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Scott topologies via b-Structures Designed on Continuous Mappings

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Abstract: *New topological spaces can be defined in many different ways, like extending the set theory by deriving a new set from an old one, or by introducing new classes of continuity of functions. In this work, we try to use these methods together to modified the theory of topological spaces. Here established the notion of b-Scott topology as a generalization of a new class of topological spaces. Firstly, investigate their basics and study continuous mappings between them. Then, we deal with some topologies like strong b-Scott topology, admissible topologies, and Isbell topologies, designed via b-open sets and b-closed sets. Relations among these topologies are discussed and illustrated.*

Keywords: Scott topology, b- open set, Strong topology, Isbell topology .

Introduction:

Idea of Scott topology proposed by Dana Scott [1] as a new mathematical tool to treatment some applications in some areas, like domain theory, computation theory, and category theory. He answered many computational problems topologically via terms that had been interpreted and considered despite continuous functions on a certain space, that so-called the domain. Then, many new techniques appeared to construct such domains as objects in category theory. However, Scott's method insight could be determined as continuity to stay experimenting with a decisive influence. In general topology, set theory plays a good role. Many studies on these sets have been discussed in different spaces; one of these sets is a b- open set that was first proposed by Njastad in 1965[2]. The same topology of preopen sets can be defined by these sets. This work deals with introducing and investigating many classes of continuity in many objects meats with topology, and raises the question of determining which topologies these mappings are continuous. Furthermore, many tools have been established to define these topologies, these topologies often considered in the analysis of mappings, especially in linear spaces. Scott topologies and strong Scott topologies can be considered important tools to construct important topologies on spaces of continuous mappings. Here modified and constructed new topologies defined by b- open subsets of topological spaces, which are called b- Scott topologies.

Scott topologies can be defined as topologies in which any open subsets are precisely functions that preserve directed joins. Indeed this, implies that, the continuity between preorders with these topologies is adverb functions between them that preserving directed joins.

Let $\mathfrak{A}, \mathcal{V}$ and $\mathfrak{D}$ are topological spaces, by $O_{\mathfrak{A}}, O_{\mathcal{V}}$ and $O_{\mathfrak{D}}$ we mean the set of all open subsets of $\mathfrak{A}$, $\mathcal{V}$ and $\mathfrak{D}$ respectively, and by $C(\mathcal{V}, \mathfrak{D})$ we refer to the set of all continuous maps from $\mathcal{V}$ to $\mathfrak{D}$. Let $S_{\mathcal{V}}$ be a sub collection of elements of $O_{\mathcal{V}}$ satisfying the axioms; if $\mathcal{V}_1 \in S_{\mathcal{V}}$, $\mathcal{V}_2 \in O_{\mathcal{V}}$ and $\mathcal{V}_1 \subseteq \mathcal{V}_2$ then $\mathcal{V}_2 \in S_{\mathcal{V}}$ and; for any collection of $O_{\mathcal{V}}$ with union included to $S_{\mathcal{V}}$, there exist many finitely elements of this collection with union also in $S_{\mathcal{V}}$. Then $S_{\mathcal{V}}$ forms a topology on $O_{\mathcal{V}}$,which is called Scott topology. An admissible topology on $C(\mathcal{V}, \mathfrak{D})$ can be described as a set $O_{\mathcal{V}\mathfrak{D}} = \{\mathcal{J}^{-1}(U) \in O_{\mathcal{V}}: \mathscr{J} \in C(\mathcal{V}, \mathfrak{D})$ and $U \in O_{\mathfrak{D}}\}$[6], i.e., a topology τ on $C(\mathcal{V}, \mathfrak{D})$ is admissible if the continuity of a map $\omega : \tau \to C_\tau(\mathcal{V}, \mathfrak{D})$. implies the continuity of the map $\beta: \mathfrak{A} \times \boldsymbol{\mathcal{V}} \to \boldsymbol{\mathfrak{D}}$ or equivalently the evaluation map $e : C_\tau(\mathcal{V}, \mathfrak{D}) \times \mathcal{V} \to \mathfrak{D}$, defined by $e(g, v) = g(v)$ for every $(g, v) \in C(\mathcal{V}, \mathfrak{D}) \times \boldsymbol{\mathcal{V}}$, is continuous. A Scott topology is called a strong Scott topology $SS_{\mathcal{V}}$ on $O_{\mathcal{V}}$ if we replace the word collection by the word open cover in the second condition of the definition of a Scott topology[7]. The Isbell topology on $C(\mathcal{V}, \mathfrak{D})$, denoted here by τ_{Is}, is the topology which generated by a subbasis of the form $\{ g \in C(\mathcal{V}, \mathfrak{D}): g^{-1}(D) \in V \}$, where $V \in S_{\mathcal{V}}$ and $D \in O_{\mathfrak{D}}$. τ_{Is} is called the strong Isbell topology [8]on $C(\mathcal{V}, \mathfrak{D})$,denoted by $S\tau_{Is}$, is the topology which has as a subbasis the family of all sets of the form $\{ f \in C(\mathcal{V}, \mathfrak{D}): g^{-1}(D) \in V \}$, where $V \in SS_{\mathcal{V}Y}$ and $D \in O_{\mathfrak{D}}$.

1. Preliminaries

Definition 1.1[2]: Let $\mathfrak{A}$ be a topological space, a subset $\mathcal{P}$ of $\mathfrak{A}$ is called $b-open$ if $\mathcal{P} \subset cl(int\, \mathcal{P}) \cup int(cl\, \mathcal{P})$.

Example 1.2. In usual topology, let $\mathcal{P} = [0,1] \cup ((1,2) \cap Q)$ where" Q stand for the set of rational numbers". Then $\mathcal{P}\ is\ b-open$.

Proposition 1.3. [2]

1) $\bigcup \mathcal{P}_i$ is again $b-open$ set.

2) $\bigcap \mathcal{P}_i$ is a $b-open$ set.

Definition 1.4. [2]: Let $\mathcal{P}$ be a subset of $\mathfrak{A}$. Then it is $b-closed\ if\ \mathfrak{A}\backslash\mathcal{P}\ is\ b-open\ and\ it\ is\ b-closed$ if only if $int(cl\ \mathcal{P}) \cap cl(int\ \mathcal{P}) \subset \mathcal{P}$.

Definition 1.5. [3]: A subset $\mathcal{P}$ of $\mathfrak{A}$, the $b-interior\ of\ \mathcal{P}$, denoted by $b_{int}\ \mathcal{P}$, is the largest $b-open$ in $\mathcal{P}$, the $b-closure\ of\ \mathcal{P}$, denoted by $b_{cl}\ \mathcal{P}$, is the smallest $b-closed$ set contains $\mathcal{P}$.

Proposition 1.6.[4]

Assume a subset $\mathcal{P}$ of $\mathfrak{A}$. Then

1) $b_{cl}\ \mathcal{P} = scl\ \mathcal{P} \cap pcl\ \mathcal{P}$;
2) $b_{int}\ \mathcal{P} = sint\ \mathcal{P} \cup pint\ \mathcal{P}$.

Collection of all $b-open$ (resp. $b-closed$) sets in $\mathfrak{A}$ is denoted by $b_{O(\mathfrak{A})}$ (resp. $b_{C(\mathfrak{A})}$). In fact $b_{O(\mathfrak{A})}$ is not a topology on $\mathfrak{A}$, but it can defines a topology by the set $T(b_{O(\mathfrak{A})}) = \{u \subset \mathfrak{A}, u \cap s \in b_{O(\mathfrak{A})}\ where\ s \in b_{O(\mathfrak{A})}\}$.

2. Admissibility of topologies via of b- continuous mappings

Let $bC_{\mathcal{V}\mathfrak{D}}$ be the set of b-continuous mappings from $\boldsymbol{\mathcal{V}}$ to $\boldsymbol{\mathfrak{D}}$ with topology τ, denoted by $bC_{\tau_{\mathcal{V}\mathfrak{D}}}$. This part deals with studying new types of Scott topologies defined on $bC_{\mathcal{V}\mathfrak{D}}$. We define b-Scott topologies on $bO_{\mathcal{V}}$ and we establish admissible topologies on $bC_{\mathcal{V}\mathfrak{D}}$ where $bO_{\mathcal{V}\mathfrak{D}} = \{f^{-1}(U) \in bO_{\mathcal{V}}\ : f \in bC_{\mathcal{V}\mathfrak{D}}$ and $U \in \mathfrak{D}_Z$. Many relations between these spaces and the spaces which defined in the first chapter are studied.

Let p: $\mathcal{U} \times \boldsymbol{\mathcal{V}} \rightarrow \boldsymbol{\mathfrak{D}}$ be a b- continuous map. Assume that γ: $\mathcal{U} \rightarrow bC_{\mathcal{V}\mathfrak{D}}$, such that $\gamma(u)(v) = p\ (u, v)$, for every $u \in \mathcal{U}$ and $v \in \boldsymbol{\mathcal{V}}$. Let $\hbar$: $\mathcal{U} \times \boldsymbol{\mathcal{V}} \rightarrow bC_{\mathcal{V}\mathfrak{D}}$ and β: $\mathcal{U} \times \boldsymbol{\mathcal{V}}$ to $\boldsymbol{\mathfrak{D}}$, such that $\beta\ (u, v) = \hbar\ (u)(v)$, for every $(u, v) \in \mathcal{U} \times \boldsymbol{\mathcal{V}}$.

Definition 2.1.

A topology τ on $bC_{\mathcal{V}\mathfrak{D}}$ is called b-admissible, if the continuity of $:\ \mathcal{U} \rightarrow bC_{\tau_{\mathcal{V}\mathfrak{D}}}$. implicit to continuity of $\mathcal{U} \times \boldsymbol{\mathcal{V}}$ to $\boldsymbol{\mathfrak{D}}$ or equivalently the evaluation map e : $bC_{\mathcal{V}\mathfrak{D}} \times \boldsymbol{\mathcal{V}} \rightarrow \boldsymbol{\mathfrak{D}}$, defined by $e(g, v) = g(v)$ for every $(g, v) \in bC_{\mathcal{V}\mathfrak{D}} \times \mathcal{V}$, is b- continuous.

Definition 2.2.

Let $bS_{\mathcal{V}}$ be a sub collection of elements of $O_{\mathcal{V}}$ satisfying the next axioms,

1. $\mathcal{V}_1 \in bS_{\mathcal{V}}$, $\mathcal{V}_2 \in O_{\mathcal{V}}$ and $\mathcal{V}_1 \subseteq \mathcal{V}_2$ imply $\mathcal{V}_2 \in bS_{\mathcal{V}}$ and;
2. Elements of $O_{\mathcal{V}}$, with union in $bS_{\mathcal{V}}$, there exist many finitely elements $O_{\mathcal{V}}$ whose union also in $bS_{\mathcal{V}}$. Then $bS_{\mathcal{V}}$ form a topology on $O_{\mathcal{V}}$ and is called Scott topology.

Definition 2.3.
A b-Scott topology is called strong b-Scott topology $bSS_{\mathcal{V}}$ on $O_{\mathcal{V}}$ if we replace the word collection by the word open cover in the second condition of definition above.

topology defined by the set:
$\{\ g \in C(\mathcal{V}, \mathfrak{D})$: $g^{-1}(D)\ \in V\ \}$, where $V \in bS_{\mathcal{V}}$ and $D \in O_{\mathfrak{D}}$. Is called b-Isbell on $C(\mathcal{V}, \mathfrak{D})$, and denoted by $b\tau_{Is}$,

Definition 2.3.[6]
The strong b-Isbell topology $bS\tau_{Is}$ on $C(\mathcal{V}, \mathfrak{D})$ is the topology defined by a subbasis of the form:
$\{\ f \in C(\mathcal{V}, \mathfrak{D})$: $g^{-1}(D)\ \in V\ \}$, where $V \in bSS_{\mathcal{V}\mathfrak{D}}$ and $D \in O_{\mathfrak{D}}$.

Theorem 2.4.

Any b- admissible topology is admissible.

Proof:

Let τ be a topology on $bC_{\mathcal{V}\mathfrak{D}}$, then for every space $\mathfrak{U}$, continuity $\gamma : \mathfrak{U} \to bC_{\tau_{\mathcal{V}\mathfrak{D}}}$. implies the continuity of the map β: $\mathfrak{U} \times \boldsymbol{\mathcal{V}} \to \boldsymbol{\mathfrak{D}}$ or equivalently the map $e : bC_{\tau_{\mathcal{V}\mathfrak{D}}} \times \boldsymbol{\mathcal{V}} \to \mathfrak{D}$, defined by $e(g, v) = g(v)$ for every $(g, v) \in bC_{\mathcal{V}\mathfrak{D}} \times \boldsymbol{\mathcal{V}}$, is regular continuous. Since e is continuous, then, τ is admissible.

Theorem 2.5.

Any b- Scott topology is Scott topology

Proof:

Let $bS_{\mathcal{V}}$ be a b- Scott topology on $bO_{\mathcal{V}}$. Assume $\mathscr{A}$ a sub collection of $bO_{\mathcal{V}}$ put $A \in \mathscr{A}$ and Ɓ $\in bO_{\mathcal{V}}$ $\ni \mathscr{A} \subseteq$ Ɓ then we get Ɓ $\in \mathscr{A}$ and for every b-open set of $\boldsymbol{\mathcal{V}}$,with union $\mathscr{A}$, exist many finitely elements of $\mathscr{A}$ with union also in $\mathscr{A}$. Since any b-open open, we get that $bS_{\mathcal{V}}$ is Scott topology.

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