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$\alpha -$ Scott Topologies Desighned on Continuous Mappings

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Abstract: *The paper deals with giving the definition of some new topologies designed by the set of all α-continuous mappings between the topological spaces like a α- Scott topology and a α- strong Scott topology; many theorems are proved and discussed relations between these topologies and the older one.*

Keywords: Scott topology, α- open set, Admissible space,Strong topology, Isbell topology

Introduction:

The notion of set theory is one of the important concepts in topological spaces. It plays a good role in general topology. There are many tools in set theory and their applications in many different topological spaces. An α- open set is one of these tools. It was first studied by O. Njasted in 1965. These sets form a topology on X which is finer than a topology τ on X, see ([6], [7]). The paper aim is to introduce and investigate a new class of continuous mappings in a topological spaces or in any other subjects joints with topology, one need to know what is the topology in which these mappings are continuous, however, there many methods to define and constructing these topologies, for examples, the methods of projective limit topology and the inductive limit topology that construct topologies called projective topology and inductive topology respectively, these methods are always used in Mappings and analysis, especially in topological vector spaces topics. Many authors used good methods to define topologies on spaces of continuous mappings between the topological spaces and called them the Scott topologies on these spaces and the strong Scott topologies. Here, we modify and introduce new topologies by the set of all α open subsets of topological spaces and we call them α Scott topologies. In this work, new types of Scott topologies on the set of all α- open subsets of a topological space Y are studied. Let Y and Z be two given topological spaces, by α *O(Y)* we denoted, the set of all α - open subsets of Y and by *O(Z)* we denoted, the set of all open subsets of Z and by α *C(Y,Z)* we mean, the set of all α - continuous maps from Y to Z. We study α -Scott type topologies on α *O(Y)* which we call it α - Scott topologies and we construct admissible topologies on α *C(Y,Z)* and $\alpha O_z(Y)$= $\{f^{-1}(U) \in \alpha$ *O(Y)* : $f \in \alpha$ *C(Y,Z)* and $U \in$ *O(Z)*. Many relations between these spaces are investigated and studied.

1. Basic Definitions and Concepts

Definition 1.1[2]

A set A is an α- open, if A⊆ int(cl (int(A)) and it's complement is an α - closed [6]. Also.The family of all α - open set (resp., α – closed), denoted by αO(X) (resp., αC(X))

Definition 1.2[7]

The union(resp., intersection) for each α-open(resp., α-closed)sets contained in(resp., contained) A is named an α-interior(resp., α-closure) of A denoted by α-int(A)(resp., α-cl(A))

Proposition 1.3[7]

Assume that A is any set. We have:

(i) α-intX = X , α-int∅ = ∅.
(ii) (ii) α-int(A)⊆ A.
(iii) (iii) A is a α-open set if and only if A=α-int(A).

(iv) α-int(A) = α-int(α-int(A)).

(iv) If A⊆B , then, α-int(A) ⊆α-int(B).

(v) (vi) α-int(A⋂ B) = α-int(A) ⋂ α-int(B).

Remark 1.4[7]

For any set A. We have: int(A) ⊆α-int(A) ⊆α-cl(A) ⊆ cl(A).

Theorem 1.5[7]

Let X_1, X_2 be spaces .Then U, V are α-open sets in X_1, X_2 respectively if and only if $U \times V$ is an α- open set in $X_1 \times X_2$.

Remark 1.6[7]

Every open is α-open set.

Definition 1.7[3]
A mapping f between two topological spaces *X* and *Y* is said to be α-continuous if the inverse image by *f* of every open set in *Y* is α- open in *X*.

Example 1.8[3]
Let f be a continuous mapping from any topology on a set *X* to a topology (*X*, T). Then it is α-continuous since any open set is α-open also by the same reason any continuous function is α-continuous.

Definition 1.9[3]

let *Y* and *Z* be two given topological spaces, *O(Y)* (respectively, *O (Z)*) the set of all open subsets of *Y* (respectively, *Z*), and *C(Y,Z)* the set of all continuous maps from *Y* to *Z*. We study Scott type topologies on *O(Y)* and we construct admissible topologies on *C(Y,Z)* and $O_z(Y)$= $\{f^{-1}(U) \in O(Y) : f \in C(Y,Z)$ and $U \in O(Z)\}$, introducing new problems in the field.

Definition 1.10[3]
We denote by Let Y and *Z* are two fixed topological spaces and by *C(Y,Z)* the set of all continuous maps from *Y* to *Z*. If *T* is a topology on *C(Y,Z)*, then the corresponding
topological space is denoted by C_T *(Y,Z).*

Definition 1.11[3]

Let *X* be a topological space, by *O(X)* we denote, the family of all open subsets of *X* under a given topology and by $O_z(Y)$ the family $\{f^{-1}(U) \in O(Y) : f \in C(Y,Z)$ and $U \in O(Z)\}$.

Definition 1.12[3]

Let *X* be a topological space and let *F*: *X* × *Y* → *Z* be a continuous map. By δ we denote the map from *X* to the set *C(Y,Z)*, such that $\delta(x)(y) = F(x, y)$, for every $x \in X$ and $y \in Y$. Let *G* be a map from *X* to *C(Y,Z)*. By β we denote the map from *X* × *Y* to *Z*, such that $\beta(x, y) = G(x)(y)$, for every $(x, y) \in X \times Y$.

Definition 1.13[1, 2]

A topology *T* on *C(Y,Z)* is called admissible, if for every space *X*, the continuity of the map $G : X \to C_T(Y,Z)$, implies the continuity of the map β: *X*×*Y* → *Z* or equivalently the evaluation map $e : C_T(Y,Z) \times Y \to Z$, defined by $e(f, y) = f(y)$ for every $(f, y) \in C(Y,Z) \times Y$, is continuous.

Definition 1.14[3]

The Scott topology *ω(Y)* on *O(Y)* is defined as follows:
a sub collection *H* of *O(Y)* belongs to *ω(Y)* if:

1. $U \in H$, $V \in O(Y)$, and $U \subseteq V$ imply $V \in H$ and;

2. For every collection of open sets of Y, whose union belongs to *H*, there are finitely many elements of this collection whose union also belongs to *H*.

Definition 1.15[4]
The strong Scott topology $s\omega(Y)$ on *O(Y)* is defined as follows: a sub collection *H* of *O(Y)* belongs to $s\omega(Y)$ if:
1. $U \in H$, $V \in O(Y)$, and $U \subseteq V$ imply $V \in H$ and;
2. For every open cover of *Y* , there are finitely many elements of this cover
whose union belongs to *H*.

Definition 1.16[4, 5]
The Isbell topology on *C(Y, Z)*, denoted here by T_{Is}, is the topology which has as a subbasis the family of all sets of the form:
$(H,U) = \{ f \in C(Y,Z) : f^{-1}(U) \in H \}$, where $H \in \omega(Y)$ and $U \in O(Z)$.

Definition 1.17[4, 5]
The strong Isbell topology on *C(Y, Z)*, denoted here by sT_{Is}, is the topology which has as a subbasis the family of all sets of the form:
$(H,U) = \{ f \in C(Y,Z) : f^{-1}(U) \in H \}$, where $H \in s\omega(Y)$ and $U \in O(Z)$.

2. Admissible topologies on a space of α- continuous mappings

We denote by *Y* and *Z* two fixed topological spaces and by $\alpha C(Y,Z)$ the set of all α continuous maps from *Y* to *Z*. If *T* is a topology on $\alpha C(Y,Z)$, then the corresponding topological space is denoted by $\alpha C_T(Y,Z)$.

Let *X* be a topological space and let $F: X \times Y \rightarrow Z$ be a α - continuous map. By δ we denote the map from *X* to the set $\alpha\ C(Y,Z)$, such that $\delta(x)(y) = F(x, y)$, for every $x \in X$ and $y \in Y$. Let *G* be a map from *X* to $\alpha\ C(Y,Z)$. By β we denote the map from $X \times Y$ to *Z*, such that $\beta(x, y) = G(x)(y)$, for every $(x, y) \in X \times Y$.

Definition 2.1.

A topology *T* on $\alpha\ C(Y,Z)$ is called α -admissible, if for every topological space *X*, the continuity of the map $G : X \rightarrow bC_T(Y,Z)$. implies the continuity of the map $\beta: X \times Y \rightarrow Z$ or equivalently the evaluation map $e : \alpha C_T(Y,Z) \times Y \rightarrow Z$, defined by $e(f, y) = f(y)$ for every $(f, y) \in \alpha C(Y,Z) \times Y$, is α - continuous.

Definition 2.2.

The α -Scott topology $\alpha\omega(Y)$ on $\alpha\ O(Y)$ is defined as follows:
a sub collection *H* of $\alpha\ O(Y)$ belongs to $\alpha\omega(Y)$ if:
3. $U \in H$, $V \in \alpha\ O(Y)$, and $U \subseteq V$ imply $V \in H$ and
4. For every collection of α open sets of Y, whose union belongs to *H*, there are finitely many elements of this collection whose union also belongs to *H*.

Definition 2.3.
The strong α -Scott topology $\alpha s\omega(Y)$ on $\alpha\ O(Y)$ is defined as follows:
a sub collection *H* of $\alpha\ O(Y)$ belongs to $\alpha s\omega(Y)$ if:
5. $U \in H$, $V \in \alpha\ O(Y)$, and $U \subseteq V$ imply $V \in H$ and
6. For every open cover of *Y* , there are finitely many elements of this cover
whose union belongs to *H*.

Definition 2.4.

The α Isbell topology on *bC(Y, Z)*, denoted here by bT_{Is}, is the topology which has as a subbasis the family of all sets of the form:
$(H,U) = \{ f \in bC(Y,Z) : f^{-1}(U) \in H \}$, where $H \in b\omega(Y)$ and $U \in O(Z)$.

Definition 2.5.
The strong α -Isbell topology on α *C(Y,Z)*, denoted here by αsT_{Is}, is the topology which has as a subbasis the family of all sets of the form:
$(H,U) = \{ f \in \alpha\ C(Y,Z) : f^{-1}(U) \in H \}$, where $H \in \alpha s\omega(Y)$ and $U \in O(Z)$.

Theorem 2.6.
Any admissible topology is α -admissible.

Proof:
Let T be a topology on *C(Y,Z)* which is admissible, then for every topological space X, the continuity of the map $G : X \rightarrow bC_T$ *(Y,Z).* implies the continuity of the map β: $X\times Y \rightarrow Z$ or equivalently the evaluation map e : C_T *(Y,Z)*×Y → Z, defined by *e(f, y)* = *f(y)* for every *(f, y)* ∈ $bC(Y,Z) \times Y$, is α -continuous. Since the evolution map is continuous then it is α -continuous and, therefore, T is α -admissible.

Theorem 2.7.

Any Scott topology α - Scott topology is

Proof:
Let Y be a topological space and let $\omega(Y)$ be a Scott topology on $O(Y)$. Let H be a sub collection of α $O(Y)$ and let $U \in H$, V ∈ α $O(Y)$such that $U \subseteq V$ then by definition we have V ∈ H and for every collection of α open sets of Y , whose union belongs to H, there are finitely many elements of this collection whose union also belongs to H. Since any open set is an α open set, we have that $\omega(Y)$ is α -Scott topology.

Remark 2.8.
By the same method of the proof of theorem 2.6 and 2.7 we can prove that any strong Scott topology is $\alpha -$ strong Scott topology and any Isbell topology is α -Isbell topology and finally any strong Isbell topology is α -strong Isbell topology.

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