Ш}_i$ is a cubic fuzzy Bd-ideal of X.

Remark 3.4

The P-intersection of any sets of cubic fuzzy Bd-ideal need not be a cubic fuzzy Bd-ideal, for Example

Example 3.5

let $X = \{0, a, b, c, d, e\}$ be a set with the following table

+	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	0	a	a	a	a
b	b	b	0	a	a	a
c	c	c	c	0	c	c
d	d	d	d	d	0	a
e	e	e	e	e	e	0

Then $(X;+,0)$ is an Bd-algebra. It is easy to show $I_1 = \{0, a, b\}$ and $I_2 = \{0, e\}$ are Bd-ideals of X we deftined two cubic Set $\mathrm{Ш}_1 = < \mu_{\mathrm{Ш}_1}, \lambda_{\mathrm{Ш}_1} >$ and

$\mathrm{Ш}_2 =< \mu_{\mathrm{Ш}_2}, \lambda_{\mathrm{Ш}_2} >$ of X by:

$$\tilde{\mu}_{\mathrm{Ш}_1}(x) = \begin{cases} [0.5,0.8] & ,if\ x \in I_1 \\ [0.4,0.7] & ,if\ x \in \{c,e\}, \\ [0.3,0.4] & ,otherwise \end{cases} \lambda_{\mathrm{Ш}_1}(x) = \begin{cases} 0.2 & ,if\ x \in I_1 \\ 0.4 & ,if\ x \in \{c,e\} \\ 0.6 & ,otherwise \end{cases}$$

$$\tilde{\mu}_{\mathrm{Ш}_2}(x) = \begin{cases} [0.4,0.9] & ,if\ x \in I_2 \\ [0.3,0.7] & ,otherwise \end{cases}, \lambda_{\mathrm{Ш}_2}(x) = \begin{cases} 0.1 & ,if\ x \in I_2 \\ 0.3 & ,otherwise \end{cases}$$

Then $\mathrm{Ш}_1$ and $\mathrm{Ш}_2$ are cubic fuzzy Bd-ideal of X, but P-intersection

are not cubic fuzzy Bd-ideals of X.Since

$\left(\tilde{\mu}_{\mathrm{Ш}_i} \cap \tilde{\mu}_{\mathrm{Ш}_2}\right)(d) = [0.3,0.4] \not\leqslant [0.3,0.7]$

$$\min\left\{\left(\tilde{\mu}_{\mathrm{Ш}_1} \cap \tilde{\mu}_{\mathrm{Ш}_2}\right)(c+a), \left(\tilde{\mu}_{\mathrm{Ш}_1} \cap \tilde{\mu}_{\mathrm{Ш}_2}\right)(a)\right\}$$

And

$\left(\lambda_{\mathrm{Ш}_1} \wedge \lambda_{\mathrm{Ш}_2}\right)(d) = \min[0.6,0.3] = 0.3 \not\leq 0.2$

$$= \min\left\{\left(\lambda_{\mathrm{Ш}_1} \wedge \lambda_{\mathrm{Ш}_2}\right)(c), \left(\lambda_{\mathrm{Ш}_1} \wedge \lambda_{\mathrm{Ш}_2}\right)(a)\right\}$$

$$= \min\{\min\{0.4,0.3\}, \min\{0.2,0.3\}$$

Proposition 3.6

Let $\mathrm{Ш} =< \tilde{\mu}_{\mathrm{ш}_i}, \lambda_{\mathrm{ш}_i} >$ be a cubic fuzzy Bd-ideal of Bd-algebra $(X; +,0)$, where

$i \in \Lambda, \inf\{\max\{\lambda_{\mathrm{ш}_i}(x), \lambda_{\mathrm{ш}_i}(y)\} = \max\{\inf \lambda_{\mathrm{ш}_i}(x), \lambda_{\mathrm{ш}_i}(y)\}$,

for all $x \in X$, then the P-intersection of $\mathrm{Ш}_i$ is also a cubic fuzzy Bd-ideal of X.

Proof:

Let $\mathrm{Ш}_i =< \tilde{\mu}_{\mathrm{ш}_i}, \lambda_{\mathrm{ш}_i} >$ where $i \in \Lambda$, be a set of cubic fuzzy

Bd-ideals of X, for all $x \in X$, then $x, y \in X$,

$(\cap \tilde{\mu}_{\mathrm{ш}_i})(0) = r \inf(\tilde{\mu}_{\mathrm{ш}_i}(0) \geqslant r \inf\left(\tilde{\mu}_{\mathrm{ш}_i}(x)\right) = (\cap \tilde{\mu}_{\mathrm{ш}_i})(x)$

$((\wedge \lambda_{\mathrm{ш}_i})(0) = \inf \lambda_{\mathrm{ш}_i}(0) \leq inf\ \lambda_{\mathrm{ш}_i}(y) \leq (\wedge \lambda_{\mathrm{ш}_i})(y)$

$(\cap \tilde{\mu}_{\mathrm{ш}_i})(x) = r \inf \mu_{\tilde{\mathrm{ш}}_i}(x)$

$\geqslant r \inf\{r \min\{\tilde{\mu}_{\mathrm{ш}_i}(x + y), \tilde{\mu}_{\mathrm{ш}_i}(y)\}\}$

$= r \min\{r \inf \tilde{\mu}_{\mathrm{ш}_i}(x + y), r \inf \tilde{\mu}_{\mathrm{ш}_i}(y)\}$

$= r \min\{(\cap \tilde{\mu}_{\mathrm{ш}_i})(x + y), (\cap \tilde{\mu}_{\mathrm{ш}_i})(y)\}$

And $(\wedge \lambda_{\mathrm{ш}_i})(x) = \inf \lambda_{\mathrm{ш}_i}(x)$

$\leq \inf\{\max\{\lambda_{\mathrm{ш}_i}(x + y), \lambda_{\mathrm{ш}_i}(y)\}\}$

$= \max\{\inf \lambda_{\mathrm{ш}_i}(x + y), \inf \lambda_{\mathrm{ш}_i}(y)\}$

$= \max\{(\wedge \lambda_{\mathrm{ш}_i})(x + y), (\wedge \lambda_{\mathrm{ш}_i})(y)\}$

$\leq \min\{(\wedge \lambda_{\mathrm{ш}_i})(x + y), (\wedge \lambda_{\mathrm{ш}_i})(y)\}$

And

$(\cap \tilde{\mu}_{\mathrm{ш}_i})(x + y) = r \inf \tilde{\mu}_{\mathrm{ш}_i}(x + y)$

$\geqslant r \inf\{r \min\{\tilde{\mu}_{\mathrm{ш}_i}(x), \tilde{\mu}_{\mathrm{ш}_i}(y)\}\}$

$= r \min\{r \inf \tilde{\mu}_{\mathrm{ш}_i}(x), r \inf \tilde{\mu}_{\mathrm{ш}_i}(y)\}$

$= r \min\{(\cap \tilde{\mu}_{\mathrm{ш}_i})(x), (\cap \tilde{\mu}_{\mathrm{ш}_i})(y)\}$

And $(\wedge \lambda_{\mathrm{ш}_i})(x + y) = \inf \lambda_{\mathrm{ш}_i}(x + y)$

$\leq \inf\{\max\{\lambda_{\mathrm{ш}_i}(x), \lambda_{\mathrm{ш}_i}(y)\}\}$

$= \max\{\inf \lambda_{\mathrm{ш}_i}(x), \inf \lambda_{\mathrm{ш}_i}(y)\}$

$= \max\{(\wedge \lambda_{\mathrm{W}_i})(x), (\wedge \lambda_{\mathrm{W}_i})(y)\}$

$\leq \min\{(\wedge \lambda_{\mathrm{W}_i})(x), (\wedge \lambda_{\mathrm{W}_i})(y)\}$

Hence, P-intersection of W_i is a cubic fuzzy Bd-ideal of X. ■

Remark 3.7

The R-union of any sets of cubic fuzzy Bd-ideal need not be a cubic Bd-ideal, for Example

Example 3.8

let $X = \{0, a, b, c, d, e\}$ be a set with the following table

+	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	0	a	a	a	a
b	b	b	0	a	a	a
c	c	c	c	0	c	c
d	d	d	d	d	0	a
e	e	e	e	e	e	0

Then $(X; +, 0)$ is an Bd-algebra. It is easy to show $I_1 = \{0, a, b\}$ and $I_2 = \{0, e\}$ are Bd-ideals of X we deftined two cubic Set $\mathrm{W}_1 = < \mu_{\mathrm{W}_1}, \lambda_{\mathrm{W}_1} >$ and

$\mathrm{W}_2 =< \mu_{\mathrm{W}_2}, \lambda_{\mathrm{W}_2} >$ of X by:

$$\tilde{\mu}_{\mathrm{W}_1}(x) = \begin{cases} [0.5,0.8] & ,if\ x \in I_1 \\ [0.4,0.7] & ,if\ x \in \{c, e\}, \\ [0.3,0.4] & ,otherwise \end{cases} \lambda_{\mathrm{W}_1}(x) = \begin{cases} 0.2 & ,if\ x \in I_1 \\ 0.4 & ,if\ x \in \{c, e\} \\ 0.6 & ,otherwise \end{cases}$$

$$\tilde{\mu}_{\mathrm{W}_2}(x) = \begin{cases} [0.4,0.9] & ,if\ x \in I_2 \\ [0.3,0.7] & ,otherwise \end{cases}, \lambda_{\mathrm{W}_2}(x) = \begin{cases} 0.1 & ,if\ x \in I_2 \\ 0.3 & ,otherwise \end{cases}$$

Then W_1 and W_2 are cubic fuzzy Bd-ideal of X, but R-union are not cubic fuzzy Bd-ideals of X.Since

$(\tilde{\mu}_{\mathrm{W}_i} \cup \tilde{\mu}_{\mathrm{W}_2})(d) = [0.3,0.7] \not\leq$

$$\min\{(\tilde{\mu}_{\mathrm{W}_1} \cup \tilde{\mu}_{\mathrm{W}_2})(c + a), (\tilde{\mu}_{\mathrm{W}_1} \cup \tilde{\mu}_{\mathrm{W}_2})(a)\}$$

$= [0.5,0.8]$

And

$(\lambda_{\mathrm{W}_1} \wedge \lambda_{\mathrm{W}_2})(d) = \min[0.6,0.3] = 0.3 \not\leq$

$$= \min\{(\lambda_{\mathrm{W}_1} \wedge \lambda_{\mathrm{W}_2})(c), (\lambda_{\mathrm{W}_1} \wedge \lambda_{\mathrm{W}_2})(a)\}$$

$= \min\{\min\{0.4,0.3\}, \min\{0.2,0.3\}$

$= 0.2$

Proposition 3.9

Let $\mathrm{Ш} =< \grave{\mu}_{\mathrm{Ш}_i}, \lambda_{\mathrm{Ш}_i} >$ be a cubic fuzzy Bd-ideal of Bd-algebra $(X; +,0)$, where

$i \in \Lambda, r\sup\{r\min\{\tilde{\mu}_{\mathrm{Ш}_i}(x), \tilde{\mu}_{\mathrm{Ш}_i}(y)\} = r\min\{r\sup\tilde{\mu}_{\mathrm{Ш}_i}(x), \tilde{\mu}_{\mathrm{Ш}_i}(y)\}$, for all $x \in X$, then the P-union of $\mathrm{Ш}_i$ is also a cubic fuzzy Bd-ideal of X.

Proof:

Let $\mathrm{Ш}_i = \{< x, \tilde{\mu}_{\mathrm{Ш}_i}(x), \lambda_{\mathrm{Ш}_i}(x) > | x \in X\}$ where $i \in \Lambda$, be a set of cubic fuzzy

Bd-ideal of X ,then $\forall\, x, y \in X$,

$(\cup\, \tilde{\mu}_{\mathrm{Ш}_i})(0) = \sup(\tilde{\mu}_{\mathrm{Ш}_i}(0) \geqslant r\, sup\left(\tilde{\mu}_{\mathrm{Ш}_i}(x)\right) = (\cup\, \tilde{\mu}_{\mathrm{Ш}_i})(x)$

$$(\vee\lambda_{\mathrm{Ш}_i})(0) = \sup(\lambda_{\mathrm{Ш}_i}(0) \leq sup\left(\lambda_{\mathrm{Ш}_i}(y)\right) = (\vee\tilde{\mu_{\mathrm{Ш}_i}})(y)$$

$(\cup\, \tilde{\mu}_{\mathrm{Ш}_i})(x) = r\sup\tilde{\mu}_{\mathrm{Ш}_i}(x)$

$\geqslant r\sup\{r\min\tilde{\mu}_{\mathrm{Ш}_i}(x+y), \tilde{\mu}_{\mathrm{Ш}_i}(y)\}$

$= r\min\{r\sup\tilde{\mu}_{\mathrm{Ш}_i}(x+y), r\sup\tilde{\mu}_{\mathrm{Ш}_i}(y)\}$

$= r\min(\cup\, \tilde{\mu}_{\mathrm{Ш}_i})(x+y), (\cup\, \tilde{\mu}_{\mathrm{Ш}_i})(y)$ and

$(\vee\, \lambda_{\mathrm{Ш}_i})(x) = \sup\lambda_{\mathrm{Ш}_i}(x)$

$\leq \sup\{\max\{\lambda_{\mathrm{Ш}_i}(x+y), \lambda_{\mathrm{Ш}_i}(y)\}\}$

$= \max\{\sup\lambda_{\mathrm{Ш}_i}(x+y), \sup\lambda_{\mathrm{Ш}_i}(y)\}$

$= \max\{(\vee\, \lambda_{\mathrm{Ш}_i})(x+y), (\vee\, \lambda_{\mathrm{Ш}_i})(y)\}$

And

$(\cup\, \tilde{\mu}_{\mathrm{Ш}_i})(x+y) = r\sup\tilde{\mu}_{\mathrm{Ш}_i}(x+y)$

$\geqslant r\sup\{r\min\tilde{\mu}_{\mathrm{Ш}_i}(x), \tilde{\mu}_{\mathrm{Ш}_i}(y)\}$

$= r\min\{r\sup\tilde{\mu}_{\mathrm{Ш}_i}(x), r\sup\tilde{\mu}_{\mathrm{Ш}_i}(y)\}$

$= r\min\{(\cup\, \tilde{\mu}_{\mathrm{Ш}_i}(x), (\cup\, \tilde{\mu}_{\mathrm{Ш}_i})(y)\}$ and

$(\vee\, \lambda_{\mathrm{Ш}_i})(x+y) = \sup\lambda_{\mathrm{Ш}_i}(x+y)$

$\leq \sup\{\max\{\lambda_{\mathrm{Ш}_i}(x), \lambda_{\mathrm{Ш}_i}(y)\}\}$

$$= \max\{\sup \lambda_{\mathbb{W}_i}(x), \sup \lambda_{\mathbb{W}_i}(y)\}$$

$$= \max\{(\vee \lambda_{\mathbb{W}_i})(x), (\vee \lambda_{\mathbb{W}_i})(y)\}$$

Hence, P-union of $\mathbb{W}_i$ is a cubic fuzzy Bd-ideal of X. ■

Proposition 3.10

Let $\mathbb{W}_i = < \tilde{\mu}_{\mathbb{W}_i}, \lambda_{\mathbb{W}_i} >$ be a cubic fuzzy Bd-ideal of Bd-algebra Where $i \in \Lambda, r\sup\{r\min\{\tilde{\mu}_{\mathbb{W}_i}(x), \tilde{\mu}_{\mathbb{W}_i}(y)\} = r\min\{r\sup\tilde{\mu}_{\mathbb{W}_i}(x), r\sup\tilde{\mu}_{\mathbb{W}_i}(y)\}$, and $\inf\{\max\{\lambda_{\mathbb{W}_i}(x), \lambda_{\mathbb{W}_i}(y)\} = \max\{\inf\lambda_{\mathbb{W}_i}(x), \lambda_{\mathbb{W}_i}(y)\}$ for all $x \in X$ then the

R-union of $\mathbb{W}_i$ is also a cubic fuzzy Bd-ideal of X.

Proof:

Let $\mathbb{W}_i = \{< \tilde{\mu}_{\mathbb{W}_i}(x), \lambda_{\mathbb{W}_i}(x) > | x \in X\}$ where $i \in \Lambda$, be a set of cubic fuzzy Bd-ideal of X, then for $x, y \in X$,

$(\cup \tilde{\mu}_{\mathbb{W}_i})(0) = r\sup(\tilde{\mu}_{\mathbb{W}_i}(0)) \geqslant r\sup(\tilde{\mu}_{\mathbb{W}_i}(x)) = (\cup \tilde{\mu}_{\mathbb{W}_i})(x)$ and

$$(\wedge \lambda_{\mathbb{W}_i})(0) = \inf\lambda_{\mathbb{W}_i}(0) \leq inf\ \lambda_{\mathbb{W}_i}(y) \leq (\wedge \lambda_{\mathbb{W}_i})(y)$$

$$(\cup \tilde{\mu}_{\mathbb{W}_i})(x) = r\sup\tilde{\mu}_{\mathbb{W}_i}(x)$$

$$\geqslant r\sup\{r\min\{\tilde{\mu}_{\mathbb{W}_i}(x+y), \tilde{\mu}_{\mathbb{W}_i}(y)\}\}$$

$$= r\min\{r\sup\tilde{\mu}_{\mathbb{W}_i}(x+y), r\sup\tilde{\mu}_{\mathbb{W}_i}(y)\}$$

$$= r\min\{(\cup \tilde{\mu}_{\mathbb{W}_i})(x+y), (\cup \tilde{\mu}_{\mathbb{W}_i})(y)\}$$

And $(\wedge \lambda_{\mathbb{W}_i})(x) = \inf\lambda_{\mathbb{W}_i}(x)$

$$\leq \inf\{\max\{\lambda_{\mathbb{W}_i}(x+y), \lambda_{\mathbb{W}_i}(y)\}\}$$

$$= \max\{\inf\lambda_{\mathbb{W}_i}(x+y), \inf\lambda_{\mathbb{W}_i}(y)\}$$

$$= \max\{(\wedge \lambda_{\mathbb{W}_i})(x+y), (\wedge \lambda_{\mathbb{W}_i})(y)\}$$

And

$(\cup \tilde{\mu}_{\mathbb{W}_i})(x+y) = r\sup\tilde{\mu}_{\mathbb{W}_i}(x+y)$

$$\geqslant r\sup\{r\min\{\tilde{\mu}_{\mathbb{W}_i}(x), \tilde{\mu}_{\mathbb{W}_i}(y)\}\}$$

$$= r\min\{r\sup\tilde{\mu}_{\mathbb{W}_i}(x), r\sup\tilde{\mu}_{\mathbb{W}_i}(y)\}$$

$$= r\min\{(\cup \tilde{\mu}_{\mathbb{W}_i})(x), (\cup \tilde{\mu}_{\mathbb{W}_i})(y)\}$$

And $(\wedge \lambda_{\mathbb{W}_i})(x+y) = \inf\lambda_{\mathbb{W}_i}(x+y)$

$$\leq \inf\{\max\{\lambda_{\mathbb{W}_i}(x), \lambda_{\mathbb{W}_i}(y)\}\}$$

$$= \max\{\inf \lambda_{\mathrm{W}_i}(x), \inf \lambda_{\mathrm{W}_i}(y)\}$$

$$\leq \max\{(\wedge \lambda_{\mathrm{W}_i})(x), (\wedge \lambda_{\mathrm{W}_i})(y)\}$$

Hence, R-union of W_i is a cubic fuzzy Bd-ideal of X. ■

Proposition 3.11

Let $(X; +, 0)$ be an Br-algebra. If a cubic subset $\mathrm{W} = < \tilde{\mu}_{\mathrm{W}_i}, \lambda_{\mathrm{W}_i} >$ of X, then W is a cubic fuzzy Bd-ideal of X, then for any $\tilde{t} \in D[0,1]$ and $\gamma \in [0,1]$, the set $\tilde{U}(\mathrm{W}; \tilde{t}, s)$ is an Bd-ideal of X.

Proof:

Assume that $\mathrm{W} = < \tilde{\mu}_{\mathrm{W}}, \lambda_{\mathrm{W}} >$ is a cubic fuzzy Bd-ideal of X and let $\tilde{t} \in D[0,1]$ and $\gamma \in [0,1]$, such that $\tilde{U}(\mathrm{W}; \tilde{t}, \gamma) \neq \emptyset$, and let $x, y \in X$ such that $x + y, y \in \tilde{U}(\mathrm{W}; \tilde{t}, \gamma)$, then $\tilde{\mu}_{\mathrm{W}}(x + y) \geqslant t, \tilde{\mu}_{\mathrm{W}}(y) \geqslant t$ and $\lambda_{\mathrm{W}}(x + y) \leq \gamma, \lambda_{\mathrm{W}}(y) \leq \gamma$.

Since W is a cubic fuzzy Bd-ideal of X, we get

$\tilde{\mu}_{\mathrm{W}}(x) \geqslant r \min\{\tilde{\mu}_{\mathrm{W}}(x + y), \tilde{\mu}_{\mathrm{W}}(y)\} \geqslant t$ and

$\lambda_{\mathrm{W}}(x) \leq \max\{\lambda_{\mathrm{W}}(x + y), \lambda_{\mathrm{W}}(y)\} \leq s$

Also

$x, y \in \tilde{U}(\mathrm{W}; \tilde{t}, s)$, then $\tilde{\mu}_{\mathrm{W}}(x) \geqslant t, \tilde{\mu}_{\mathrm{W}}(y) \geqslant t$ and $\lambda_{\mathrm{W}}(x) \leq \gamma, \lambda_{\mathrm{W}}(y) \leq \gamma$. Since W is a cubic fuzzy Bd-ideal of X, W get

$\tilde{\mu}_{\mathrm{W}}(x + y) \geqslant \min\{\tilde{\mu}_{\mathrm{W}}(x), \tilde{\mu}_{\mathrm{W}}(y)\} \geqslant t$ and

$\lambda_{\mathrm{W}}(x + y) \leq \max\{\lambda_{\mathrm{W}}(x), \lambda_{\mathrm{W}}(y)\} \leq s$.

Hence the set $\tilde{U}(\mathrm{W}; \tilde{t}, s)$ is an Bd-ideal of X. ■

Proposition 3.12

Let $(X; +, 0)$ be an Bd-algebra. If for all $\tilde{t} \in D[0,1]$ and $\gamma \in [0,1]$, the set $\tilde{U}(\mathrm{W}; \tilde{t}, \gamma)$ is an Bd-ideal of X. then $\mathrm{W} = < \tilde{\mu}_{\mathrm{W}}, \lambda_{\mathrm{W}} >$ is a cubic fuzzy Bd ideal of X.

Proof:

Suppose that $\tilde{U}(\mathrm{W}; \tilde{t}, \gamma)$ is an Bd-ideal of X and let $x, y \in X$ such that

$\tilde{\mu}_{\mathrm{W}}(x) \prec r \min\{\tilde{\mu}_{\mathrm{W}}(x + y), \tilde{\mu}_{\mathrm{W}}(y)\}$, and

$\lambda_{\mathrm{W}}(x) > \max\{\lambda_{\mathrm{W}}(x + y), \lambda_{\mathrm{W}}(y)\}$

Consider

$$\tilde{\beta} = 1/2 \ \{\tilde{\mu}_{\mathrm{W}}(x) + r \min\{\tilde{\mu}_{\mathrm{W}}(x + y), \tilde{\mu}_{\mathrm{W}}(y)\}\}$$

And

$$\beta = 1/2 \ \{\lambda_{\mathrm{W}}(x) + \max\{\lambda_{\mathrm{W}}(x + y), \lambda_{\mathrm{W}}(y)\}\}$$

We have $\tilde{\beta} \in D[0,1]$ and $\beta \in [0,1]$ and

$\tilde{\mu}_{\text{Ш}}(x) \prec \tilde{\beta} \prec r\,min\{\tilde{\mu}_{\text{Ш}}(x+y), \tilde{\mu}_{\text{Ш}}(y)\}$, and

$\lambda_{\text{Ш}}(x) > \beta > \max\{\lambda_{\text{Ш}}(x+y), \lambda_{\text{Ш}}(y)\}$

Also

$\tilde{\mu}_{\text{Ш}}(x+y) \prec r \min\{\tilde{\mu}_{\text{Ш}}(x), \tilde{\mu}_{\text{Ш}}(y)\}$, and $\lambda_{\text{Ш}}(x+y) > \max\{\lambda_{\text{Ш}}(x), \lambda_{\text{Ш}}(y)\}$

Consider

$$\tilde{\beta} = 1/2\ \{\tilde{\mu}_{\text{Ш}}(x+y) + r \min\{\tilde{\mu}_{\text{Ш}}(x), \tilde{\mu}_{\text{Ш}}(y)\}\}$$

And

$$\beta = 1/2\ \{\lambda_{\text{Ш}}(x+y) + \max\{\lambda_{\text{Ш}}(x), \lambda_{\text{Ш}}(y)\}\}$$

We have $\tilde{\beta} \in D[0,1]$ and $\beta \in [0,1]$ and

$\tilde{\mu}_{\text{Ш}}(x+y) \prec \tilde{\beta} \prec r\,min\{\tilde{\mu}_{\text{Ш}}(x), \tilde{\mu}_{\text{Ш}}(y)\}$ and

$\lambda_{\text{Ш}}(x+y) > \beta > \max\{\lambda_{\text{Ш}}(x), \lambda_{\text{Ш}}(y)\}$

It follow s that $x+y, x, y \in \tilde{U}(\text{Ш}; \tilde{t}, \gamma)$, and $X \notin \tilde{U}(\text{Ш}; \tilde{t}, s), x+y \notin \tilde{U}(\text{Ш}; \tilde{t}, s)$. This is a contradiction and therefore $\text{Ш} =< \tilde{\mu}_{\text{Ш}}, \lambda_{\text{Ш}} >$ is a cubic fuzzy Bd-ideal of X.

Theorem 3.13

Cubic set $\text{Ш} =< \tilde{\mu}_{\text{Ш}_i}, \lambda_{\text{Ш}} >$ is a cubic fuzzy Bd-ideal of X if and only if, $\mu^{-}_{\text{Ш}}$ and $\mu^{+}_{\text{Ш}}$ be fuzzy Bd-ideal of X and $\lambda_{\text{Ш}}$ be anti-fuzzy Bd-ideals of X.

Proof:

Let $\mu^{-}_{\text{Ш}}$ and $\mu^{+}_{\text{Ш}}$be fuzzy Bd-ideals of X and $\lambda_{\text{Ш}}$ be anti-fuzzy Bd-ideals of X and $x, y \in X$, then

$\mu^{-}_{\text{Ш}}(0) \geq \mu^{-}_{\text{Ш}}(x), \mu^{+}_{\text{Ш}}(0) \geq \mu^{+}_{\text{Ш}}(x)$ and $\lambda_{\text{Ш}}(0) \leq \lambda_{\text{Ш}}(x)$

$\mu^{-}_{\text{Ш}}(x) \geq \min\{\mu^{-}_{\text{Ш}}(x+y), \mu^{-}_{\text{Ш}}(y)\}, \mu^{+}_{\text{Ш}}(x) \geq \min\{\mu^{+}_{\text{Ш}}(x+y), \mu^{+}_{\text{Ш}}(y)\}$ and

$\lambda_{\text{Ш}}(x) \leq \max\{\lambda_{\text{Ш}}(x+y), \lambda_{\text{Ш}}(y)\}$. Now

$\tilde{\mu}_{\text{Ш}}(x) = [\mu^{-}_{\text{Ш}}(x), \mu^{+}_{\text{Ш}}(y)] \geqslant [\min\{\mu^{-}_{\text{Ш}}(x+y), \mu^{-}_{\text{Ш}}(y)], [\mu^{+}_{\text{Ш}}(x+y), \mu^{+}_{\text{Ш}}(y)]\}$

$=\min\{\mu^{-}_{\text{Ш}}(x+y), \mu^{+}_{\text{Ш}}(x+y)], [\mu^{-}_{\text{Ш}}(y), \mu^{+}_{\text{Ш}}(y)]\}$

$= r \min\{\tilde{\mu}_{\text{Ш}}(x+y), \tilde{\mu}_{\text{Ш}}(y)\}$, therefore

$\tilde{\mu}_{\text{Ш}}(x) \geqslant r \min\{\tilde{\mu}_{\text{Ш}}(x+y), \tilde{\mu}_{\text{Ш}}(y)\}$. And $\lambda_{\text{Ш}}(x) \leq \max\{\lambda_{\text{Ш}}(x+y),, \lambda_{\text{Ш}}(y)\}$. Now,

$\tilde{\mu}\,(x+y) = [\mu^{-}_{\text{Ш}}(x+y), \mu^{+}_{\text{Ш}}(x+y)] \geqslant [\min\{\mu^{-}_{\text{Ш}}(x), \mu^{-}_{\text{Ш}}(y)], [\mu^{+}_{\text{Ш}}(x), \mu^{+}_{\text{Ш}}(y)]\}$

$=\text{r}\min\{[\{\mu^{-}_{\text{Ш}}(x), \mu^{+}_{\text{Ш}}(x)], [\mu^{-}_{\text{Ш}}(y), \mu^{+}_{\text{Ш}}(y)]$

$= r \min\{\tilde{\mu}_{\mathrm{Ш}}(x), \tilde{\mu}_{\mathrm{Ш}}(y)\}$, therefore

$\tilde{\mu}_{\mathrm{Ш}}(x+y) \geqslant r \min\{\tilde{\mu}_{\mathrm{Ш}}(x), \tilde{\mu}_{\mathrm{Ш}}(y)\}$. And $\lambda_{\mathrm{Ш}}(x+y) \leq \max\{\lambda_{\mathrm{Ш}}(x),, \lambda_{\mathrm{Ш}}(y)\}$

Hence Ш is a cubic fuzzy Bd-ideal of X.

Conversely, assume that Ш is a cubic fuzzy Bd-ideal of X, for any $x, y \in X$, $\tilde{\mu}_{\mathrm{Ш}}(0) \geqslant \tilde{\mu}_{\mathrm{Ш}}(0)$ and $\lambda_{\mathrm{Ш}}(0) \leq \lambda_{\mathrm{Ш}}(x)$, and

$[\mu_{\mathrm{Ш}}^{-}(x), \mu_{\mathrm{Ш}}^{+}(x)] = \tilde{\mu}_{\mathrm{Ш}}(x) \geqslant r \min\{\tilde{\mu}_{\mathrm{Ш}}(x+y), \tilde{\mu}_{\mathrm{Ш}}(y)\}$

$r\,min\{[\mu_{\mathrm{Ш}}^{-}(x+y), \mu_{\mathrm{Ш}}^{+}(x+y)], [\mu_{\mathrm{Ш}}^{-}(y), \mu_{\mathrm{Ш}}^{+}(y)]\}$

$= r \min\{[\mu_{\mathrm{Ш}}^{-}(x+y), \mu_{\mathrm{Ш}}^{-}(y)], [\mu_{\mathrm{Ш}}^{+}(x+y), \mu_{\mathrm{Ш}}^{+}(y)]\}$

Thus

$\mu_{\mathrm{Ш}}^{-}(x) \geq \{\mu_{\mathrm{Ш}}^{-}(x+y), \mu_{\mathrm{Ш}}^{-}(y)\}, \mu_{\mathrm{Ш}}^{-}(x) \geq \{\mu_{\mathrm{Ш}}^{+}(x+y), \mu_{\mathrm{Ш}}^{+}(y)\}$ and

$\lambda_{\mathrm{Ш}}(x) \leq \max\{\lambda_{\mathrm{Ш}}(x+y), \lambda_{\mathrm{Ш}}(y)\}$

$[\mu_{\mathrm{Ш}}^{-}(x+y), \mu_{\mathrm{Ш}}^{+}(x+y)] = \tilde{\mu}_{\mathrm{Ш}}(x+y)$

$\geqslant r \min\{\tilde{\mu}_{\mathrm{Ш}}(x), \tilde{\mu}_{\mathrm{Ш}}(y)\}$

$= r \min\{\mu_{\mathrm{Ш}}^{-}(x), \mu_{\mathrm{Ш}}^{+}(x)\}, \{\mu_{\mathrm{Ш}}^{-}(y), \mu_{\mathrm{Ш}}^{+}(y)\}$

Thus

$\mu_{\mathrm{Ш}}^{-}(x+y) \geq \{\mu_{\mathrm{Ш}}^{-}(x), \mu_{\mathrm{Ш}}^{-}(y)\}, \mu_{\mathrm{Ш}}^{+}(x+y) \geq \{\mu_{\mathrm{Ш}}^{+}(x), \mu_{\mathrm{Ш}}^{+}(y)\}$ and

$\lambda_{\mathrm{Ш}}(x+y) \leq \max\{\lambda_{\mathrm{Ш}}(x), \lambda_{\mathrm{Ш}}(y)\}$,

Therefore $\mu_{\mathrm{Ш}}^{-}$ and $\mu_{\mathrm{Ш}}^{+}$ be fuzzy Bd-ideal of X and $\lambda_{\mathrm{Ш}}$ be anti fuzzy Bd-ideal of X ■

Theorem 3.14

Every cubic fuzzy Bd-ideal of Bd-algebra $(X; +, 0)$ is a cubic fuzzy Bd-subalgebra of X.

Proof:

let $(X; +, 0)$ be an Bd-algebra and $\mathrm{Ш} = < \tilde{\mu}_{\mathrm{Ш}}(x), \lambda_{\mathrm{Ш}}(x) >$ is a cubic fuzzy Bd-ideal of X. since Ш is an cubic fazzy Bd-ideal of X, then by proposition ( ), for every $\gamma \in [0,1], \tilde{t} \in D[0,1]$,

$\tilde{U}(\mathrm{Ш}; \tilde{t}, \gamma) = \{x \in X | \tilde{\mu}_{\mathrm{Ш}}(x) \geq \tilde{t}\, \lambda_{\mathrm{Ш}}(x) \leq \ \}$, is ideal of X. By proposition ( ), for every $\delta \in (0,1], \tilde{t} \in D[0,1]$ and $\delta \in [0,1]$, $\tilde{U}(\mathrm{Ш}; \tilde{t}, s)$ is Bd-subalgebra of X. Hence Ш is a cubic fuzzy Bd-subalgebra of X by Proposition ( ). ■

Remark 3.15

The converse of Theorem ( ) is not true as the following Example:

Example 3.16

let $X = \{0, a, b, c\}$ be a set with the following table:

+	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	b
c	c	c	c	0

hen $(X;+,0)$ is an Bd-algebra. Define a cubic set $\mathrm{Ш} =< \tilde{\mu}_{\mathrm{Ш}}, \lambda_{\mathrm{Ш}} >$ of X is fuzzy subset $\mu: X \to [0,1]$ by

$\tilde{\mu}_{\mathrm{Ш}}(x) = \begin{cases} [0.3,0.9] & ,if\ x = \{0,c\} \\ [0.1,0.6] & ,otherwise \end{cases}$ and $\lambda_{\mathrm{Ш}} = \begin{cases} 0.1 & if\ x = \{0,c\} \\ 0.6 & otherwise \end{cases}$

The cubic set $\mathrm{Ш} =< \tilde{\mu}_{\mathrm{Ш}}(x), \lambda_{\mathrm{Ш}}(x) >$ is a cubic fuzzy Bd-subalgebra of X.

Note that $\lambda_{\mathrm{Ш}}$ is not an anti-fuzzy Bd-ideal of X since

$\lambda_{\mathrm{Ш}}(3) = 0.6 > 0.1 = \max\{\lambda_{\mathrm{Ш}}(b+a), \lambda_{\mathrm{Ш}}(a)\}$

$= \max\{\lambda_{\mathrm{Ш}}(b), \lambda_{\mathrm{Ш}}(b)\}.$

Hence $\mathrm{Ш}$ is not cubic fuzzy Bd-ideal of X.

Theorem 3.17

If a cubic fuzzy subset $\mathrm{Ш} =< \tilde{\mu}_{\mathrm{Ш}}, \lambda_{\mathrm{Ш}} >$ is a cubic fuzzy Bd-ideal of X, then the upper $[\gamma_1, \gamma_2]$-level and Lower t-level of $\mathrm{Ш}$ are Bd-ideals of X.

Proof:

Let $(x+y); (y) \in U(\tilde{\mu}_{\mathrm{Ш}}|[\gamma_1, \gamma_2])$, then $\tilde{\mu}_{\mathrm{Ш}}(x+y) \geqslant [\gamma_1, \gamma_2]$ and

$\tilde{\mu}_{\mathrm{Ш}}(y) \geqslant [\gamma_1, \gamma_2]$. It follows that

$\tilde{\mu}_{\mathrm{Ш}}(x) \geqslant r \min\{\tilde{\mu}_{\mathrm{Ш}}(x+y), \tilde{\mu}_{\mathrm{Ш}}(y)\} \geqslant [\gamma_1, \gamma_2]$ so that

$(x) \in U(\tilde{\mu}_{\mathrm{Ш}}|[\gamma_1, \gamma_2])$. Hence $U(\tilde{\mu}_{\mathrm{Ш}}|[\gamma_1, \gamma_2])$ is an Bd-ideal of X.

Let $(x+y)(y) \in L(\lambda_{\mathrm{Ш}}|t)$, then $\lambda_{\mathrm{Ш}}(x+y) \leq t$ and $\lambda_{\mathrm{Ш}}(y) \leq t$.

It follows that $\lambda_{\mathrm{Ш}}(x) \leq \max\{\lambda_{\mathrm{Ш}}(x+y), \lambda_{\mathrm{Ш}}(y)\} \leq t$, so that $(x) \in L(\lambda_{\mathrm{Ш}}|t)$.

And

$(x), (y) \in U(\tilde{\mu}_{\mathrm{Ш}}|[\gamma_1, \gamma_2])$, then $\tilde{\mu}_{\mathrm{Ш}}(x) \geqslant [\gamma_1, \gamma_2]$ and $\tilde{\mu}_{\mathrm{Ш}}(y) \geqslant [\gamma_1, \gamma_2]$,

It follows, that

$\tilde{\mu}_{\mathrm{Ш}}(x+y) \geqslant r \min\{\tilde{\mu}_{\mathrm{Ш}}(x), \tilde{\mu}_{\mathrm{Ш}}(y)\} \geqslant [\gamma_1, \gamma_2]$, so that $x \in U(\tilde{\mu}_{\mathrm{Ш}}|[\gamma_1, \gamma_2])$.

Hence $U(\tilde{\mu}_{\mathrm{Ш}}|[\gamma_1, \gamma_2])$ is an Bd-ideal of X.

Let $(x), (y) \in L(\lambda_{\mathrm{Ш}}|t)$, then $\lambda_{\mathrm{Ш}}(x) \leq t$ and $\lambda_{\mathrm{Ш}}(y) \leq t$. It follows that $\lambda_{\mathrm{Ш}}(x+y) \leq \max\{\lambda_{\mathrm{Ш}}(x), \lambda_{\mathrm{Ш}}(y)\} \leq t$, so that $(x+y) \in L(\lambda_{\mathrm{Ш}}|t)$. Hence $L(\lambda_{\mathrm{Ш}}|t)$ is an Bd-ideal of X. ■

Corollary 3.18

let $\mathrm{Ш} =< \tilde{\mu}_{\mathrm{Ш}}, \lambda_{\mathrm{Ш}} >$ be a cubic fuzzy Bd-ideal of Y, then

$\mathrm{Ш}([\gamma_1, \gamma_2]; t) = U(\tilde{\mu}_{\mathrm{Ш}}|[\gamma_1, \gamma_2]) \cap L(\lambda_{\mathrm{Ш}}|t) = \{x \in X|\tilde{\mu}_{\mathrm{Ш}}(x) \succcurlyeq [\gamma_1, \gamma_2],$

$\lambda_{\mathrm{Ш}}(x) \leq t\}$ is a cubic fuzzy Bd-ideal of X.

The following example shows that the converse of corollary ( ) is not valid

Example 3.19

let $X = \{0, a, b, c, d, e\}$ be Bd-algebra in example ( ) and cubic set $\mathrm{Ш} = (\tilde{\mu}_{\mathrm{Ш}}, \lambda_{\mathrm{Ш}})$ of X by

$$\tilde{\mu}_{\mathrm{Ш}}(x) = \begin{cases} [0.6,0.8] & ,if\ x = 0 \\ [0.5,0.6] & ,if\ x \in \{c,d,e\}, \\ [0.3,0.4] & ,if\ x \in \{a,b\} \end{cases} \lambda_{\mathrm{Ш}_1}(x) = \begin{cases} 0.1 & ,if\ x = 0 \\ 0.3 & ,if\ x \in \{b,c,d\} \\ 0.8 & ,if\ x \in \{a,e\} \end{cases}$$

$\mathrm{Ш}([\gamma_1, \gamma_2]; t) = U(\tilde{\mu}_{\mathrm{Ш}}|[\gamma_1, \gamma_2]) \cap L(\lambda_{\mathrm{Ш}}|t)$

$= \{x \in X|\tilde{\mu}_{\mathrm{Ш}}(x) \succcurlyeq [\gamma_1, \gamma_2], \lambda_{\mathrm{Ш}}(x) \leq t\}$

$= \{0, c, d, e\} \cap \{0, , b, c, d, \} = \{0, c, d\}$

Bd-ideal of X, but $\mathrm{Ш} =< \tilde{\mu}_{\mathrm{Ш}}, \lambda_{\mathrm{Ш}} >$ is not a cubic fuzzy Bd-ideal since

$\tilde{\mu}_{\mathrm{Ш}}(a) \not\succcurlyeq r \min\{\tilde{\mu}_{\mathrm{Ш}}(d+b), \tilde{\mu}_{\mathrm{Ш}}(b)\}$ and $\lambda_{\mathrm{Ш}}(a) \not\leq \max\{\lambda_{\mathrm{Ш}}(d+b), \lambda_{\mathrm{Ш}}(b)\}$

Theorem 3.20

Let $\mathrm{Ш} = (\tilde{\mu}_{\mathrm{Ш}}, \lambda_{\mathrm{Ш}})$ be a cubic fuzzy subset of X such that the sets

$U(\tilde{\mu}_{\mathrm{Ш}}|[\gamma_1, \gamma_2])$ and $L(\lambda_{\mathrm{Ш}}|t)$ are Bd-ideals of X for every $[\gamma_1, \gamma_2] \in D[0,1]$ and

$t \in [0,1]$, then $\mathrm{Ш} =< \tilde{\mu}_{\mathrm{Ш}}, \lambda_{\mathrm{Ш}} >$ is a cubic fuzzy Bd-ideal of X.

Proof:

Let $U(\tilde{\mu}_{\mathrm{Ш}}|[\gamma_1, \gamma_2])$ and $L(\lambda_{\mathrm{Ш}}|t)$ are ideals of X, for every $[\gamma_1, \gamma_2] \in D[0,1]$ and $t \in [0,1]$ on the contrary, let $x_0, y_0 \in X$ such that $\tilde{\mu}_{\mathrm{Ш}}(x_0) \prec r \min\{\tilde{\mu}_{\mathrm{Ш}}(x_0 + y_0), \tilde{\mu}_{\mathrm{Ш}}(y)\}$.

Let $\tilde{\mu}_{\mathrm{Ш}}(x_0 + y_0) = [\eta_1, \eta_2]$ and $\tilde{\mu}_{\mathrm{Ш}}(y_0) = [\eta_3, \eta_4]$ and

$\tilde{\mu}_{\mathrm{Ш}}(x_0) = [\gamma_1, \gamma_2]$. Then

$[\gamma_1, \gamma_2] \prec r \min\{[\eta_1, \eta_2], [\eta_3, \eta_4]\}$

$= \{\min[\eta_1, \eta_2], \min[\eta_3, \eta_4]\}.$

So $\gamma_1 < \min\{\eta_1, \eta_3\}$ and $\gamma_2 < \min\{\eta_2, \eta_4\}$. Let μ_s consiten

$[\rho_1, \rho_2] = \frac{1}{2}[\{\tilde{\mu}_{\text{Ш}}(x_0) + r\min\{\tilde{\mu}_{\text{Ш}}(x_0 + y_0), \tilde{\mu}_{\text{Ш}}(y_0)\}\}]$

$= \frac{1}{2}[[\gamma_1, \gamma_2] + [\min\{\eta_1, \eta_3\}, \min\{\eta_2, \eta_4\}].$

Therefore $\min\{\eta_1, \eta_3\} > \rho_1 = \frac{1}{2}(\gamma_1 + \min\{\eta_1, \eta_3\}) > \gamma_1$ and

$\min\{\eta_2, \eta_4\} > \rho_2 = \frac{1}{2}(\gamma_1 + \min\{\eta_1, \eta_3\}) > \gamma_1$

Hence $[\min\{\eta_1, \eta_3\}, \min\{\eta_2, \eta_4\}] \succ [\rho_1, \rho_2] \succ [\gamma_1, \gamma_2]$, so that

$(x_0) \notin U(\tilde{\mu}_{\text{Ш}}|[\gamma_1, \gamma_2])$ which is a contradiction, since

$\tilde{\mu}_{\text{Ш}}(x_0) = [\eta_1, \eta_2] \succ [\min\{\eta_1, \eta_3\}, \min\{\eta_2, \eta_4\} \succ [\rho_1, \rho_2]$

and $\tilde{\mu}_{\text{Ш}}(y_0) = [\eta_3, \eta_4] \succ [\min\{\eta_1, \eta_3\}, \min\{\eta_2, \eta_4\}] \succ [\rho_1, \rho_2]$ this implies

$(x_0) \in U(\tilde{\mu}_{\text{Ш}}|[\gamma_1, \gamma_2])$.

Thus $\tilde{\mu}_{\text{Ш}}(x) =\geq r\min\{\tilde{\mu}_{\text{Ш}}(x + y), \tilde{\mu}_{\text{Ш}}(y)\}$ for all $x, y \in X$. And, Let $x_0, y_0 \in X$ such that $\lambda_{\text{Ш}}(x_0) > \max\{\lambda_{\text{Ш}}(x_0 + y_0), \lambda_{\text{Ш}}(y_0)\}$.

Let $\lambda_{\text{Ш}}(x_0 + y_0) = \nu_1$,$\lambda_{\text{Ш}}(y_0) = \nu_2$ and $\lambda_{\text{Ш}}(x_0) = t$, then

$t > \max\{\nu_1, \nu_2\}$

Let Ш is consider, $t_1 = \frac{1}{2}[\lambda_{\text{Ш}}(x_0) + \max\{\nu_1, \nu_2\} + \max\{\lambda_{\text{Ш}}(x_0 + y_0), \lambda_{\text{Ш}}(y_0)\}]$

we got that $t_1 = \frac{1}{2}(t + \max\{\nu_1, \nu_2\})$, therefore,

$\nu_1 < t_1 = \frac{1}{2}(t + \max\{\nu_1, \nu_2\} < t$ and $\nu_2 < t_2 = \frac{1}{2}(t + \max\{\nu_1, \nu_2\}) < t$

Hence, $\max\{\nu_1, \nu_2\} < t_1 < t = \lambda_{\text{Ш}}(x_0)$.

So that $x_0 + y_0 \notin L(\lambda_{\text{Ш}}|t)$ which is a contradiction since

$\lambda_{\text{Ш}}(x_0) = \nu_1 \leq \max\{\nu_1, \nu_2\} < t_1$ and $\lambda_{\text{Ш}}(y_0) = \nu_2 \leq \max\{\nu_1, \nu_2\} < t_1$, this implies $(x_0) \in L(\lambda_{\text{Ш}}|t)$ this implies $\lambda_{\text{Ш}}(x) \leq \max\{\lambda_{\text{Ш}}(x + y), \lambda_{\text{Ш}}(y)\}$ for all

$x, y \in X$.

$\tilde{\mu}_{\text{Ш}}(x_0 + y_0) \prec r\min\{\tilde{\mu}_{\text{Ш}}(x_0), \tilde{\mu}_{\text{Ш}}(y)\}$.

Let $\tilde{\mu}_{\text{Ш}}(x_0) = [\eta_1, \eta_2]$ and $\tilde{\mu}_{\text{Ш}}(y_0) = [\eta_3, \eta_4]$ and

$\tilde{\mu}_{\text{Ш}}(x_0 + y_0) = [\gamma_1, \gamma_2]$. Then

$[\gamma_1, \gamma_2] \prec r\min\{[\eta_1, \eta_2], [\eta_3, \eta_4]\}$

$= \{\min[\eta_1, \eta_2], \min[\eta_3, \eta_4]\}.$

So $\gamma_1 < \min\{\eta_1, \eta_3\}$ and $\gamma_2 < \min\{\eta_2, \eta_4\}$. Let μ_s consiten

$[\rho_1, \rho_2] = \frac{1}{2}[\{\tilde{\mu}_{\text{Ш}}(x_0 + y_0) + r\min\{\tilde{\mu}_{\text{Ш}}(x_0), \tilde{\mu}_{\text{Ш}}(y_0)\}\}]$

$= \frac{1}{2}[[\gamma_1, \gamma_2] + [\min\{\eta_1, \eta_3\}, \min\{\eta_2, \eta_4\}]$.

Therefore $\min\{\eta_1, \eta_3\} > \rho_1 = \frac{1}{2}(\gamma_1 + \min\{\eta_1, \eta_3\}) > \gamma_1$ and

$\min\{\eta_2, \eta_4\} > \rho_2 = \frac{1}{2}(\gamma_1 + \min\{\eta_1, \eta_3\}) > \gamma_2$

Hence $[\min\{\eta_1, \eta_3\}, \min\{\eta_2, \eta_4\}] \succ [\rho_1, \rho_2] \succ [\gamma_1, \gamma_2]$, so that

$(x_0) \notin U(\tilde{\mu}_{\mathrm{Ш}} | [\gamma_1, \gamma_2])$ which is a contradiction, since

$\tilde{\mu}_{\mathrm{Ш}}(x_0) = [\eta_1, \eta_2] \succ [\min\{\eta_1, \eta_3\}, \min\{\eta_2, \eta_4\} \succ [\rho_1, \rho_2]$

and $\tilde{\mu}_{\mathrm{Ш}}(y_0) = [\eta_3, \eta_4] \succ [\min\{\eta_1, \eta_3\}, \min\{\eta_2, \eta_4\}] \succ [\rho_1, \rho_2]$ this implies $(x_0) \in U(\tilde{\mu}_{\mathrm{Ш}} | [\gamma_1, \gamma_2])$.

Thus $\tilde{\mu}_{\mathrm{Ш}}(x + y) \succcurlyeq r \min\{\tilde{\mu}_{\mathrm{Ш}}(x), \tilde{\mu}_{\mathrm{Ш}}(y)\}$ for all $x, y \in X$. And, Let $x_0, y_0 \in X$ such that $\lambda_{\mathrm{Ш}}(x_0 + y_0) > \max\{\lambda_{\mathrm{Ш}}(x_0), \lambda_{\mathrm{Ш}}(y_0)\}$.

Let $\lambda_{\mathrm{Ш}}(x_0) = \nu_1$,$\lambda_{\mathrm{Ш}}(y_0) = \nu_2$ and $\lambda_{\mathrm{Ш}}(x_0 + y_0) = t$, then

$t > \max\{\nu_1, \nu_2\}$

Let Ш is consider, $t_1 = \frac{1}{2}[\lambda_{\mathrm{Ш}}(x_0 + y_0) + \max\{\nu_1, \nu_2\} + \max\{\lambda_{\mathrm{Ш}}(x_0), \lambda_{\mathrm{Ш}}(y_0)\}]$

we got that $t_1 = \frac{1}{2}(t + \max\{\nu_1, \nu_2\})$, therefore,

$\nu_1 < t_1 = \frac{1}{2}(t + \max\{\nu_1, \nu_2\} < t$ and $\nu_2 < t_2 = \frac{1}{2}(t + \max\{\nu_1, \nu_2\}) < t$

Hence, $\max\{\nu_1, \nu_2\} < t_1 < t = \lambda_{\mathrm{Ш}}(x_0 + y_0)$.

So that $x_0 + y_0 \notin L(\lambda_{\mathrm{Ш}} | t)$ which is a contradiction since

$\lambda_{\mathrm{Ш}}(x_0) = \nu_1 \leq \max\{\nu_1, \nu_2\} < t_1$ and $\lambda_{\mathrm{Ш}}(y_0) = \nu_2 \leq \max\{\nu_1, \nu_2\} < t_1$, this implies $(x_0 + y_0) \in L(\lambda_{\mathrm{Ш}} | t)$ this implies $\lambda_{\mathrm{Ш}}(x + y) \leq \max\{\lambda_{\mathrm{Ш}}(x), \lambda_{\mathrm{Ш}}(y)\}$

Hence, $\mathrm{Ш} = < \tilde{\mu}_{\mathrm{Ш}}, \lambda_{\mathrm{Ш}} >$ is a cubic fuzzy Bd-ideal of X. ■

Homomorphism of Cubic Fuzzy Bd-ideals of Bd-algebra

In this section, the researcher presents some results on images and preimages of cubic fuzzy Bd-subalgebras and fuzzy Bd-ideals of Bd-algebras.

Theorem 4.1

A homomorphic preimage of cubic fuzzy Bd-subalgebra is also cubic fuzzy

Bd-subalgebra.

Proof:

Let $f: (X; +, 0) \to (Y; +', 0')$ be homomorphis from an Bd-algebra X into an Bd-algebra. If $\beta = < \tilde{\mu}_\beta, \lambda_\beta >$ is cubic fuzzy Bd-subalgebra of Y and

$\mathrm{Ш} = < \tilde{\mu}_{\mathrm{Ш}}, \lambda_{\mathrm{Ш}} >$ the pre-image of β under f, then

$\tilde{\mu}_{f^{-1}(\beta)}(x) = \tilde{\mu}_{\text{Ш}}(f(x)), \lambda_{f^{-1}(\beta)}(x) = \lambda_{\text{Ш}}(f(x))$ for all $x \in X$.

Let $x \in X$, then $\left(\tilde{\mu}_{f^{-1}(\beta)}(0)\right) = \tilde{\mu}_{\text{Ш}}(f(0)) \geq \tilde{\mu}_{\text{Ш}}(f(x)) = \tilde{\mu}_{f^{-1}(\beta)}(x)$, and

$\left(\lambda_{f^{-1}(\beta)}\right)(0) = \lambda_{\text{Ш}}(f(0)) \leq \lambda_{\text{Ш}}(f(x)) = \lambda_{f^{-1}(\beta)}(x)$.

Now let $x, y \in X$, then

$\tilde{\mu}_{f^{-1}(\beta)}(x + y) = \tilde{\mu}_{\text{Ш}}(f(x + y) = \tilde{\mu}_{\text{Ш}}(f(x) +' f(y))$

$\geqslant r \min\{\tilde{\mu}_{\text{Ш}}(f(x)), \tilde{\mu}_{\text{Ш}}(f(y))\}$

$= r \min\{\tilde{\mu}_{f^{-1}(\beta)}(x), \tilde{\mu}_{f^{-1}(\beta)}(y)\}$ and

$\lambda_{f^{-1}(\beta)}(x + y) = \lambda_{\text{Ш}}(f(x + y) = \lambda_{\text{Ш}}(f(x) +' f(y))$

$\geq \max\{\lambda_{\text{Ш}}(f(x)), \lambda_{\text{Ш}}(f(y))\}$

$= \max\{\lambda_{f^{-1}(\beta)}(x), \lambda_{f^{-1}(\beta)}(y)\}$. ■

Theorem 4.2

Let $f: (X; +, 0) \to (Y; +', 0')$ be an epimorphism from an Bd-algebra X into

Bd-algebra Y. For every cubic fuzzy Bd-subalgebra $\text{Ш} =< \mu_{\tilde{\text{Ш}}}, \lambda_{\text{Ш}} >$ of X With sup and inf properties, then $f(\text{Ш})$ is a cubic fuzzy Bd-subalgebra of Y.

Proof:

Since $f(\tilde{\mu}_{\text{Ш}})(y)' = r \sup \tilde{\mu}_{\text{Ш}}(x)$ and

$f(\lambda_{\text{Ш}})(y') = \inf_{x \in f^{-1}(y')} \lambda_{\text{Ш}}(x)$ for all $y' \in Y$ and

$r \sup(\emptyset) = [0,0]$ and $\inf(\emptyset) = 1$, we have prove that

$f(\tilde{\mu}_{\text{Ш}})(x' + y') \geqslant r \min\{f(\tilde{\mu}_{\text{Ш}})(x'), \tilde{\mu}_{\text{Ш}}(y)\}$, and

$f(\lambda_{\text{Ш}})(x' + y') \leq r \max\{f(\lambda_{\text{Ш}})(x'), f(\lambda_{\text{Ш}})(y')\}$, for all $x', y' \in X$.

$f(\tilde{\mu}_{\text{Ш}})(x' +' y') = r \sup_{t \in f^{-1}(x' + y')} \tilde{\mu}_{\text{Ш}}(t) = \tilde{\mu}_{\text{Ш}}(x_0 + y_0)$

$\geqslant r \min\{\tilde{\mu}_{\text{Ш}}(x_0), \tilde{\mu}_{\text{Ш}}(y_0)\}$

$= r \min\{r \sup_{t \in f^{-1}(x')} \tilde{\mu}_{\text{Ш}}(t), r \sup_{t \in f^{-1}(y')} \tilde{\mu}_{\text{Ш}}(t)\}$

$= r \min\{f(\tilde{\mu}_{\text{Ш}})(x'), f(\tilde{\mu}_{\text{Ш}})(y')\}$ and

$f(\lambda_{\text{Ш}})(x' +' y') = \inf_{t \in f^{-1}(x' + y')} \lambda_{\text{Ш}}(t) = \lambda_{\text{Ш}}(x_0 + y_0)$

$\leq \max\{\lambda_{\text{Ш}}(x_0), \lambda_{\text{Ш}}(y_0)\}$

$$= \max\{\inf_{t\in f^{-1}(x')} \lambda_{\text{Ш}}(t), \inf_{t\in f^{-1}(y')} \lambda_{\text{Ш}}(t)\}$$

$$= \max\{f(\lambda_{\text{Ш}})(x'), f(\lambda_{\text{Ш}})(y')\}$$

Hence, $f(\text{Ш})$ is a cubic fuzzy Bd-subalgebra of X. ■

Theorem 4.3

A homomorphism Pre-image of cubic fuzzy Bd-ideal is also cubic fuzzy Bd-ideal.

Proof:

Let $f = (X; +,0) \to (Y; +', 0')$ be homomorphism from an Bd- algebra X into an Bd-algebra Y if $\beta =< \tilde{\mu}_\beta, \lambda_\beta >$ is a cubic ideal of Y and $\text{Ш} =< \tilde{\mu}_{\text{Ш}}, \lambda_{\text{Ш}} >$ the pre-image of β under f. then $\tilde{\mu}_{\text{Ш}}(x) = \tilde{\mu}_\beta(f(x)), \lambda_{\text{Ш}}(x) = \lambda_\beta(f(x))$ for all $x \in X$, then

$\tilde{\mu}_{\text{Ш}}(0) = \tilde{\mu}_\beta(f(0)) \geq \tilde{\mu}_\beta(f(x)) = \tilde{\mu}_{\text{Ш}}(x)$

And $\lambda_{\text{Ш}}(0) = \lambda_\beta(f(0)) \leq \lambda_\beta(f(x)) = \lambda_{\text{Ш}}(x)$.

Now, let $x, y \in X$ then

$\tilde{\mu}_{\text{Ш}}(x) = \tilde{\mu}_\beta(f(x))$

$\geq r \min\{\tilde{\mu}_\beta(f(x+y), \tilde{\mu}_\beta(f(y))\}$

$\geq r \min\{\tilde{\mu}_{\text{Ш}}(f(x+y), \tilde{\mu}_{\text{Ш}}(f(y))\}$ and

$\lambda_{\text{Ш}}(x) = \lambda_\beta(f(x))$

$\leq \max\{\lambda_\beta(f(x+y)), \lambda_\beta(y)\}$

$= \max\{\lambda_{\text{Ш}}(x+y), \lambda_{\text{Ш}}(y)\}$

And

$\tilde{\mu}_{\text{Ш}}(x+y) = \tilde{\mu}_\beta(f(x+y))$
$\geq r \min\{\tilde{\mu}_\beta(f(x)), \tilde{\mu}_\beta(f(y))\}$
$= r \min\{\tilde{\mu}_{\text{Ш}}(x), \tilde{\mu}_{\text{Ш}}(y)\}$ and
$\lambda_{\text{Ш}}(x+y) = \lambda_\beta(f(x+y))$
$\leq \max\{\lambda_\beta(f(x+y)), \lambda_\beta(f(y))\}$

$= \max\{\lambda_{\text{Ш}}(x), \lambda_{\text{Ш}}(y)\}$. ■

Theorem 4.4

let $f: (X; +,0) \to (Y; +', 0')$ be an epimorphism from an Bd-algebra X into an

Bd-algebra Y. For every cubic fuzzy Bd-ideal $\text{Ш} =< \mu_{\tilde{\text{Ш}}}, \lambda_{\text{Ш}} >$ of X with sup and inf properties, then $f(\text{Ш})$ cubic fuzzy Bd-ideal of Y.

Proof:

Since $\tilde{\mu}_\beta(x'+'y') = f(\tilde{\mu}_\text{Ш})(x'+'y') = \underset{x+y\in f^{-1}(x'+'y')}{\text{rsup}}\ \tilde{\mu}_\text{Ш}(x+y)$ and

And $\lambda_\beta(x'+'y') = f(\lambda_\text{Ш})(x'+'y') = \underset{x+y\in f^{-1}(x'+'y')}{\inf}\ \lambda_\text{Ш}(x+y)$ and

$\tilde{\mu}_\beta(y') = f(\tilde{\mu}_\text{Ш})(y') = \underset{y\in f^{-1}(y')}{\text{rsup}}\ \tilde{\mu}_\text{Ш}(y)$

$\lambda_\text{Ш}(y') = f(\lambda_\text{Ш})(y') = \underset{y\in f^{-1}(y')}{\inf}\ \lambda_\text{Ш}(y)$ for all $x', y' \in Y$ and

$r\sup(\emptyset) = [0,0]$ and $\inf(\emptyset) = 0$. we have prove that

$\tilde{\mu}_\beta(x') \geqslant r\min\{\tilde{\mu}_\beta(x'+y'), \tilde{\mu}_\beta(y')\}$, and

$\lambda_\beta(x') \leq \max\{\lambda_\beta(x'+y'), \lambda_\beta(y')\}$, for all $x', y' \in Y$.

Let $f\colon (X;+,0) \to (Y;+',0')$ be epimorphism of Bd-algebras, $\text{Ш} =< \tilde{\mu}_\text{Ш}, \lambda_\text{Ш} >$

is a cubic fuzzy Bd-ideal of X has sup and inf properties and $\beta =< \tilde{\mu}_\beta, \lambda_\beta >$ the image of $\text{Ш} =< \tilde{\mu}_\text{Ш}, \lambda_\text{Ш} >$ under f. Since $\text{Ш} =< \tilde{\mu}_\text{Ш}, \lambda_\text{Ш} >$ is a cubic fuzzy

Bd -ideal of X, we have $(\tilde{\mu}_\text{Ш})(0) \geqslant \tilde{\mu}_\text{Ш}(x)$ and $(\lambda_\text{Ш})(0) \leq \lambda_\text{Ш}(x)$, for all $x \in X$. Note that, $0 \in f(0')$ where $0,0'$ are the Zero of x and Y, respectively. Thus

$\tilde{\mu}_\beta(0') = r\underset{t\in f^{-1}(x)}{\sup}\ \tilde{\mu}_\text{Ш}(t) = \tilde{\mu}_\text{Ш}(0) \geqslant \tilde{\mu}_\text{Ш}(x) = \underset{t\in f^{-1}(x)}{\text{rsup}}\ \tilde{\mu}_\text{Ш}(t) = \tilde{\mu}_\beta(x')$

$\lambda_\beta(0') = \underset{t\in f^{-1}(0')}{\inf}\ \lambda_\text{Ш}(t) = \lambda_\text{Ш}(0) \leq \lambda_\text{Ш}(x) = \underset{t\in f^{-1}(x')}{\inf}\ \lambda_\text{Ш}$, for all $x \in X$.

Which implies that $\tilde{\mu}_\beta(0') \geqslant \tilde{\mu}_\beta(x')$, and $\lambda_\beta(0') \leq \lambda_\beta(x')$, for all $x' \in Y$.

For any $x', y' \in Y$, let $x_0 \in f^{-1}(x'), y_0 \in f^{-1}(y')$ be such that

$\tilde{\mu}_\text{Ш}(x_0+y_0) = r\underset{t\in f^{-1}(x'+'y')}{\sup}\ \tilde{\mu}_\text{Ш}(t), \tilde{\mu}_\text{Ш}(y_0) = r\underset{t\in f^{-1}(x'+'y')}{\sup}$ and

$\tilde{\mu}_\text{Ш}(x_0) = \tilde{\mu}_\beta(x') = \underset{t\in f^{-1}(x')}{\text{rsup}}$ and

$\lambda_\text{Ш}(x_0+y_0) = \underset{t\in f^{-1}(x'+'y')}{\inf}\ \lambda_\text{Ш}(t), \lambda_\text{Ш}(y_0) = \underset{t\in f^{-1}(y')}{\inf}\ \lambda_\text{Ш}(t)$ and

$\lambda_\text{Ш}(x_0) = \underset{t\in f^{-1}(x')}{\inf}\ \lambda_\text{Ш}(t)$. Also, $\tilde{\mu}_\beta(x') = \underset{t\in f'(x')}{\text{rsup}}$

$\tilde{\mu}_\text{Ш}(t) = \tilde{\mu}_\text{Ш}(x_0) \geqslant r\min\{\tilde{\mu}_\text{Ш}(x_0+y_0), \tilde{\mu}_\text{Ш}(y_0)\}$

$= r\min\{r\underset{t\in f^{-1}(x'+y')}{sup}\ \tilde{\mu}_\text{Ш}(t), r\underset{t\in f^{-1}(y')}{sup}\ \tilde{\mu}_\text{Ш}(t)\}$

$= r\min\{\tilde{\mu}_\beta(x'+'y'), \tilde{\mu}_\beta(y')\}$ and

$\lambda_\text{Ш}(x') = \underset{x_0\in f^{-1}(x')}{\inf}\ \lambda_\text{Ш}(x_0) \leq \max\{\lambda_\text{Ш}(x_0+y_0), \lambda_\text{Ш}(y_0)\}$

$= \max\{\underset{t\in f^{-1}(x'+y')}{inf}\ \lambda_\text{Ш}(t), \underset{t\in f^{-1}(y')}{inf}\ \lambda_\text{Ш}(t)\}$

$= \max\{\lambda_{\mathrm{III}}(x'+'y'), \lambda_{\mathrm{III}}(y')\}$

And

$\tilde{\mu}_{\mathrm{III}}(x_0) = \underset{t\in f^{-1}(x')}{\mathrm{rsup}}\, \tilde{\mu}_{\mathrm{III}}(t), \tilde{\mu}_{\mathrm{III}}(y_0) = \underset{t\in f^{-1}(y')}{\mathrm{rsup}}\, \tilde{\mu}_{\mathrm{III}}(t)$

And $\tilde{\mu}_{\mathrm{III}}(x_0 + y_0) = \tilde{\mu}_{\beta}(x'+'y') = \underset{t\in f^{-1}(x'+'y')}{\mathrm{rsup}}$

$\lambda_{\mathrm{III}}(x_0) = \underset{t\in f^{-1}(x')}{\inf} \lambda_{\mathrm{III}}(t), \lambda_{\mathrm{III}}(y_0) = \underset{t\in f^{-1}(y')}{\inf} \lambda_{\mathrm{III}}(t)$

$\lambda_{\mathrm{III}}(x_0 + y_0) = \underset{t\in f^{-1}(x'+'y')}{\inf} \lambda_{\mathrm{III}}(t)$

Also $\tilde{\mu}_{\beta}(x' + y') = \underset{t\in f^{-1}(x'+y')}{\mathrm{rsup}}\, \tilde{\mu}_{\mathrm{III}}(t) = \tilde{\mu}_{\mathrm{III}}(x_0 + y_0)$

$\geqslant r \min\{\tilde{\mu}_{\mathrm{III}}(x_0), \tilde{\mu}_{\mathrm{III}}(y_0)\},$

$= r \min\{\underset{t\in f^{-1}(x')}{sup}\, \tilde{\mu}_{\mathrm{III}}(t), r \underset{t\in f^{-1}(y')}{sup}\, \tilde{\mu}_{\mathrm{III}}(t)\}$

$\geqslant r \min\{\tilde{\mu}_{\beta}(x'), \tilde{\mu}_{\beta}(y')\}$ and

$\lambda_{\mathrm{III}}(x' + y') = \underset{(x_0+y_0)\in f^{-1}(x'+y')}{\inf} \lambda_{\mathrm{III}}(x_0 + y_0)$

$\leq \max\{\lambda_{\mathrm{III}}(x_0), \lambda_{\mathrm{III}}(y_0)\},$

$= \max\{\underset{t\in f^{-1}(x')}{inf}\, \lambda_{\mathrm{III}}(t), \underset{t\in f^{-1}(y')}{inf}\, \lambda_{\mathrm{III}}(t)\}$

$= \max\{\lambda_{\mathrm{III}}(x'), \lambda_{\mathrm{III}}(y')\}$

Henece, β is a cubic fuzzy Bd-ideal of Y. ■

4. CARTESIAN PRODUCT CUBIC OF BD - ALGEBRA

In the Section, we will introduce the notions of cartesian product of cubic

Bd-subalgebras and cubic fuzzy ideals in a Bd-algebra.

Definition 5.1

Let $\mathrm{III}_1 = < \tilde{\mu}_{\mathrm{III}_1}, \lambda_{\mathrm{III}_1} >$ be a cubic fuzzy subset of X and $\mathrm{III}_2 = < \tilde{\mu}_{\mathrm{III}_2}, \lambda_{\mathrm{III}_2} >$ be a cubic fuzzy subset of Y. The cartesian product of III_1 and III_2 is defined as $\mathrm{III}_1 \times \mathrm{III}_2 = \left(X \times Y, \tilde{\mu}_{\mathrm{III}_1} \times \tilde{\mu}_{\mathrm{III}_2}, \lambda_{\mathrm{III}_1} \times \lambda_{\mathrm{III}_2}\right)$ where,

$\tilde{\mu}_{\mathrm{III}_1} \times \tilde{\mu}_{\mathrm{III}_2}: X \times Y \to [0,1]$ and

$\lambda_{\mathrm{III}_1} \times \lambda_{\mathrm{III}_2} = X \times Y \to [0,1]\ \forall x \in X, y \in Y$, such that

$\left(\tilde{\mu}_{\mathrm{III}_1} \times \tilde{\mu}_{\mathrm{III}_2}\right)(x, y) = r \min\{\tilde{\mu}_{\mathrm{III}_1}(x), \tilde{\mu}_{\mathrm{III}_2}(y)\}$ and

$\left(\lambda_{\mathrm{III}_1} \times \lambda_{\mathrm{III}_2}\right)(x, y) = \max\{\lambda_{\mathrm{III}_1}(x), \lambda_{\mathrm{III}_2}(y)\}$

Theorem 5.2

Let $\mathrm{Ш}_1 =< \tilde{\mu}_{\mathrm{Ш}_1}, \lambda_{\mathrm{Ш}_1} >$ be a cubic fuzzy Bd-subalgebra of X and

$\mathrm{Ш}_2 =< \tilde{\mu}_{\mathrm{Ш}_2}, \lambda_{\mathrm{Ш}_2} >$ be a cubic fuzzy Bd-subalgebra of Y then $\mathrm{Ш}_1 \times \mathrm{Ш}_2$ is a cubic fuzzy Bd-subalgebra of $X \times Y$.

Proof:

Let $(x_1, y_1) \in X \times Y$ and $(x_2, y_2) \in X \times Y$, then

$$\left(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2}\right)\left((x_1, y_1) + (x_2, y_2)\right) = \left(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2}\right)\left((x_1 + x_2), (y_1 + y_2)\right)$$

$$= \min\left\{\tilde{\mu}_{\mathrm{Ш}_1}(x_1 + x_2), \tilde{\mu}_{\mathrm{Ш}_2}(y_1 + y_2)\right\}$$

$$\geqslant \min\{r \min\{\tilde{\mu}_{\mathrm{Ш}_1}(x_1), \tilde{\mu}_{\mathrm{Ш}_2}(x_2), r \min\left(\tilde{\mu}_{\mathrm{Ш}_2}(y_2), \tilde{\mu}_{\mathrm{Ш}_2}(y_2)\right)$$

$$= r \min\{\min\{\tilde{\mu}_{\mathrm{Ш}_1}(x_1), \tilde{\mu}_{\mathrm{Ш}_2}(y_1), \min\left(\tilde{\mu}_{\mathrm{Ш}_1}(x_2), \tilde{\mu}_{\mathrm{Ш}_2}(y_2)\right)\}$$

$$= r \min\{\left(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2}\right)(x_1, y_1), \left(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2}\right)(x_2, y_2)\}$$

And

$$\left(\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2}\right)\left((x_1, y_1) + (x_2, y_2)\right) = \left(\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2}\right)\left((x_1 + x_2), (y_1 + y_2)\right)$$

$$= \max\left\{\lambda_{\mathrm{Ш}_1}(x_1 + x_2), \lambda_{\mathrm{Ш}_2}(y_1 + y_2)\right\}$$

$$\leq \max\{\max\left\{\lambda_{\mathrm{Ш}_1}(x_1), \lambda_{\mathrm{Ш}_2}(x_2)\right\},$$

$$= \max\{\max\left\{\lambda_{\mathrm{Ш}_1}(x_1), \lambda_{\mathrm{Ш}_2}(y_1)\right\}, \max\left\{\lambda_{\mathrm{Ш}_1}(x_2), \lambda_{\mathrm{Ш}_2}(y_2)\right\}\}$$

$$= \max\{\left(\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2}\right)(x_1, y_1), \left(\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2}\right)(x_2, y_2)\}$$

Hence $\mathrm{Ш}_1 \times \mathrm{Ш}_2$ is a cubic fuzzy Bd-subalgebra of $X \times Y$. ■

Theorem 5.3

Let $\mathrm{Ш}_1 =< \tilde{\mu}_{\mathrm{Ш}_1}, \lambda_{\mathrm{Ш}_1} >$ be a cubic fuzzy Bd-subset of X and

$\mathrm{Ш}_2 =< \tilde{\mu}_{\mathrm{Ш}_2}, \lambda_{\mathrm{Ш}_2} >$ be a cubic fuzzy Bd-subset of Y. If $\mathrm{Ш}_1 \times \mathrm{Ш}_2$ is a cubic fuzzy Bd-subalgebra of $X \times Y$, then $\mathrm{Ш}_2$ is a cubic fuzzy Bd-subalgebra of Y.

Proof:

Assume that $\mathrm{Ш}_1 \times \mathrm{Ш}_2$ is a cubic fuzzy Bd-subalgebra of $X \times Y$, then

$$\left(\left(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2}\right)\right)\left((x_1, y_1) + (x_2, y_2)\right) \geqslant \min\{\left(\left(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2}\right)((x_1, y_1), \left(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2}\right)(x_2, y_2)\} \ldots (1)$$

Putting $x_1 = x_2 = 0$ in (1) we get,

$$\left(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2}\right)(0, y_1) + (0, y_2) \geqslant r \min\left\{\left(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2}\right)(0, y_1), \left(\tilde{\mu}_{\mathrm{Ш}_1} \times \ \tilde{\mu}_{\mathrm{Ш}_2}\right)(0, y_2)\right\},$$

$$\left(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2}\right)(0, y_1 + y_2) \geqslant r \min\left\{\left(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2}\right)(0, y_1), \left(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2}\right)(0, y_2)\right\}$$

then we have

$\tilde{\mu}_{\mathrm{Ш}_2}(y_1 + y_2) \geq r \min\{\tilde{\mu}_{\mathrm{Ш}_2}(y_1), \tilde{\mu}_{\mathrm{Ш}_2}(y_2)\}$

Hence $\tilde{\mu}_{\mathrm{Ш}_2}$ is a fuzzy Bd-subalgebra of X.

Also,

$\left(\left(\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2}\right)\right)((x_1, y_1) + (x_2, y_2)) \leq \max\{(\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2})(x_1, y_1), (\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2})(x_2, y_2)\}$ … (2)

Putting $x_1 = x_2 = 0$ in (2) we get,

$$(\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2})(0, y_1)(0, y_2) \leq \max\{(\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2})(0, y_1), (\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2})(0, y_2)\}$$

$(\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2})(0, y_1 + y_2) \leq \max\{(\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2})(0, y_1), (\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2})(0, y_2)\}$

then we have

$\lambda_{\mathrm{Ш}_2}(y_1 + y_2) \leq \max\{\lambda_{\mathrm{Ш}_2}(y_1), \lambda_{\mathrm{Ш}_2}(y_2)\}$,

Then Hence $\lambda_{\mathrm{Ш}_2}$ is anti-fuzzy Bd-subalgebra of Y.

Hence $\mathrm{Ш}_2$ is a cubic fuzzy Bd-subalgebra of Y. ■

Theorem 5.4

Let $\mathrm{Ш}_1 =< \tilde{\mu}_{\mathrm{Ш}_1}, \lambda_{\mathrm{Ш}_1} >$ he a cubic fuzzy Bd-subset of X and $\mathrm{Ш}_2 =< \tilde{\mu}_{\mathrm{Ш}_2}, \lambda_{\mathrm{Ш}_2} >$ be a cubic fuzzy Bd-subset of Y. If $\mathrm{Ш}_1 \times \mathrm{Ш}_2$ is a cubic fuzzy Bd- Ideal of $X \times Y$.

Proof

For any $X = (x_1, x_2) \in X \times X$, we have

$(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2})(0) = (\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2})(0,0) = r \min\{\tilde{\mu}_{\mathrm{Ш}_1}(0), \tilde{\mu}_{\mathrm{Ш}_2}(0)\}$

$\geqslant r \min\left(\tilde{\mu}_{\mathrm{Ш}_1}(x_1), \tilde{\mu}_{\mathrm{Ш}_2}(x_2)\right)$

$= (\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2})(x_1, x_2) = (\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2})(x)$

$(\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2})(0) = (\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2})(0,0) = \max\left(\lambda_{\mathrm{Ш}_1}(0), \lambda_{\mathrm{Ш}_2}(0)\right)$

$\leq \max\left(\lambda_{\mathrm{Ш}_1}(x_1), \lambda_{\mathrm{Ш}_2}(x_2)\right) = (\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2})(x_1, x_2)$

$= (\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2})(x)$.

Let $x = (x_1, x_2), y = (y_1, y_2)$

$(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2})(x) = (\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2})(x_1, x_2)$

$\geqslant r\,min\{rmin\ \tilde{\mu}_{\mathrm{Ш}_1}(x_1 + y_1), \tilde{\mu}_{\mathrm{Ш}_1}(\ y_1)\}, r\,min\{\tilde{\mu}_{\mathrm{Ш}_2}(x_2 + y_2), \tilde{\mu}_{\mathrm{Ш}_2}(y_2)\}$

$$= r\min\{r\min\{\tilde{\mu}_{\mathrm{Ш}_1}(x_1,y_1),\tilde{\mu}_{\mathrm{Ш}_2}(x_2,y_2)\},r\min\{\tilde{\mu}_{\mathrm{Ш}_1}(y_1),\tilde{\mu}_{\mathrm{Ш}_2}(y_2)\}\}$$

$$= r\min\{(\tilde{\mu}_{\mathrm{Ш}_1}\times\tilde{\mu}_{\mathrm{Ш}_2})((x_1,x_2)+(y_1,y_2)),r(\tilde{\mu}_{\mathrm{Ш}_1}\times\tilde{\mu}_{\mathrm{Ш}_2})(y_1,y_2)\}$$

$$= r\min\{(\tilde{\mu}_{\mathrm{Ш}_1}\times\tilde{\mu}_{\mathrm{Ш}_2})(x+y),(\tilde{\mu}_{\mathrm{Ш}_1}\times\tilde{\mu}_{\mathrm{Ш}_2})(y)\},$$

$$(\lambda_{\mathrm{Ш}_1}\times\lambda_{\mathrm{Ш}_2})(x) = (\lambda_{\mathrm{Ш}_1}\times\lambda_{\mathrm{Ш}_2})(x_1,x_2)$$

$$\leq \max\{\max\{\lambda_{\mathrm{Ш}_1}(x_1+y_1),\lambda_{\mathrm{Ш}_2}(y_1)\},\max\{\lambda_{\mathrm{Ш}_2}(x_2+y_2),\lambda_{\mathrm{Ш}_2}(y_2)\}\}$$

$$= \max\{\max\{\lambda_{\mathrm{Ш}_1}(x_1+y_1),\lambda_{\mathrm{Ш}_2}(x_2+y_2)\},\max\{\lambda_{\mathrm{Ш}_1}(y_1),\lambda_{\mathrm{Ш}_2}(y_2)\}\}$$

$$= \max\{(\lambda_{\mathrm{Ш}_1}\times\lambda_{\mathrm{Ш}_2})((x_1,x_2)+(y_1,y_2)),(\lambda_{\mathrm{Ш}_1}\times\lambda_{\mathrm{Ш}_2})(y_1,y_2)\}$$

$$= \max\{(\lambda_{\mathrm{Ш}_1}\times\lambda_{\mathrm{Ш}_2})(x+y),(\lambda_{\mathrm{Ш}_1}\times\lambda_{\mathrm{Ш}_2})(y)\},$$

And

$$(\tilde{\mu}_{\mathrm{Ш}_1}\times\tilde{\mu}_{\mathrm{Ш}_2})(x+y) = (\tilde{\mu}_{\mathrm{Ш}_1}\times\tilde{\mu}_{\mathrm{Ш}_2})(x_1,x_2),(y_1,y_2)$$

$$= (\tilde{\mu}_{\mathrm{Ш}_1}\times\tilde{\mu}_{\mathrm{Ш}_2})(x_1,x_2),(y_1,y_2)$$

$$= (\tilde{\mu}_{\mathrm{Ш}_1}\times\tilde{\mu}_{\mathrm{Ш}_2})(x_1+y_1,x_2+y_2)$$

$$\geqslant r\min\{\tilde{\mu}_{\mathrm{Ш}_1}(x_1+y_1),\tilde{\mu}_{\mathrm{Ш}_2}(x_2+y_2)\}$$

$$\geqslant r\min\{r\min\{\tilde{\mu}_{\mathrm{Ш}_1}(x_1),\tilde{\mu}_{\mathrm{Ш}_1}(y_1),r\min\{\tilde{\mu}_{\mathrm{Ш}_2}(x_2),\tilde{\mu}_{\mathrm{Ш}_2}(y_2)\}\}\}$$

$$= r\min\{r\min\{\tilde{\mu}_{\mathrm{Ш}_1}(x_1),\tilde{\mu}_{\mathrm{Ш}_2}(x_2),r\min\{\tilde{\mu}_{\mathrm{Ш}_1}(y_1),\tilde{\mu}_{\mathrm{Ш}_2}(y_2)\}\}\}$$

$$= r\min\{(\tilde{\mu}_{\mathrm{Ш}_1}\times\tilde{\mu}_{\mathrm{Ш}_2})(x_1,x_2),(\tilde{\mu}_{\mathrm{Ш}_1}\times\tilde{\mu}_{\mathrm{Ш}_2})(y_1,y_2)\}$$

$$= r\min\{(\tilde{\mu}_{\mathrm{Ш}_1}\times\tilde{\mu}_{\mathrm{Ш}_2})(x),(\tilde{\mu}_{\mathrm{Ш}_1}\times\tilde{\mu}_{\mathrm{Ш}_2})(y)\}$$

$$(\lambda_{\mathrm{Ш}_1}\times\lambda_{\mathrm{Ш}_2})(x+y) = (\lambda_{\mathrm{Ш}_1}\times\lambda_{\mathrm{Ш}_2})(x_1,x_2)+(y_1,y_2)$$

$$= (\lambda_{\mathrm{Ш}_1}\times\lambda_{\mathrm{Ш}_2})(x_1+x_2),(y_1+y_2)$$

$$= \max\{\lambda_{\mathrm{Ш}_1}(x_1+y_1),\lambda_{\mathrm{Ш}_2}(x_2+y_2)\}$$

$$\leq \max\{\max\{\lambda_{\mathrm{Ш}_1}(x_1),\lambda_{\mathrm{Ш}_1}(y_1),\max\{\lambda_{\mathrm{Ш}_2}(x_2),\lambda_{\mathrm{Ш}_2}(y_2)\}\}\}$$

$$= \max\{\max\{\lambda_{\mathrm{Ш}_1}(x_1),\lambda_{\mathrm{Ш}_2}(x_2),\max\{\lambda_{\mathrm{Ш}_1}(y_1),\lambda_{\mathrm{Ш}_2}(y_2)\}\}\}$$

$$= \max\{(\lambda_{\mathrm{Ш}_1}\times\lambda_{\mathrm{Ш}_2})(x_1,x_2),(\lambda_{\mathrm{Ш}_1}\times\lambda_{\mathrm{Ш}_2})(y_1,y_2)\}$$

$$= \max\{(\lambda_{\mathrm{Ш}_1}\times\lambda_{\mathrm{Ш}_2})(x),(\lambda_{\mathrm{Ш}_1}\times\lambda_{\mathrm{Ш}_2})(y)\}$$

Hence $\mathrm{Ш}_1\times\mathrm{Ш}_2$ is a cubic fuzzy Bd-ideal of $X\times Y$. ■

Theorem 5.6

Let $\mathrm{Ш}_1 =< \tilde{\mu}_{\mathrm{Ш}_1}, \lambda_{\mathrm{Ш}_1} >$ be a cubic fuzzy Bd-subset of X and

$\mathrm{Ш}_2 =< \tilde{\mu}_{\mathrm{Ш}_2}, \lambda_{\mathrm{Ш}_2} >$ be a cubic fuzzy Bd-subset of Y. If $\mathrm{Ш}_1 \times \mathrm{Ш}_2$ is a cubic fuzzy Bd-ideal of $X \times Y$, then $\mathrm{Ш}_1$ is a cubic fuzzy Bd-ideal of X.

Proof:

Let $x = (x_1, x_2), y = (y_1, y_2) \in X \times X$

$$\left(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2}\right)((0,0),(0,0)) = r\min\left\{\left(\tilde{\mu}_{\mathrm{Ш}_1}\right)(0,0), \left(\tilde{\mu}_{\mathrm{Ш}_2}\right)(0,0)\right\}$$

$$\geqslant r\min\left\{\left(\tilde{\mu}_{\mathrm{Ш}_1}\right)(x_1, y_1), \left(\tilde{\mu}_{\mathrm{Ш}_2}\right)(x_2, y_2)\right\}$$

Putting $y = (y_1, y_2) = (0,0)$, we have $\tilde{\mu}_{\mathrm{Ш}_1}(0,0) \geqslant \tilde{\mu}_{\mathrm{Ш}_1}(x_1, x_2)$,

$$\left(\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2}\right)((0,0),(0,0)) = \max\left\{\left(\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_1}\right)(0,0), \left(\lambda_{\mathrm{Ш}_1}, \lambda_{\mathrm{Ш}_2}\right)(0,0)\right\}$$

$$\leq \max\left\{\left(\lambda_{\mathrm{Ш}_1}\right)(x_1, x_2), \left(\lambda_{\mathrm{Ш}_2}\right)(y_1, y_2)\right\}$$

Putting $y = (y_1, y_2) = (0,0)$, we have $\lambda_{\mathrm{Ш}_1}(0,0) \leq \lambda_{\mathrm{Ш}_2}(x_1, x_2)$

Assume that $\mathrm{Ш}_1 \times \mathrm{Ш}_2$ is a cubic fuzzy Bd-ideal of $X \times Y$, then

$$\left(\left(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2}\right)\right)\left((x_1, x_2)\right) \geqslant r\min\left\{\left(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2}\right)(x_1, x_2) + (y_1, y_2)\right.$$

$$\left., \left(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2}\right)(y_1, y_2)\right\} \ldots (1)$$

Putting $x_2 = y_2 = 0$, then we have

$\left(\tilde{\mu}_{\mathrm{Ш}_1}\right)(x_1, 0) \geqslant r\min\left\{\left(\tilde{\mu}_{\mathrm{Ш}_1}\right)\left((x_1, 0) + (y_1, 0)\right), \left(\tilde{\mu}_{\mathrm{Ш}_1}\right)(y_1, 0)\right\}$, thus

$\left(\tilde{\mu}_{\mathrm{Ш}_1}\right)(x_1) \geqslant r\min\left\{\left(\tilde{\mu}_{\mathrm{Ш}_1}\right)(x_1 + y_1), \left(\tilde{\mu}_{\mathrm{Ш}_1}\right)(y_1)\right\}$. And

$$\left(\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2}\right)\left((x_1, x_2)\right) \leq \max\left\{\left(\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2}\right)\left((x_1, y_1) + (y_1, y_2)\right), \left(\lambda_{\mathrm{Ш}_1} \times \lambda_{\mathrm{Ш}_2}\right)(y_1, y_2)\right\} \ldots (2)$$

Putting $x_2 = y_2 = 0$, the we have

$\left(\lambda_{\mathrm{Ш}_1}\right)(x_1, 0) \leq \max\left\{\left(\lambda_{\mathrm{Ш}_1}\right)\left((x_1, 0) + (y_1, 0)\right), \left(\lambda_{\mathrm{Ш}_1}\right)(y_1, 0)\right\}$, thus

$\lambda_{\mathrm{Ш}_1}(x_1) \leq \max\left\{\lambda_{\mathrm{Ш}_1}(x_1 + y_1), \lambda_{\mathrm{Ш}_1}(y_1)\right\}$,

And

$$\left(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2}\right)\left((x_1, x_2) + (y_1, y_2)\right) \geq r\min\left\{\left(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2}\right)(x_1, x_2), \left(\tilde{\mu}_{\mathrm{Ш}_1} \times \tilde{\mu}_{\mathrm{Ш}_2}\right)(y_1, y_2)\right\} \ldots (3)$$

Putting $x_2 = y_2 = 0$, then we have

$$\left(\tilde{\mu}_{\mathrm{Ш}_1}\right)(x_1, 0) + (y_1, 0) \geq r\min\left\{\tilde{\mu}_{\mathrm{Ш}_1}(x_1, 0), \tilde{\mu}_{\mathrm{Ш}_1}(y_1, 0)\right\}$$

Thus

$\tilde{\mu}_{\mathrm{W}_1}(x_1 + y_1) \geq r \min\{\tilde{\mu}_{\mathrm{W}_1}(x_1), \tilde{\mu}_{\mathrm{W}_1}(y_1)\}$. And

$$(\lambda_{\mathrm{W}_1} \times \lambda_{\mathrm{W}_2})((x_1, x_2), (y_1, y_2)) \leq \max\left\{(\lambda_{\mathrm{W}_1} \times \lambda_{\mathrm{W}_2})\left((x_1, x_2), (\lambda_{\mathrm{W}_1} \times \lambda_{\mathrm{W}_2})(y_1, y_2)\right)\right\} \ldots (4)$$

Putting $x_2 = y_2 = 0$, the we have

$$(\lambda_{\mathrm{W}_1})(x_1, 0) + (y_1, 0) \leq \max\{(\lambda_{\mathrm{W}_1})(x_1, 0), (\lambda_{\mathrm{W}_1})(y_1)\}$$

Thus

$$\lambda_{\mathrm{W}_1}(x_1 + y_1) \leq \max\{\lambda_{\mathrm{W}_1}(x_1), \lambda_{\mathrm{W}_2}(y_1)\}.$$

Hence W_1 is a cubic fuzzy Bd-ideal of X. ■

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