International Journal of Academic Information Systems Research (IJAISR)
ISSN: 2643-9026
Vol. 10 Issue 4 April - 2026, Pages: 8-12

On the Second Accuracy of Generalized Approximation Spaces via Graph Neighborhood Structures

Yousief S. Eassa

Ministry of Education, Karbala Education Directorate

Karbala 56001, Iraq

yousifalmosaalmsa@gmial.com

Abstract: *This paper introduces a generalized approximation framework derived from graph structures using neighborhood and non-neighborhood relations of vertices. By associating each vertex with specific subsets of the vertex set, new graph-induced spaces are constructed which generate generalized approximation spaces. Based on these spaces, second lower and upper approximations are defined together with their boundary, positive, and negative regions. Several fundamental properties and theorems related to these approximations are established. Illustrative examples are presented to clarify the behavior of the proposed model.*

Keywords— Graph Theory; Rough Sets; Generalized Approximation Space; Neighborhood System; Lower and Upper Approximations.

1. Introduction and Preliminaries

Graph theory and topology constitute two fundamental areas of modern mathematics that have received considerable attention due to their deep theoretical foundations and their wide spectrum of applications. In recent decades, the interaction between these two disciplines has led to the development of several powerful mathematical frameworks that enable the investigation of discrete structures through topological perspectives. Such approaches provide effective tools for modeling complex relationships, analyzing structural properties of networks, and studying uncertainty in discrete systems [1–7].

Among the mathematical approaches designed to handle uncertainty and incomplete information, rough set theory occupies a central position. Originally introduced by Pawlak [8], rough set theory offers a formal mathematical framework for representing and analyzing imprecise knowledge through the concepts of lower and upper approximations. These approximations characterize the boundary between elements that certainly belong to a set and those that possibly belong to it. Due to its conceptual simplicity and analytical power, rough set theory has been successfully applied in numerous domains including data analysis, machine learning, information systems, and decision-making processes [9].

In recent years, increasing attention has been directed toward integrating rough set concepts with graph structures in order to study uncertainty within networked systems. In such settings, vertices represent objects while edges describe relationships between them. Within this context, the neighborhood structure of a vertex plays a crucial role in capturing the local connectivity of the graph and provides a natural mechanism for constructing approximation operators [10,11]. Neighborhood systems in graphs offer a flexible framework for generating various approximation spaces that can be used to define different types of lower and upper approximations. These approximations facilitate the analysis of vertex subsets and provide quantitative insight into the certainty and uncertainty associated with information represented in graph-based models. Motivated by these developments, the present work introduces new approximation operators induced by vertex neighborhood systems in graphs. The proposed approach constructs generalized approximation spaces derived from graph structures and investigates the corresponding lower and upper approximation operators. Several fundamental properties of these operators are established, and the notion of approximation accuracy is examined within the proposed framework. The obtained results contribute to strengthening the connection between graph theory, topology, and rough set theory, while providing new mathematical tools for the analysis of uncertainty in graph-based systems. Moreover, the proposed framework opens promising directions for further research in rough graph models and topological structures associated with graphs. Let $\text{I€} = \left(\mathcal{A}(\text{I€}), \mathcal{B}(\text{I€})\right)$ be a graph, where $\mathcal{A}(\text{I€})$ denotes the set of vertices and $\mathcal{B}(\text{I€})$ denotes the set of edges. An edge is an unordered pair of vertices representing a connection between them.[1] The neighborhood of $\mathcal{O} \in \mathcal{B}(\text{I€})$ is defined by $\boldsymbol{Nbh}(\boldsymbol{\mathcal{O}}) = \{\mathfrak{m} \in \boldsymbol{\mathcal{B}}(\text{I€}); (\boldsymbol{\mathcal{O}}, \mathfrak{m}) \in \boldsymbol{\mathcal{B}}(\text{I€})\}$**.**[2]. Rough set theory can be thought of as a new mathematical tool for analyzing flawed data. The hypothesis has been used in a variety of fields, including engineering, medicine, and others. The lower and upper approximations are a pair of precise sets that can be used with any rough set. The approximation of lower and upper bounds of a set is the formal classification of knowledge in the interest domain, and the approximation of spaces is the formal classification of knowledge in the interest domain. The partition characterizes a topological space, called approximation space $\mathcal{G} = (X, R)$, where X is a set called the universe and $R \subseteq X \times X$ is an indiscernibility equivalence relation[11,12]. The equivalence classes of R are also known granules, elementary sets or blocks. we will use $R_x \subseteq X$ to denoted the equivalence class containing $x \in X$.

Definition 1.1. [8] Let $\mathcal{G} = (X, R)$ be an approximation space and $A \subseteq X$, then the lower approximation (resp . upper approximation) of A is denoted by $L(A)$ (resp . $U(A)$) and is defined by :

$$L(A) = \{x \in X; R_x \subseteq X\}$$

(resp . $U(A) = \{x \in X; R_x \cap A \neq \emptyset\}$)

Based on the lower and upper approximations of a set $A \subseteq X$, the universe X can be divided into three disjoint regions , the positive region (briefly $POS_R(A)$) , negative region (briefly $NEG_R(A)$) and boundary region (briefly $Bd_R(A)$) . Boundary , positive and negative regions are defined by :

$Bd_R(A) = U(A) - L(A)$,
$POS_R(A) = L(A)$ and
$NEG_R(A) = X - L(A)$.

The difference between the upper and lower approximations defines the set's boundary region. The definitions of the upper and lower approximations are shown in Figuree 1.1 [9].

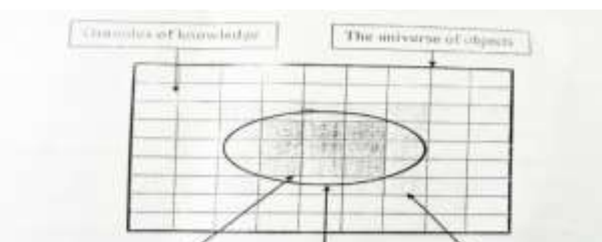

Figure 1.1: upper and lower approximations

An approximation space $\mathcal{G} = (X, R)$, if A and B are two subsets of X , then directly from the definitions of upper and lower and approximations , we can get the following properties of lower and upper approximations [8] :

(1) $L(A) \subseteq A \subseteq U(A)$,

(2) $L(\emptyset) = \emptyset, L(X) = X$ and $U(\emptyset) = \emptyset, U(X) = X$,

(3) $U(A \cup B) = U(A) \cup U(B)$,

(4) $L(A \cap B) = L(A) \cap L(B)$,

(5) If $A \subseteq B$, then $L(A) \subseteq L(B)$,

(6) If $A \subseteq B$, then $U(A) \subseteq U(B)$,

(7) $L(A \cup B) \supseteq L(A) \cup L(B)$,

(8) $U(A \cap B) \subseteq U(A) \cap U(B)$,

(9) $L(A^c) = [U(A)]^c$,

(10) $U(A^c) = [L(A)]^c$,

(11) $L(L(A) = U(L(A)) = L(A)$ and

(12) $U(U(A)) = L(U(A)) = U(A)$.

2. Main Results

Definition 2.1. Let $\text{IЄ} = (\mathcal{A}(\text{IЄ}), \mathcal{B}(\text{IЄ}))$ be a graph and a vertex $\mathcal{O} \in \mathcal{A}(\text{IЄ})$.

A)The neighborhood set of $\mathcal{O}$ is denoted by $\mathcal{N}\hbar\mathcal{S}(\mathcal{O})$ and defined by:

$$\mathcal{N}\hbar\mathcal{S}(\mathcal{O}) = \{b \in \mathcal{A}(\text{IЄ}) |\ \mathcal{O}b \in \mathcal{B}(\text{IЄ})\}.$$

B)The non-neighborhood set of $\mathcal{O}$ is denoted by $\mathcal{N}\hbar\mathcal{S}(\mathcal{O})^{\mathcal{C}}$ and defined by:

$$(\mathcal{N}\hbar\mathcal{S}(\mathcal{O}))^{\mathcal{C}} = \{b \in \mathcal{A}(\text{IЄ}) |\ \mathcal{O}b \notin \mathcal{B}(\text{IЄ})\}.$$

Definition 2.2. Let $\text{IЄ} = (\mathcal{A}(\text{IЄ}), \mathcal{B}(\text{IЄ}))$ be a graph and let $\mathcal{P}(\mathcal{A}(\text{IЄ}))$ denote the power set of $\mathcal{A}(\text{IЄ})$. Assume that with each vertex $\mathcal{O} \in \mathcal{A}(\text{IЄ})$ there are associated two subsets $\boldsymbol{\mathcal{N}\hbar\mathcal{S}(\mathcal{O})} \subseteq \mathcal{A}(\text{IЄ})$ and $(\boldsymbol{\mathcal{N}\hbar\mathcal{S}(\mathcal{O})})^{\boldsymbol{\mathcal{C}}} \subseteq \mathcal{A}(\text{IЄ})$. Define a mapping:

$\psi^{*T}: \mathcal{A}(\text{IЄ}) \longrightarrow \mathcal{P}\left(\mathcal{P}(\mathcal{A}(\text{IЄ}))\right)$ by: $\psi^{*T}(\mathcal{O}) = \left\{\mathcal{N}\hbar\mathcal{S}(\mathcal{O}), (\boldsymbol{\mathcal{N}\hbar\mathcal{S}(\mathcal{O})})^{\boldsymbol{\mathcal{C}}}\right\}$, $\forall \mathcal{O} \in \mathcal{A}(\text{IЄ})$.

Moreover, consider the following two mappings:

$\psi^{1T}: \mathcal{A}(\text{IЄ}) \longrightarrow \mathcal{P}(\mathcal{A}(\text{IЄ}))$, $\psi^{1T}(\mathcal{O}) = \mathcal{N}\hbar\mathcal{S}(\mathcal{O})$,

$\psi^{2T}: \mathcal{A}(\text{IЄ}) \longrightarrow \mathcal{P}(\mathcal{A}(\text{IЄ}))$,$\psi^{2T}(\mathcal{O}) = (\mathcal{N}\hbar\mathcal{S}(\mathcal{O}))^{\mathcal{C}}$

Accordingly, the following graph spaces are induced:

1) The pair $(\text{IЄ}, \psi^{*T})$ is called $* T$ −space,
2) The pair $(\text{IЄ}, \psi^{1T})$ is called $1T$ −space,
3) The pair $(\text{IЄ}, \psi^{2T})$ is called $2T$ −space.

Definition 2.3. Let $\text{IЄ} = (\mathcal{A}(\text{IЄ}), \mathcal{B}(\text{IЄ}))$ be a graph, and for each $\mathcal{O} \in \mathcal{A}(\text{IЄ})$, and let $\psi^{*T} \subseteq \mathcal{P}\left(\mathcal{P}(\mathcal{A}(\text{IЄ}))\right)$. Define a binary relation $\mathcal{R}_{*T} \subseteq \mathcal{A}(\text{IЄ}) \times \mathcal{A}(\text{IЄ})$ by: $(\mathcal{O}_1, \mathcal{O}_2) \in \mathcal{R}_{*T} \Leftrightarrow \mathcal{O}_1 \in K$ where $K \in \psi^{*T}(\mathcal{O}_2)$. Then the ordered pair $\mathcal{S}_{*T} = (\mathcal{A}(\text{IЄ}), \mathcal{R}_{*T})$ is called the generalized approximation space induced by the $* T$ −space.

Definition 2.4. Let $\text{IЄ} = (\mathcal{A}(\text{IЄ}), \mathcal{B}(\text{IЄ}))$ be a graph, and for each $\mathcal{O} \in \mathcal{A}(\text{IЄ})$, and let $\psi^{1T} \subseteq \mathcal{P}(\mathcal{A}(\text{IЄ}))$. Define a binary relation $\mathcal{R}_{1T} \subseteq \mathcal{A}(\text{IЄ}) \times \mathcal{A}(\text{IЄ})$ by: $(\mathcal{O}_1, \mathcal{O}_2) \in \mathcal{R}_{1T} \Leftrightarrow \mathcal{O}_1 \in \psi^{1T}(\mathcal{O}_2)$. Then the ordered pair $\mathcal{S}_{1T} = (\mathcal{A}(\text{IЄ}), \mathcal{R}_{1T})$ is called the generalized approximation space induced by the $1T$ −space.

Definition 2.4. Let $\text{IЄ} = (\mathcal{A}(\text{IЄ}), \mathcal{B}(\text{IЄ}))$ be a graph, and for each $\mathcal{O} \in \mathcal{A}(\text{IЄ})$, and let $\psi^{2T} \subseteq \mathcal{P}(\mathcal{A}(\text{IЄ}))$. Define a binary relation $\mathcal{R}_{2T} \subseteq \mathcal{A}(\text{IЄ}) \times \mathcal{A}(\text{IЄ})$ by: $(\mathcal{O}_1, \mathcal{O}_2) \in \mathcal{R}_{2T} \Leftrightarrow \mathcal{O}_1 \in \psi^{2T}(\mathcal{O}_2)$. Then the ordered pair $\mathcal{S}_{2T} = (\mathcal{A}(\text{IЄ}), \mathcal{R}_{2T})$ is called the generalized approximation space induced by the $2T$ −space.

Definition 2.5. Let $\mathcal{S}_{jT}$, where $j \in \{1,2,*\}$ be a generalized approximation space. For any $\mathcal{J} \subseteq \text{IЄ}$, then:

1) The second jT-lower and jT-upper approximations of $\mathcal{J}$ where $j \in \{1,2,*\}$ are defined respectively by:

$L^2_{jT}(\mathcal{A}(\mathcal{J})) = \{\mathcal{O} \in \mathcal{A}(\mathcal{J}) : \psi^{jT}(\mathcal{O}) \subseteq \mathcal{A}(\mathcal{J})\}$,

International Journal of Academic Information Systems Research (IJAISR)
ISSN: 2643-9026
Vol. 10 Issue 4 April - 2026, Pages: 8-12

$U_{jT}^2(\mathcal{A}(\mathcal{J})) = \mathcal{A}(\mathcal{J}) \cup \{\mathcal{O} \in \mathcal{A}(\text{Ҝ}) \setminus \mathcal{A}(\mathcal{J})$
$: \psi^{jT}(\mathcal{O}) \cap \mathcal{A}(\mathcal{J}) \neq \emptyset\}$,

2) The second jT- boundary, jT- positive and jT- negative regions of $\mathcal{J}$ where $j \in \{1,2,*\}$ are defined respectively by:

$Bnd_{jT}^2(\mathcal{A}(\mathcal{J})) = U_{jT}^2(\mathcal{A}(\mathcal{J})) \setminus L_{jT}^2(\mathcal{A}(\mathcal{J}))$,

$POS_{jT}^2(\mathcal{A}(\mathcal{J})) = L_{jT}^2(\mathcal{A}(\mathcal{J}))$,

$NEg_{jT}^2(\mathcal{A}(\mathcal{J})) = \mathcal{A}(\text{Ҝ}) - U_{jT}^2(\mathcal{A}(\mathcal{J}))$.

Definition 2.6. Let $\mathcal{S}_{jT}$, where $j \in \{1,2,*\}$ be a generalized approximation space. For any $\mathcal{J} \subseteq \text{Ҝ}$, then the second accuracy of the approximation of $\mathcal{J}$ is defined by:

$$\eta_{jT}^2(\mathcal{A}(\mathcal{J})) = \frac{|\mathcal{A}(\text{Ҝ}) - Bnd_{jT}^2(\mathcal{A}(\mathcal{J}))|}{|\mathcal{A}(\text{Ҝ})|}$$

It is obvious that $0 \leq \eta_{jT}^2(\mathcal{A}(\mathcal{J})) \leq 1$, Moreover, if $\eta_{jT}^2(\mathcal{A}(\mathcal{J})) = 1$ then $\mathcal{J}$ is called $\mathcal{J}$-definable ($\mathcal{J}$-exact) graph otherwise, it is called $\mathcal{J}$-rough.

Example 2.7. Let $\text{Ҝ} = (\mathcal{A}(\text{Ҝ}), E(\text{Ҝ}))$ such that $\mathcal{A}(\text{Ҝ}) = \{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_5\}$ and

$\mathcal{B}(\text{Ҝ}) = \{(\mathcal{O}_1, \mathcal{O}_2), (\mathcal{O}_1, \mathcal{O}_4), (\mathcal{O}_2, \mathcal{O}_2), (\mathcal{O}_2, \mathcal{O}_3),$

$(\mathcal{O}_2, \mathcal{O}_4), (\mathcal{O}_2, \mathcal{O}_5), (\mathcal{O}_3, \mathcal{O}_4), (\mathcal{O}_4, \mathcal{O}_5), (\mathcal{O}_5, \mathcal{O}_5)\}$.

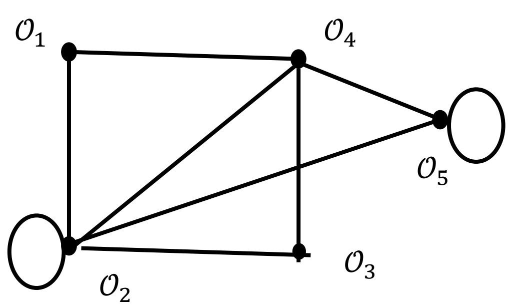

Figure 2.1: graph Ҝ given in Example (2.7).

We get:

$\psi^{1T}(\mathcal{O}_1) = \{\mathcal{O}_2, \mathcal{O}_4\}$,
$\psi^{1T}(\mathcal{O}_2) = \mathcal{A}(\text{Ҝ})$,
$\psi^{1T}(\mathcal{O}_3) = \{\mathcal{O}_2, \mathcal{O}_4\}$,
$\psi^{1T}(\mathcal{O}_4) = \{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_5\}$,
$\psi^{1T}(\mathcal{O}_5) = \{\mathcal{O}_2, \mathcal{O}_4, \mathcal{O}_5\}$.

Also we have:

$\psi^{2T}(\mathcal{O}_1) = \{\mathcal{O}_3, \mathcal{O}_5\}$,
$\psi^{2T}(\mathcal{O}_2) = \emptyset$,
$\psi^{2T}(\mathcal{O}_3) = \{\mathcal{O}_1, \mathcal{O}_5\}$,
$\psi^{2T}(\mathcal{O}_4) = \emptyset$,
$\psi^{2T}(\mathcal{O}_5) = \{\mathcal{O}_1, \mathcal{O}_3\}$.

Then we obtain:

$\psi^{*T}(\mathcal{O}_1) = \{\{\mathcal{O}_2, \mathcal{O}_4\}, \{\mathcal{O}_3, \mathcal{O}_5\}\}$,
$\psi^{*T}(\mathcal{O}_2) = \{\mathcal{A}(\text{Ҝ}), \emptyset\}$,
$\psi^{*T}(\mathcal{O}_3) = \{\{\mathcal{O}_2, \mathcal{O}_4\}, \{\mathcal{O}_1, \mathcal{O}_5\}\}$,
$\psi^{*T}(\mathcal{O}_4) = \{\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_5\}, \emptyset\}$,
$\psi^{*T}(\mathcal{O}_5) = \{\{\mathcal{O}_2, \mathcal{O}_4, \mathcal{O}_5\}, \{\mathcal{O}_1, \mathcal{O}_3\}\}$.

Accordingly, can be obtain the following table

Table 2.1 $\eta_{jT}^2(\mathcal{A}(\mathcal{J}))$, where $j \in \{1,2,*\}$ for all $\mathcal{J} \subseteq \text{Ҝ}$.

$\mathcal{A}(\mathcal{J})$	$\eta_{jT}^2(\mathcal{A}(\mathcal{J}))$	$\eta_{jT}^2(\mathcal{A}(\mathcal{J}))$	$\eta_{jT}^2(\mathcal{A}(\mathcal{J}))$
$\{\mathcal{O}_1\}$	3/5	3/5	1
$\{\mathcal{O}_2\}$	0	1	1
$\{\mathcal{O}_3\}$	3/5	3/5	1
$\{\mathcal{O}_4\}$	1/5	1	1
$\{\mathcal{O}_5\}$	2/5	3/5	1
$\{\mathcal{O}_1, \mathcal{O}_2\}$	0	3/5	3/5
$\{\mathcal{O}_1, \mathcal{O}_3\}$	3/5	3/5	1
$\{\mathcal{O}_1, \mathcal{O}_4\}$	0	3/5	3/5
$\{\mathcal{O}_1, \mathcal{O}_5\}$	2/5	3/5	4/5
$\{\mathcal{O}_2, \mathcal{O}_3\}$	0	3/5	3/5
$\{\mathcal{O}_2, \mathcal{O}_4\}$	2/5	1	1
$\{\mathcal{O}_2, \mathcal{O}_5\}$	0	3/5	3/5
$\{\mathcal{O}_3, \mathcal{O}_4\}$	0	3 /5	3/5
$\{\mathcal{O}_3, \mathcal{O}_5\}$	2/5	3/5	4/5
$\{\mathcal{O}_4, \mathcal{O}_5\}$	0	3/5	3/5
$\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3\}$	0	3/5	3/5
$\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_4\}$	2/5	3/5	4/5
$\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_5\}$	0	3/5	3/5
$\{\mathcal{O}_1, \mathcal{O}_3, \mathcal{O}_4\}$	0	3/5	3/5
$\{\mathcal{O}_1, \mathcal{O}_3, \mathcal{O}_5\}$	2/5	1	1
$\{\mathcal{O}_1, \mathcal{O}_4, \mathcal{O}_5\}$	0	3/5	3/5
$\{\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4\}$	2/3	3/5	4/5
$\{\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_5\}$	0	3/5	3/5
$\{\mathcal{O}_2, \mathcal{O}_4, \mathcal{O}_5\}$	3/5	3/5	1
$\{\mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_5\}$	0	3/5	3/5
$\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4\}$	2/5	3/5	1
$\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3\mathcal{O}_5\}$	1/5	1	1
$\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_4\mathcal{O}_5\}$	3/5	3/5	1
$\{\mathcal{O}_1, \mathcal{O}_3, \mathcal{O}_4\mathcal{O}_5\}$	0	1	1
$\{\mathcal{O}_2, \mathcal{O}, \mathcal{O}_4\mathcal{O}_5\}$	3/5	3/5	1
$\mathcal{A}(\text{Ҝ})$	1	1	1
$\emptyset$	1	1	1

Theorem 2.8. Let $\mathcal{S}_{jT}$, where $j \in \{1,2,*\}$ be a generalized approximation space. For any $\mathcal{J} \subseteq \text{Ҝ}$,. Then:

(a) $L^2_{*T}(\mathcal{A}(\mathcal{J})) = L^2_{1T}(\mathcal{A}(\mathcal{J})) \cup L^2_{2T}(\mathcal{A}(\mathcal{J}))$

(b) $U^2_{*T}(\mathcal{A}(\mathcal{J})) = U^2_{1T}(\mathcal{A}(\mathcal{J})) \cap U^2_{2T}(\mathcal{A}(\mathcal{J}))$

(c) $\boldsymbol{Bnd^2_{*T}(\mathcal{A}(\mathcal{J})) = Bnd^2_{1T}(\mathcal{A}(\mathcal{J})) \cap Bnd^2_{2T}(\mathcal{A}(\mathcal{J}))}$

(d) $\boldsymbol{\eta^2_{*T}(\mathcal{A}(\mathcal{J})) \geq max\{\eta^2_{2T}(\mathcal{A}(\mathcal{J})), \eta^2_{1T}(\mathcal{A}(\mathcal{J}))\}}$.

Theorem 2.9. Let $\mathcal{S}_{jT}$, where $j \in \{1,2,*\}$ be a generalized approximation spaces. For any $\mathcal{J}, \mathcal{M} \subseteq$ Ҝ. Then:

(L_1) $L^2_j(\mathcal{A}(Ҝ)) = \mathcal{A}(Ҝ)$,

(L_2) if $\mathcal{A}(\mathcal{J}) \subseteq \mathcal{A}(\mathcal{M})$,then $L^2_j(\mathcal{A}(\mathcal{J})) \subseteq L^2_j(\mathcal{A}(\mathcal{M}))$

(L_3) $L^2_j(\mathcal{A}(\mathcal{J}) \cap \mathcal{A}(\mathcal{M})) \subseteq L^2_j(\mathcal{A}(\mathcal{J})) \cap L^2_j(\mathcal{A}(\mathcal{M}))$,

(L_4) $L^2_j(\mathcal{A}(\mathcal{J}) \cup \mathcal{A}(\mathcal{M})) \supseteq L^2_j(\mathcal{A}(\mathcal{J})) \cup L^2_j(\mathcal{A}(\mathcal{M}))$,

(L_5) $L^2_j(\mathcal{A}(\mathcal{J})) = \mathcal{A}(Ҝ) - [U^2_j(\mathcal{A}(Ҝ) - \mathcal{A}(\mathcal{J}))]$,

(U_6) $U^2_j(\emptyset) = \emptyset$,

(U_7) if $\mathcal{A}(\mathcal{J}) \subseteq \mathcal{A}(\mathcal{M})$,then $U^2_j(\mathcal{A}(\mathcal{J})) \subseteq U^2_j(\mathcal{A}(\mathcal{M}))$,

(U_8) $U^2_j(\mathcal{A}(\mathcal{J}) \cap \mathcal{A}(\mathcal{M})) \subseteq U^2_j(\mathcal{A}(\mathcal{J})) \cap U^2_j(\mathcal{A}(\mathcal{M}))$,

(U_9) $U^2_j(\mathcal{A}(\mathcal{J}) \cup \mathcal{A}(\mathcal{M})) \supseteq U^2_j(\mathcal{A}(\mathcal{J})) \cup U^2_j(\mathcal{A}(\mathcal{M}))$,

(U_{10}) $U^2_j(\mathcal{A}(\mathcal{J})) = \mathcal{A}(Ҝ) - [L^2_j(\mathcal{A}(Ҝ) - \mathcal{A}(\mathcal{J}))]$

Remark 2.10. Let $\mathcal{S}_{jT}$, where $j \in \{1,2,*\}$ be a generalized approximation space. For any $\mathcal{J}, \mathcal{M} \subseteq$ Ҝ. Then the following statements are not necessarily true:

(L_0) $L^2_j(\mathcal{A}(\mathcal{J})) \subseteq \mathcal{A}(\mathcal{J})$,

(L_{11}) $L^2_j(\emptyset) = \emptyset$,

(L_{12}) $L^2_j(\mathcal{A}(\mathcal{J})) = L^2_j(L^2_j(\mathcal{A}(\mathcal{J})))$,

(L_{13}) $L^2_j(\mathcal{A}(\mathcal{J})) = U^2_j(L^2_j(\mathcal{A}(\mathcal{J})))$,

(L_{14}) $\mathcal{A}(\mathcal{J}) \subseteq L^2_j(U^2_j(\mathcal{A}(\mathcal{J})))$,

(L_{15}) $L^2_j(\mathcal{A}(\mathcal{J})) \subseteq L^2_j(L^2_j(\mathcal{A}(\mathcal{J})))$,

(L_{16}) $L^2_j(\mathcal{A}(\mathcal{J}) \cup \mathcal{A}(\mathcal{M})) = L^2_j(\mathcal{A}(\mathcal{J})) \cup L^2_j(\mathcal{A}(\mathcal{M}))$,

(U_{17}) $\mathcal{A}(\mathcal{J}) \subseteq U^2_j(\mathcal{A}(\mathcal{J}))$,

(U_{18}) $U^2_j(\mathcal{A}(Ҝ)) = \mathcal{A}(Ҝ)$,

(U_{19}) $U^2_j(\mathcal{A}(\mathcal{J})) = U^2_j(U^2_j(\mathcal{A}(\mathcal{J})))$,

(U_{20}) $U^2_j(\mathcal{A}(\mathcal{J})) = L^2_j(U^2_j(\mathcal{A}(\mathcal{J})))$,

(U_{21}) $\mathcal{A}(\mathcal{J}) \supseteq U^2_j(L^2_j(\mathcal{A}(\mathcal{J})))$,

(U_{22}) $U^2_j(\mathcal{A}(\mathcal{J})) \supseteq U^2_j(U^2_j(\mathcal{A}(\mathcal{J})))$ and

(U_{23}) $U^2_j(\mathcal{A}(\mathcal{J}) \cup \mathcal{A}(\mathcal{M})) = U^2_j(\mathcal{A}(\mathcal{J})) \cup U^2_j(\mathcal{A}(\mathcal{M}))$,

(LU) $L^2_j(\mathcal{A}(\mathcal{J})) \subseteq U^2_j(\mathcal{A}(\mathcal{J}))$.

The following example is applied to show this remark.

Example 2.11. Let Ҝ $= (\mathcal{A}(Ҝ), \mathcal{B}(Ҝ))$ such that $\mathcal{A}(Ҝ) = \{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4\}$ and
$\mathcal{B}(Ҝ) = \{(\mathcal{O}_2, \mathcal{O}_3), , (\mathcal{O}_2, \mathcal{O}_4)\}$.
We get

$\psi^{1T}(\mathcal{O}_1) = \emptyset, \psi^{1T}(\mathcal{O}_2) = \{\mathcal{O}_3, \mathcal{O}_4\}, \psi^{1T}(\mathcal{O}_3) = \{\mathcal{O}_2\}, \psi^{1T}(\mathcal{O}_4) = \{\mathcal{O}_2\}$,Also we have $\psi^{2T}(\mathcal{O}_1) = \{\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4\}, \psi^{2T}(\mathcal{O}_2) = \{\mathcal{O}_1\}, \psi^{2T}(\mathcal{O}_3) = \{\mathcal{O}_1, \mathcal{O}_4\}, \psi^{2T}(\mathcal{O}_4) = \{\mathcal{O}_1, \mathcal{O}_3\}$. And

$$\psi^{*T}(\mathcal{O}_1) = \{\emptyset, \{\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4\}\},$$

$$\psi^{*T}(\mathcal{O}_2) = \{\{\mathcal{O}_3, \mathcal{O}_4\}, \{\mathcal{O}_1\}\},$$

$$\psi^{*T}(\mathcal{O}_3) = \{\{\mathcal{O}_2\}, \{\mathcal{O}_1, \mathcal{O}_4\}\},$$

$$\psi^{*T}(\mathcal{O}_4) = \{\{\mathcal{O}_2\}, \{\mathcal{O}_1, \mathcal{O}_3\}\}.$$

Accordingly ,we have when $j = *$.

(L_0) Let $\mathcal{J} = (\mathcal{A}(\mathcal{J}), \mathcal{B}(\mathcal{J}))$ such that $\mathcal{A}(\mathcal{J}) = \{\mathcal{O}_1, \mathcal{O}_2\}$ and $\mathcal{B}(\mathcal{J}) = \emptyset$. Then $L^2_j(\mathcal{A}(\mathcal{J})) = \mathcal{A}(Ҝ)$.Therefore, $L^2_j(\mathcal{A}(\mathcal{J})) \nsubseteq \mathcal{A}(\mathcal{J})$.

(L_{11}) Let $\mathcal{J} = (\mathcal{A}(\mathcal{J}), \mathcal{B}(\mathcal{J}))$ such that $\mathcal{A}(\mathcal{J}) = \emptyset$and $\mathcal{B}(\mathcal{J}) = \emptyset$. Then $L^2_j(\mathcal{A}(\mathcal{J})) = \{\mathcal{O}_1\}$.Therefore, $L^2_j(\emptyset) \neq \emptyset$.

(L_{12}) Let $\mathcal{J} = (\mathcal{A}(\mathcal{J}), \mathcal{B}(\mathcal{J}))$ such that $\mathcal{A}(\mathcal{J}) = \{\mathcal{O}_2, \mathcal{O}_3\}$ and $\mathcal{B}(\mathcal{J}) = \emptyset$. Then $L^2_j(\mathcal{A}(\mathcal{J})) = \{\mathcal{O}_1, \mathcal{O}_3, \mathcal{O}_4\}$,$L^2_j(L^2_j(\mathcal{A}(\mathcal{J})) = \mathcal{A}(Ҝ)$.Therefore, $L^2_j(\mathcal{A}(\mathcal{J})) \neq L^2_j(L^2_j(\mathcal{A}(\mathcal{J})))$.

(L_{13}) Let $\mathcal{J} = (\mathcal{A}(\mathcal{J}), \mathcal{B}(\mathcal{J}))$ such that $\mathcal{A}(\mathcal{J}) = \{\mathcal{O}_2, \mathcal{O}_4\}$ and $\mathcal{B}(\mathcal{J}) = \emptyset$. Then $L^2_j(\mathcal{A}(\mathcal{J})) = \{\mathcal{O}_1, \mathcal{O}_3, \mathcal{O}_4\}$, $U^2_j(L^2_j(\mathcal{A}(\mathcal{J})) = \{\mathcal{O}_2\}$. Therefore, $L^2_j(\mathcal{A}(\mathcal{J})) \neq U^2_j(L^2_j(\mathcal{A}(\mathcal{J})))$.

(L_{14}) Let $\mathcal{J} = (\mathcal{A}(\mathcal{J}), \mathcal{B}(\mathcal{J}))$ such that $\mathcal{A}(\mathcal{J}) = \{\mathcal{O}_3, \mathcal{O}_4\}$ and $\mathcal{B}(\mathcal{J}) = \emptyset$. Then $L^2_j(U^2_j(\mathcal{A}(\mathcal{J}))) = \{\mathcal{O}_1\}$. Therefore, $\mathcal{A}(\mathcal{J}) \nsubseteq L^2_j(U^2_j(\mathcal{A}(\mathcal{J})))$.

(L_{15}) Accordingly ,to example (2.7). Let $\mathcal{J} = (\mathcal{A}(\mathcal{J}), \mathcal{B}(\mathcal{J}))$ such that $\mathcal{A}(\mathcal{J}) = \{\mathcal{O}_2\}$ and $\mathcal{B}(\mathcal{J}) = \emptyset$. Then $L^2_j(\mathcal{A}(\mathcal{J})) = \{\mathcal{O}_4\}$, $L^2_j(L^2_j(\mathcal{A}(\mathcal{J})) = \{\mathcal{O}_3\}$. Therefore, $L^2_j(\mathcal{A}(\mathcal{J})) \nsubseteq L^2_j(L^2_j(\mathcal{A}(\mathcal{J})))$.

(L_{16}) Accordingly ,to example (2.7). let $\mathcal{J} = (\mathcal{A}(\mathcal{J}), \mathcal{B}(\mathcal{J}))$ such that $\mathcal{A}(\mathcal{J}) = \{\mathcal{O}_1\}$ and $\mathcal{B}(\mathcal{J}) = \emptyset$ and $\mathcal{M} = (\mathcal{A}(\mathcal{M}), \mathcal{B}(\mathcal{M}))$ such that $\mathcal{A}(\mathcal{M}) = \{\mathcal{O}_3\}$ and $\mathcal{B}(\mathcal{M}) =$

$\emptyset$Then $L_j^2(\mathcal{A}(\mathcal{J})) = \{\mathcal{O}_2\}$ and $L_j^2(\mathcal{A}(\mathcal{M})). = \{\mathcal{O}_1\}$ But, $\mathcal{J} \cup \mathcal{M} = (\mathcal{A}(\mathcal{J}) \cup \mathcal{A}(\mathcal{M}), \mathcal{B}(\mathcal{J}) \cup \mathcal{B}(\mathcal{M}))$ Such that $\mathcal{A}(\mathcal{J}) \cup \mathcal{A}(\mathcal{M}) = \{\mathcal{O}_1, \mathcal{O}_3\}$ and $\mathcal{B}(\mathcal{J}) \cup \mathcal{B}(\mathcal{M}) = \{(\mathcal{O}_2, \mathcal{O}_3)\}$ Then, $L_j^2(\mathcal{A}(\mathcal{J}) \cup \mathcal{A}(\mathcal{M})) = \{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_4\}$ .Therefore, $L_j^2(\mathcal{A}(\mathcal{J}) \cup \mathcal{A}(\mathcal{M})) \neq L_j^2(\mathcal{A}(\mathcal{J})) \cup L_j^2(\mathcal{A}(\mathcal{M}))$.

(U_{17}) Let $\mathcal{J} = (\mathcal{A}(\mathcal{J}), \mathcal{B}(\mathcal{J}))$ such that $\mathcal{A}(\mathcal{J}) = \{\mathcal{O}_3\}$ and $\mathcal{B}(\mathcal{J}) = \emptyset$. Then $U_j^2(\mathcal{A}(\mathcal{J})) = \{\mathcal{O}_1\}$Therefore, $\mathcal{A}(\mathcal{J}) \nsubseteq U_j^2(\mathcal{A}(\mathcal{J}))$.

(U_{18}) Let $\mathcal{J} = (\mathcal{A}(\mathcal{J}), \mathcal{B}(\mathcal{J}))$ such that $\mathcal{A}(\mathcal{J}) = \mathcal{A}(\text{Ꞓ})$and $\mathcal{B}(\mathcal{J}) = \mathcal{B}(\text{Ꞓ})$. Then $U_j^2(\mathcal{A}(\text{Ꞓ})) = \{\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4\}$.Therefore, $U_j^2(\mathcal{A}(\text{Ꞓ})) \neq \mathcal{A}(\text{Ꞓ})$.

(U_{19}) Let $\mathcal{J} = (\mathcal{A}(\mathcal{J}), \mathcal{B}(\mathcal{J}))$ such that $\mathcal{A}(\mathcal{J}) = \{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3\}$ and $\mathcal{B}(\mathcal{J}) = \emptyset$. Then $U_j^2(\mathcal{A}(\mathcal{J})) = \{\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4\}$, $U_j^2(U_j^2(\mathcal{A}(\mathcal{J})) = \{\mathcal{O}_3, \mathcal{O}_4\}$ Therefore, $U_j^2(\mathcal{A}(\mathcal{J})) \neq U_j^2(U_j^2(\mathcal{A}(\mathcal{J})))$.

(U_{20}) Let $\mathcal{J} = (\mathcal{A}(\mathcal{J}), \mathcal{B}(\mathcal{J}))$ such that $\mathcal{A}(\mathcal{J}) = \{\mathcal{O}_1, \mathcal{O}_2\}$ and $\mathcal{B}(\mathcal{J}) = \emptyset$. Then $U_j^2(\mathcal{A}(\mathcal{J})) = \{\mathcal{O}_3, \mathcal{O}_4\}$, $L_j^2(U_j^2(\mathcal{A}(\mathcal{J})) = \emptyset$. Therefore, $U_j^2(\mathcal{A}(\mathcal{J})) \neq L_j^2(U_j^2(\mathcal{A}(\mathcal{J})))$.

(U_{21}) Let $\mathcal{J} = (\mathcal{A}(\mathcal{J}), \mathcal{B}(\mathcal{J}))$ such that $\mathcal{A}(\mathcal{J}) = \{\mathcal{O}_1, \mathcal{O}_3\}$ and $\mathcal{B}(\mathcal{J}) = \emptyset$. Then $U_j^2(L_j^2(\mathcal{A}(\mathcal{J}))) = \{\mathcal{O}_3, \mathcal{O}_4\}$. Therefore, $\mathcal{A}(\mathcal{J}) \nsupseteq U_j^2(L_j^2(\mathcal{A}(\mathcal{J})))$.

(U_{22}) Accordingly ,to example (2.7). Let $\mathcal{J} = (\mathcal{A}(\mathcal{J}), \mathcal{B}(\mathcal{J}))$ such that $\mathcal{A}(\mathcal{J}) = \{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3\}$ and $\mathcal{B}(\mathcal{J}) = \emptyset$. Then $U_j^2(\mathcal{A}(\mathcal{J})) = \{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_4\}$, $U_j^2(U_j^2(\mathcal{A}(\mathcal{J})) = \{\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4\}$.Therefore, $U_j^2(\mathcal{A}(\mathcal{J})) \nsupseteq U_j^2(U_j^2(\mathcal{A}(\mathcal{J})))$.

(U_{23}) Accordingly ,to example (2.7). let $\mathcal{J} = (\mathcal{A}(\mathcal{J}), \mathcal{B}(\mathcal{J}))$ such that $\mathcal{A}(\mathcal{J}) = \{\mathcal{O}_1\}$ and $\mathcal{B}(\mathcal{J}) = \emptyset$ and $\mathcal{M} = (\mathcal{A}(\mathcal{M}), \mathcal{B}(\mathcal{M}))$ such that $\mathcal{A}(\mathcal{M}) = \{\mathcal{O}_4\}$ and $\mathcal{B}(\mathcal{M}) = \emptyset$Then $U_j^2(\mathcal{A}(\mathcal{J})) = \emptyset$ and $U_j^2(\mathcal{A}(\mathcal{M})) = \emptyset$ But, $\mathcal{J} \cup \mathcal{M} = (\mathcal{A}(\mathcal{J}) \cup \mathcal{A}(\mathcal{M}), \mathcal{B}(\mathcal{J}) \cup \mathcal{B}(\mathcal{M}))$ Such that $\mathcal{A}(\mathcal{J}) \cup \mathcal{A}(\mathcal{M}) = \{\mathcal{O}_1, \mathcal{O}_4\}$ and $\mathcal{B}(\mathcal{J}) \cup \mathcal{B}(\mathcal{M}) = \{(\mathcal{O}_2, \mathcal{O}_4)\}$ Then, $U_j^2(\mathcal{A}(\mathcal{J}) \cup \mathcal{A}(\mathcal{M})) = \{\mathcal{O}_2, \mathcal{O}_3\}$ .Therefore, $U_j^2(\mathcal{A}(\mathcal{J}) \cup \mathcal{A}(\mathcal{M})) \neq U_j^2(\mathcal{A}(\mathcal{J})) \cup U_j^2(\mathcal{A}(\mathcal{M}))$.

(LU) Let $\mathcal{J} = (\mathcal{A}(\mathcal{J}), \mathcal{B}(\mathcal{J}))$ such that $\mathcal{A}(\mathcal{J}) = \{\mathcal{O}_1, \mathcal{O}_3\}$ and $\mathcal{B}(\mathcal{J}) = \emptyset$. Then $L_j^2(\mathcal{A}(\mathcal{J})) = \{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_4\}$,$U_j^2(\mathcal{A}(\mathcal{J})) = \{\mathcal{O}_2\}$. Therefore, $L_j^2(\mathcal{A}(\mathcal{J})) \nsubseteq U_j^2(\mathcal{A}(\mathcal{J}))$.

References:

[1] Harary, F., Graph Theory, Addison-Wesley, 1969.
[2] Douglas B. West, Introduction to Graph Theory, 2nd ed., Upper Saddle River, NJ, USA: Prentice Hall, 2001.
[3] Fandy, H. H., and Al'Dzhabri, K. Sh., Admixture Vertex Edges System and A-Space,Journal of Discrete Mathematical Sciences and Cryptography,28(2), 2025, pp. 469–474.
[4] Fandy, H. H., and Al'Dzhabri, K. Sh., Supra Topological Space Using Admixture Vertex Edges System, AIP Conference Proceedings, Vol. 3393, Conference on Mathematical And Computational Sciences, (27 Feb. 2026), Art. No. 070090.
[5] K. S. Al'Dzhabri and et al, DG-domination topology in Digraph. Journal of Prime Research in Mathematics 2021, 17(2), pp. 93–100
[6] K. S. Al'Dzhabri, Enumeration of connected components of acyclic digraph. Journal of Discrete Mathematical Sciences and Cryptography, 2021, 24(7), pp. 2047–2058
[7] K.S. Al'Dzhabri and M. F. Hani, On Certain Types of Topological Spaces Associated with Digraphs.Journal of Physics:Conference Series 1591(2020)012055.
[8] Zdzisław Pawlak, Rough Sets: Theoretical Aspects of Reasoning about Data, Dordrecht, Netherlands: Springer, 1991.
[9] Y. Y. Yao, "Two views of the theory of rough sets in finite universes," International Journal of Approximate Reasoning, vol. 15, no. 4, pp. 291–317, 1996Munkres, J. R., Topology, 2nd Edition, Prentice Hall, 2000
[10] Hassan H. Fandy and Khalid Sh. Al'Dzhabri, "Supra Topological Space Using Admixture Vertex Edges System," Journal of Physics: Conference Series, 2022.
[11] Hassan H. Fandy and Khalid Sh. Al'Dzhabri, "First Accuracy in Generalized Approximation Space Using Admixture Vertex Edges Systems," International Journal of Academic Information Systems Research, vol. 9, no. 2, 2025.