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Chromatic Uniqueness and Equivalence of Generalized Fan Graphs

Hanan Hayder Mohammed

Department of Mathematic, Faculty of Basic Education, University of Kufa, Najaf, Iraq
E-mail: hanah.alaridhi@uokufa.edu.iq

Abstract: *This paper studies the chromatic uniqueness of the generalized fan graph* $\boldsymbol{F_{n,m} = O_n + P_m}$*, where* $\boldsymbol{n \geq 1}$ *and* $\boldsymbol{m \geq 2}$*. The objective is to determine the conditions under which* $\boldsymbol{F_{n,m}}$ *is chromatically unique and to classify its equivalence classes. By analyzing the chromatic polynomial, we show that uniqueness is restricted to the small cases:* $\boldsymbol{F_{n,m}}$*,* $\boldsymbol{n \leq 2}$*,* $\boldsymbol{m \leq 3}$*,* $\boldsymbol{with\ F_{1,3} \cong F_{2,2}}$*. For all other parameters,* $\boldsymbol{F_{n,m}}$ *is not unique, and infinite families of chromatic equivalences are derived, including* $\boldsymbol{F_{n,2} \sim F_{1,m}}$ *and* $\boldsymbol{F_{n,3} \sim F_{2,m}}$*,* $\boldsymbol{m \geq n+1}$*. This provides a complete characterization of the chromatic properties of generalized fan graphs and contributes to the broader theory of join-graph invariants.*

Keywords— ***Chromatic polynomials, chromatically equivalent, chromatically unique, join-graph, fan graphs.***

1. INTRODUCTION

All graphs considered in this paper are simple graphs. For a graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. The order and size of G are given by $|V(G)|$ and $|E(G)|$, respectively. A proper λ-coloring of G is a mapping $f: V(G) \to \{1,2,\dots,\lambda\}$ such that adjacent vertices receive distinct colors. The number of distinct λ-colorings is denoted by $P(G,\lambda)$, the *chromatic polynomial* of G. Two graphs G and H are *chromatically equivalent* (χ-equivalent), written $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. A graph G is *chromatically unique* (χ-unique) if $G \sim H$ implies $H \cong G$ [1].

The chromatic polynomial was introduced by Birkhoff [2] in 1912 as part of his approach to the four-color problem, and later generalized by Whitney [7]. Since then, many families of chromatically unique graphs have been studied [3,4,5], yet no general method for constructing such families is known. The classification of join-graphs, in particular, remains incomplete.

In this paper, we investigate the generalized fan $F_{n,m} = O_n + P_m$ where $n \geq 1$ and $m \geq 2$, defined as the join of an empty graph O_n and a path graph P_m [9]. To enrich this framework, we also introduce the generalized star-fan graph $SF_{n,m} = O_n + S_m$, where S_m is a star graph. We prove that $SF_{n,m} \sim F_{n,m}$, establishing a fundamental equivalence between these two constructions.

Our analysis shows that chromatic uniqueness in this family is rare, restricted to the small cases $F_{n,m}$, $n \leq 2$, $m \leq 3$, with $F_{1,3} \cong F_{2,2}$. For all other parameters, $F_{n,m}$ is not χ-unique, and infinite equivalence families arise, such as $F_{n,2} \sim F_{1,m}$ and $F_{n,3} \sim F_{2,m}$, $m \geq n+1$. Thus, this work provides a complete characterization of the chromatic properties of generalized fan graphs and their star-fan counterparts, contributing to the broader theory of join-graph invariants.

Beyond their theoretical significance, these results have practical implications. In statistical physics, the chromatic polynomial corresponds to the partition function of the zero-temperature Potts model [8], so the equivalence between $F_{n,m}$ and $SF_{n,m}$ indicates that structurally distinct systems may exhibit identical thermodynamic behavior. In network engineering, chromatic non-uniqueness provides flexibility in design: as noted by Biggs [1], if multiple non-isomorphic networks share the same chromatic polynomial, engineers can select configurations that are more cost-effective or robust without altering resource distribution or interference constraints.

Thus, this work completes the chromatic classification of generalized fan graphs by establishing their equivalence with star-fan graphs, identifying the rare cases of uniqueness, and mapping infinite families of chromatic equivalences. In doing so, it advances the broader theory of join-graph invariants and highlights new structural insights into the interplay between paths and stars.

2. PRELIMINARIES

This section recalls several established results and fixes the notation used throughout the paper.

Example 2.1 ([4]). Let O_n be the empty graph of order n. Let T_n, P_n *and* S_n denote a tree, a path, and a star graph on $n \geq 1$ vertices, and C_m be the cycle on $m \geq 3$ vertices, respectively. The chromatic polynomials of these fundamental graphs are $P(O_n,\lambda) = \lambda^n$, for any tree T_n (including P_n and S_n): $P(T_n,\lambda) = \lambda(\lambda-1)^{n-1} = \sum_{k=1}^{n} S(n-1,k-1)\,(\lambda)_k$, where $S(n-1,k-1)$ are the Stirling numbers of the second kind, $P(C_m,\lambda) = (\lambda-1)^m + (-1)^m(\lambda-1)$.

Theorem 2.2 ([10]). Let G_1 and G_2 be any two graphs with $P(G_i, \lambda)$ expressed in

factorial form, $i = 1,2$. Then
$P\ (G_1\ + G_2, \lambda) = P\ (G_1, \lambda) \otimes P\ (G_2, \lambda)$

Theorem 2.3 ([4]). If two graphs H_1 *and* H_2 have the same chromatic polynomial, then they have the same number of vertices and edges, the same chromatic number, the same girth, and the same block structure. Moreover, H_1 is connected (respectively, 2-connected) if and only if H_2 is connected (respectively, 2-connected).

Example 2.2 ([4]). The empty graph O_n, the path graph P_n, the complete graph K_n, and the cycle graph C_m, where $n \geq 1$ and $m \geq 3$, are χ-unique.

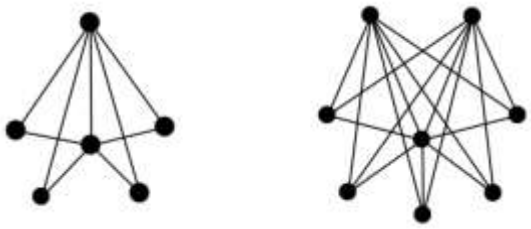
Theorem 2.4 ([6]). Let W_{n+1} denote the wheel graph on $n + 1$ vertices with $n \geq 4$. If n is even, then W_{n+1} is chromatically unique.

Theorem 2.5 ([3]). Any edge-gluing of a complete graph K_p and a cycle C_q, where $p, q \geq 3$, is χ-unique.

3. MAIN RESULTS

This section presents the main results of the paper concerning the chromatic polynomials, chromatic equivalence, and chromatic uniqueness of the generalized fan graph $F_{n,m}$.

Definition 3.1. For integers $n \geq 1$ and $m \geq 2$, the Generalized Star-Fan Graph, denoted by $SF_{n,m}$, is defined as the join of an empty graph O_n *and a* star graph S_m, such that $SF_{n,m} = O_n + S_m$. This structure consists of $n + m$ vertices and $(m-1) + nm$ edges, where every vertex of the independent set O_n is adjacent to every vertex of the star S_m.



$SF_{1,5}$ $SF_{2,6}$

Fig.1. The Generalized Star-Fan Graph $SF_{1,5}$, $SF_{2,6}$

Proposition 3.1. For the generalized fan graph $F_{n,m} = O_n + P_m$ with $n \geq 1$ *and* $m \geq 2$, the chromatic polynomial is
$$P(F_{n,m}, \lambda) = \sum_{k=1}^{m} S(m-1, k-1)\ (\lambda)_k\ (\lambda - k)^n,$$
Where $(\lambda)_k = \lambda(\lambda-1)\cdots(\lambda-k+1)$ and $S(m-1, k-1)$ are Stirling numbers of the second kind.
Proof: From Theorem 2.2. we have $P\ (G_1\ + G_2, \lambda) = P\ (G_1, \lambda) \otimes P\ (G_2, \lambda)$, and since $F_{n,m} = O_n + P_m$ *then*
$$P(F_{n,m}, \lambda) = P\ (O_n + P_m, \lambda) = P\ (O_n, \lambda) \otimes P\ (P_m, \lambda),$$
By Example 2.1 and 2.2.
$$\Longrightarrow P\ (O_n + P_m, \lambda) = \lambda^n \otimes \lambda(\lambda-1)^{m-1} = \lambda^n \otimes \sum_{k=1}^{m} S(m-1, k-1)\ (\lambda)_k$$
By the identity $\lambda^n \otimes (\lambda)_k = (\lambda)_k\ (\lambda - k)^n$ obtain
$$\Longrightarrow P(F_{n,m}, \lambda) = P\ (O_n + P_m, \lambda) = \sum_{k=1}^{m} S(m-1, k-1)\ (\lambda)_k\ (\lambda - k)^n$$

Proposition 3.2. For all integers $n \geq 1$ and $m \geq 2$, the generalized fan graph $F_{n,m}$ is chromatically equivalent to the generalized star-fan graph $SF_{n,m}$, $P(F_{n,m}, \lambda) = P(SF_{n,m}, \lambda)$.
Proof: By Proposition 3.1, the chromatic polynomial of $F_{n,m} = O_n + P_m$ is uniquely determined by the join product of its constituents: $P(F_{n,m}, \lambda) = P(O_n, \lambda) \otimes P(P_m, \lambda)$,
From Example 2.1 in the Preliminaries, it is established that all trees of the same order share an identical chromatic polynomial. Specifically, for any path P_m and star S_m on m vertices: $P(P_m, \lambda) = P(S_m, \lambda) = \lambda(\lambda-1)^{m-1}$
Consequently, P_m and S_m have identical expansions in the falling factorial basis with the same Stirling coefficients $S(m-1, k-1)$. Substituting S_m for P_m in the closed-form expression derived in Proposition 3.1:

$$P(SF_{n,m},\lambda) = P(O_n + S_m,\lambda) = \sum_{k=1}^{m} S(m-1,k-1)\,(\lambda)_k\,(\lambda-k)^n$$

It follows immediately that $P(F_{n,m},\lambda) = P(SF_{n,m},\lambda)$. Since these two non-isomorphic structures yield the same chromatic polynomial for all n and m, they are chromatically equivalent.

Proposition 3.3. The generalized fan graph $F_{n,m}$, with $n \le 2, m \le 3$ are chromatically unique.
Proof:

- For $F_{1,2}$ we have $F_{1,2} = O_1 + P_2 = C_3 \cong K_3$ As shown in Example 2.2, the cycle C_3 is chromatically unique; hence, $F_{1,2}$ is chromatically unique.
- For $F_{1,3}, F_{2,2}$, both are isomorphic to the graph $(K_4 - e)$. Structurally, $F_{1,3}$ can be characterized as the edge-gluing of a complete graph K_3 and a cycle C_3. By applying Theorem 2.5, which preserves chromatic uniqueness for gluing's of unique cliques, it follows that $F_{1,3}$ (and consequently its isomorphic counterpart $F_{2,2}$) is chromatically unique.
- For $F_{1,3}$ we have $F_{2,3} = O_2 + P_3$ is isomorphic to the wheel graph W_5, (specifically, the 5-cycle with an additional vertex joined to every other vertex, forming a hub-and-spoke structure). Since it is established in Theorem 2.4 that wheel graphs of certain orders, including W_5, are chromatically unique, we conclude that $F_{2,3}$ is chromatically unique.

Thus, all generalized fan graphs $F_{n,m}$, with $n \le 2, m \le 3$ are chromatically unique.

Proposition 3.5. For any $n \ge 1$ *and* $m \ge 4$, the generalized fan graph $F_{n,m}$ and the generalized star-fan graph $SF_{n,m}$ are not chromatically unique.
Proof: By Proposition 3.2, we have: $P(F_{n,m},\lambda) = P(SF_{n,m},\lambda)$. Thus, $F_{n,m} \sim SF_{n,m}$ (chromatically equivalent).
Let $F_{n,m} = O_n + P_m$ *and* $SF_{n,m} = O_n + S_m$. *For* $m \ge 4$*:*
1. In P_m*:* The maximum degree is $\Delta(P_m) = 2$.
2. In S_m*:* The maximum degree is $\Delta(S_m) = m - 1$.
3. Since $m \ge 4 \Rightarrow m - 1 \ge 3$, then $\Delta(S_m) > \Delta(P_m)$, hence $P_m \ncong S_m$.
For the join $G + H$, the degree of $v \in V(G)$ *is* $d_{G+H}(v) = d_G(v) + |V(H)|$. Thus*:* $\Delta(F_{n,m}) = max\{m,\ 2+n\}$, $\Delta(SF_{n,m}) = max\{m,\ (m-1)+n\}$, *For* $m \ge 4$, $(m-1)+n > n+2$, which implies $F_{n,m} \ncong SF_{n,m}$. *Since* $P(F_{n,m},\lambda) = P(SF_{n,m},\lambda)$ but $F_{n,m} \ncong SF_{n,m}$, both graphs are not chromatically unique.

Proposition 3.6. For any $n \ge 3$, the generalized fan graph $F_{n,2}$ is chromatically equivalent to $F_{1,m}$ but it is not chromatically unique for all $m \ge n+1$.
Proof. By Proposition 3.1 we compute the chromatic polynomials for small cases:
$P(F_{3,2},\lambda) = P(F_{1,4},\lambda) = \lambda^5 - 7\lambda^4 + 18\lambda^3 - 20\lambda^2 + 8\lambda$
$P(F_{4,2},\lambda) = P(F_{1,5},\lambda) = \lambda^6 - 9\lambda^5 + 32\lambda^4 - 56\lambda^3 + 48\lambda^2 - 16\lambda$
$P(F_{5,2},\lambda) = P(F_{1,6},\lambda) = \lambda^7 - 11\lambda^6 + 50\lambda^5 - 120\lambda^4 + 160\lambda^3 - 112\lambda^2 + 32\lambda$
Hence, $F_{n,2}$ $(n = 3,4,5)$ are chromatically equivalent to $F_{1,m}$ $(m = 4,5,6)$.
In general, for all $n \ge 3$ and $m \ge n+1$, we have $P(F_{n,2},\lambda) = P(F_{1,m},\lambda)$, since by Proposition 3.5, the graph $F_{1,m}$ is not chromatically unique. it follows that $F_{n,2}$ is also not chromatically unique.

Proposition 3.7. For any $n \ge 3$, the generalized fan graph $F_{n,3}$ is chromatically equivalent to $F_{2,m}$ but it is not chromatically unique for all $m \ge 4$.
Proof**.** Applying Proposition 3.1, the chromatic polynomials in small cases are:
$P(F_{3,3},\lambda) = P(F_{2,4},\lambda) = \lambda^6 - 11\lambda^5 + 49\lambda^4 - 108\lambda^3 + 115\lambda^2 - 46\lambda$
$P(F_{4,3},\lambda) = P(F_{2,5},\lambda) = \lambda^7 - 14\lambda^6 + 83\lambda^5 - 262\lambda^4 + 457\lambda^3 - 411\lambda^2 + 146\lambda$
$P(F_{5,3},\lambda) = P(F_{2,6},\lambda) = \lambda^8 - 17\lambda^7 + 126\lambda^6 - 520\lambda^5 + 1275\lambda^4 - 1838\lambda^3 + 1427\lambda^2 - 454\lambda$
These computations show that $F_{n,3}$ $(n = 3,4,5)$ share identical chromatic polynomials with $F_{2,m}$ $(m = 4,5,6)$. By the structure of the Stirling expansion, the equality $P(F_{n,3},\lambda) = P(F_{2,m},\lambda)$ extends to all $n \ge 3$ and $m \ge 4$. Since Proposition 3.5, establishes that $F_{2,m}$ is not chromatically unique, it follows that $F_{n,3}$ is likewise not chromatically unique.

To provide a clear taxonomy of the $F_{n,m}$ family, the following table summarizes the status of generalized fan graphs:

Table 1: CHROMATIC EQUIVALENCE AND UNIQUENESS OF GENERALIZED FAN GRAPHS

Graphs	**Conditions**	**χ- equivalent**	**χ-unique?**
$F_{n,m}$	$n \leq 2, m \leq 3$	$F_{n,m} \sim SF_{n,m}$	yes
$F_{n,m}$	$n \geq 3, m \geq 4$	$F_{n,m} \sim SF_{n,m}$	No
$F_{n,2}$	$n \geq 3$	$F_{n,2} \sim F_{1,m},\ m \geq n+1$	No
$F_{n,3}$	$n \geq 3$	$F_{n,3} \sim F_{2,m},\ m \geq n+1$	No

4. CONCLUSION

This study provides a full characterization of the chromatic properties of the generalized fan graph $F_{n,m}$. We proved that chromatic uniqueness is limited to the small cases with $n \leq 2, m \leq 3$, while for all other parameters the graphs are not unique. Infinite families of chromatic equivalences were identified, and the equivalence between fan and star-fan graphs was established. These results complete the classification of this family and contribute to the broader understanding of join-graph invariants.

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