International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 9 Issue 8 August - 2025, Pages: 104-105

Theory of Differential Operators

Mohammed A. Akraa[1], Amera W. Al-funjan[2], Huda A.Hadi[3],

University of Babylon, College of Education for pure Science, Iraq

mat.mohammed.akraa@uobabylon.edu.iq[1], amera.alfunjan@uobabylon.edu.iq[2], hudahdi@uobabylon.edu.iq [3]

Abstract: *The concept of differential operators between differentiable vector bundles is discussed in detail in this work. We define a differential operator as a local linear mapping and analyze its fundamental properties, such as local structure and order. This paper explains how differential operators can be locally represented by matrices of partial differential operators and introduces the concept of the principal symbol as a coordinate-independent linear operator. The idea of the transposed operator and its connection to the principal symbol are also explained in detail*

Keywords: Differential Operators, Hodge theory, complex manifolds, cohomology

1. The introduction

In differential geometry, differential operators are crucial instruments for analyzing the relationships between distinct vector bundle segments. The reference work provides a precise definition of a differential operator between two vector bundles E and F on a differentiable manifold X, as well as its fundamental properties that connect local and global analysis. The definition of the operator, its local structure, and more complicated concepts like the principle symbol and the transposed operator are all covered in this essay.

2. Definition[8]

Let E and F be vector bundles over an n-dimensional manifold X that are differentiable. Any C-linear mapping, $\boldsymbol{P}: \mathcal{E}(\boldsymbol{E}) \to \mathcal{E}(\boldsymbol{F})$, that is local, meaning supp $\boldsymbol{Pf} \subset \boldsymbol{supp\ f}$ f for each section $\boldsymbol{f} \in \mathcal{E}(\boldsymbol{E})$, is a differential operator P of type $E \to F$.
The greatest integer p such that $\boldsymbol{P(\phi^p f)(x) \neq 0}$ for some smooth function ϕ that vanishes at x and some local section f near x is the order of the operator P at a point $x \in X$, represented by $p(x)$. The operator P's order can be expressed as $\boldsymbol{sup_{x\in X} p(x)}$

2.1 A Differential Operator's Local Structure[2,8]

A matrix of partial differential operators with differentiable coefficients can be used to represent a differential operator of type $\boldsymbol{E \to F, P}$, in any coordinate neighborhood $\boldsymbol{U}$ where the bundles $\boldsymbol{E}$ and $\boldsymbol{F}$ are trivial. More specifically, $\boldsymbol{P_V}$can be expressed as follows if it is the mapping on an open set $\boldsymbol{V \subset\subset U}$:

$$\boldsymbol{P(x, D) = \sum_{|\nu|\le p} P_\nu(x) D^\nu}$$

where k and l are the ranks of the bundles E and F, respectively, and $\boldsymbol{P_\nu(x)}$ are $(l \times k)$-matrices of differentiable functions on V. This theorem establishes that the analysis of partial differential operators on open subsets of R^n is equivalent to the local analysis of differential operators between sections of vector bundles.

3. The Differential Operator Symbol

3.1 Definition [4,8]

A linear map from $\boldsymbol{E_x \to F_x}$defines the principle symbol of a differential operator, $\boldsymbol{\sigma(P) \in Hom(\pi^* E \to \pi^* F)}$, at a fixed point $\boldsymbol{(x, z) \in T^*(X)}$ in the cotangent bundle. We select a section $\boldsymbol{f \in \mathcal{E}(E)}$ such that $\boldsymbol{f(x) = e}$ and a function $\phi \in \mathcal{E}(X)$ such that $\phi(x) = 0$ and $\boldsymbol{d\phi(x) = z}$ in order to define this symbol. The definition of the symbol is then as follows:

$$\boldsymbol{\sigma(P)(x, z)e = (\sqrt{-1})^p p!\, P(\phi^p f)(x) \in F_x}$$

The selection of ϕ and f has no bearing on this definition. The symbol is provided by the following formula if the operator P is locally represented by the matrix $\boldsymbol{P(x, D) = \sum_{|\nu|\le p} P_\nu(x) D^\nu}$and $\boldsymbol{z = \sum_{j=1}^n z_j dx_j}$:

$$\boldsymbol{\sigma(P)(x, z) = \sum_{|\nu|=p} P_\nu(x)\, z^\nu}$$

3.2 The Operator Transposed[8]

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 9 Issue 8 August - 2025, Pages: 104-105

A transposed operator $\boldsymbol{P}^{\boldsymbol{T}}$ of type $\boldsymbol{F}' \to \boldsymbol{E}'$ exists for each differential operator $\boldsymbol{P}$ of type $\boldsymbol{E} \to \boldsymbol{F}$, where $\boldsymbol{E}'$ and $\boldsymbol{F}'$ are the dual bundles. The formula for the transposed operator's local representation is

$$\boldsymbol{P}^{\boldsymbol{T}}(\boldsymbol{x}, \boldsymbol{D}) = \sum_{|\boldsymbol{\nu}| \leq \boldsymbol{p}} (-\mathbf{1})^{|\boldsymbol{\nu}|} \boldsymbol{D}^{\boldsymbol{\nu}} (\boldsymbol{P}_{\boldsymbol{\nu}}(\boldsymbol{x})^{\boldsymbol{T}} \cdot)$$

With the following connection, it is simple to confirm that the symbol of the transposed operator is related to the symbol of the original operator:
$\boldsymbol{\sigma}(\boldsymbol{P}^{\boldsymbol{T}}) = (-\mathbf{1})^{p} \boldsymbol{\sigma}(\boldsymbol{P})^{\boldsymbol{T}}$.

4. Summary and Conclusion

The reference material demonstrates that matrices of partial differential operators can be used to represent the precise local structure of differential operators between differentiable vector bundles. In order to reduce the global analysis of operators to local analysis, this characteristic is essential. While the transposed operator makes the connection between the original operator and operators on dual bundles clear, the principle symbol concept offers a strong tool for researching the geometric properties of the operator, such as ellipticity. The theory of differential operators and its applications in differential geometry and analysis are based on these ideas.

References:

[1] Cartan, H. (1950). *Homological Algebra*.

[2] de Rham, G. (1955). *Variétés Differentiables*.

[3] Gugenheim, V. K. A. M., & Spencer, D. C. (1957). *Homological Algebra, with applications to differential geometry*.

[4] Guillemin, V. W., & Singer, I. M. (1974). *Differential Operators on Differentiable Manifolds*.

[5] Palamodov, V. P. (1970). *Linear Differential Operators with Constant Coefficients*.

[6] Peetre, J. (1961). *Une caractérisation abstraite des opérateurs différentiels*.

[7] Spencer, D. C. (1965). *Deformation of structures on manifolds*.

[8] Tarkhanov, N. (1995). *Complexes of Differential Operators*. Kluwer Academic Publishers, Dordrecht, Netherlands.

[9] Wells, R. O. (1980). *Differential Analysis on Complex Manifolds*.