

T_{12}^* -I-OPEN SETS

Abbas A. Jedi

College of Basic Education, Department of Mathematics, University of Al-Kufa, Najaf, Iraq
 abbasa.jedi@uokufa.edu.iq

Abstract: In classical and operator topological spaces, the concept of ideal convergence has been widely studied by many authors since 1933. In this work, we extend the notion of ideal convergence to bi-operator topological spaces equipped with T_1 and T_2 . We introduce the T_{12}^* by extending a previous one-operator work. We establish fundamental properties and examine the relationship with the classical ideal convergence. Furthermore, we will study the connection between one and bi-operator ideal convergence. We also provide an inclusion to show that bi-operator ideal convergence represents a strict generalization of bi-operator. This study results in a framework that shows the underlying relation and interaction of the bi-operator with ideal convergence.

Keywords—T-I-open set, T_{12}^* -I-open set, T-I-convergence, T_{12}^* -I-convergence, T-I-sequentially open, T_{12}^* -I-sequentially open, T-I-sequentially space, and T_{12}^* -I-sequentially space.

1. INTRODUCTION

Kuratowski introduced the concept of an ideal on a set X in his seminal 1933 work [1]. He defined it as a non-empty family I of subsets that remains closed under finite unions and stable under taking subsets. If a set belongs to I , its subsets must too. Over the decades, this elegant framework has deeply influenced general topology, serving as a cornerstone for modern structural analysis. [2], [3], [4], [5], [6], [7], [8], [9], [10], [12].

We introduce a new class (X, τ, T_1, T_2, I) be an ideal bi-operator topological space where T_1, T_2 are operators associated with the topology τ on X .

Layth M. introduced the notion of a contra- T_{12}^* bi-operator topological space. More recently, Zhou et al. (2020) employed the ideal I of the set of natural numbers $\mathbb{N}$ to formulate the concept of I -convergence [3]. Motivated by these developments, this work extends these ideas within the framework of operator topological spaces. We define the notions of T_{12}^* -I-open sets and T_{12}^* -I-convergence on T^* -open sets, and investigate their basic properties. Furthermore, additional results concerning T_{12}^* -I-sequential convergence are established.

2. T_{12}^* -I- CONVERGENCE DEFINITIONS AND PROPERTIES

We first introduce some important concepts and definitions,

Definition (2.1): A non-trivial ideal $I \subseteq \mathbb{N}$ is said to be admissible $\Leftrightarrow \{\{n\}: n \in \mathbb{N}\} \subseteq I$.

Definition (2.2): [10] Let (X, τ) be a topological space, an operator T is a function $T: P(X) \rightarrow P(X)$ s.t $\forall A \in \tau, A \subseteq T(A)$. Note that the triple (X, τ, T) is called an operator topological space.

Definition (2.3): [12] Let (X, τ) be a topological space and let T_1, T_2 be two operators associated with the topology τ on X then (X, τ, T_1, T_2) is called a bioperator topological space.

Definition (2.4): [10] [11] Let (X, τ, T) be an operator topological space, let $A \subseteq X$

1. A is called T -open if given $x \in A$, then there exists $V \in \tau$ there exists $x \in V \subseteq T(V) \subseteq A$.

2. A is called T^* -open if $A \subseteq T(A)$ (A is not necessarily open).

Definition (2.5): [12] Let (X, τ, T_1, T_2) be a bioperator topological space and let $A \subseteq X$. The set A is said to be a T_{12}^* -open if $A \subseteq T_1(A) \cup T_2(A)$, The complement of T_{12}^* -open is called T_{12}^* -closed for

Definition (2.6): Let (X, τ, T_1, T_2) be a bioperator topological space, $A \subset X$ and $x \in X$. Then the set A is called

- A neighborhood of $x \Leftrightarrow \exists B$ an open set $\ni x \in B \subset A$.
- A T -neighborhood of $x \Leftrightarrow \exists B$ a T -open set $\ni x \in B \subset A$
- A T^* -neighborhood of $x \Leftrightarrow \exists B$ a T^* -open set $\ni x \in B \subset A$
- A T_{12}^* -neighborhood of $x \Leftrightarrow \exists B$ a T_{12}^* -open set $\ni x \in B \subset A$.

Lemma (2.1): Let (X, τ, T_1, T_2) be a bioperator topological space, then the following is satisfied,

- Every T -open set is T_{12}^* -open set.
- Every T -open set is open set.
- Every open set is T_{12}^* -open set.
- Every T^* -open set is T_{12}^* -open set.

Definition (2.7): [12] Let (X, τ, T_1, T_2) be a bioperator topological space, and $A \subset X$. Then A is said to be a T_{12}^* - open if :

$$T_1(A) = \text{Cl}(\text{Int}(A)),$$

$$T_2(A) = \text{Int}(\text{Cl}(A)),$$

$$\text{Then, } A \subseteq \text{Cl}(\text{Int}(A)) \cup \text{Int}(\text{Cl}(A)),$$

this is the definition of b -open set.

Notice that every T_1^* -open (T_2^* -open) is T_{12}^* -open because if A is T_1^* -open then $A \subseteq T_1(A) \subseteq T_1(A) \cup T_2(A)$, so A will be T_{12}^* -open.

Definition (2.8) : [1] Let (X, τ) be a topological space and $I \subseteq \mathbb{N}$ be an ideal. A sequence $\{x_n\}, n \in \mathbb{N}$ is identified as I -convergent to a point x in X , if there exists a neighborhood V of x with $A_V = \{n \in \mathbb{N}: x_n \notin V\} \in I$, and is denoted as $I\text{-}\lim_{n \rightarrow \infty} x_n = x$.

Definition (2.9) : [1] (X, τ, T, I) be an ideal operator topological space and $I \subseteq \mathbb{N}$. A sequence $\{x_n\}$, $n \in \mathbb{N}$ is identified as T -I-convergent to a point x in X , if there exists a T -neighborhood V of x with $A_v = \{n \in \mathbb{N} : x_n \notin V\} \in I$, and is denoted as T -I- $\lim_{n \rightarrow \infty} x_n = x$.

Definition (2.10) : (X, τ, T_1, T_2, I) be an ideal bioperator topological space and $I \subseteq \mathbb{N}$. A sequence $\{x_n\}$, $n \in \mathbb{N}$ is identified as T_{12}^* -I-convergent to a point x in X , if there exists a T_{12}^* -neighborhood V of x with $A_v = \{n \in \mathbb{N} : x_n \notin V\} \in I$, and is denoted as T_{12}^* -I- $\lim_{n \rightarrow \infty} x_n = x$.

Lemma (2.2): Let (X, τ, T_1, T_2, I) be an ideal bioperator topological space, then the following is satisfied,

- 1) T -I-convergence implies I-convergence
- 2) T -I-convergence implies T_{12}^* -I-convergence
- 3) I-convergence implies T_{12}^* -I-convergence

Proof (3): Assume I-convergence, implies there exist A be an open set of (X, τ, T_1, T_2, I) , with T_{12}^* -I-convergence sequence $\{x_n\}$ to point x , such that $\{n \in \mathbb{N} : x_n \notin A\} \in I$. by Lemma(2.1)(3) A is also T_{12}^* -open set. Then $\{x_n\}$ is a T_{12}^* -I-convergence by Definition (2.10).

Similar proof follows for (1) and (2) using Lemma (2.2) as well as Lemma (2.1).

Lemma (2.3) : As every open set is a T_{12}^* -open, we verified that T_{12}^* -I-convergence implies I-convergence.

If we would like to find an equivalent between them, (X, τ, T_1, T_2, I) should be a topological space with τ be the discrete topology. If τ is the discrete topology, every open set is a closed set and conversely. Let A be a

T_{12}^* -open set, then

$$A \subseteq \text{Cl}(\text{Int}(A)) \cup \text{Int}(\text{Cl}(A)),$$

$$A \subseteq \text{Cl}(A) \cup \text{Int}(A),$$

$$A \subseteq A \cup A,$$

$$A = A.$$

And A is a subset of (X, τ) therefore if we have a sequence $\{x\}$ which is I-convergent, $\{x\}$ must

be T_{12}^* -I-convergent.

Definition (2.11): Let (X, τ, T_1, T_2, I) be an ideal bioperator topological space and an ideal $I \subseteq \mathbb{N}$. Then:

- 1) $B \subset X$ is called T_{12}^* -I-closed if for each sequence $\{x_n\}$ in B with T_{12}^* -I- $\lim_{n \rightarrow \infty} x_n = x \in X$. Then $x \in B$.
- 2) $B \subset X$ is called T_{12}^* -I-open if $X - B$ T_{12}^* -I-closed.

Lemma (2.4): Let (X, τ, T_1, T_2, I) be an ideal bioperator topological space, and an Ideal $I \subset \mathbb{N}$.

If a sequence $\{x_n\}$ is T_{12}^* -I-convergence to point $x \in X$. Then any other sequence in X with $n \in I$ is T_{12}^* -I-convergence to same point x .

Proof: Let $\{y_n\}$ another sequence in X , s.t., $x_n \neq y_n$, $n \in I$. For each T_{12}^* -neighborhood A of x with,

$$\{n \in \mathbb{N} : y_n \notin A\} \subseteq \{n \in \mathbb{N} : x_n \neq y_n\} \cup \{n \in \mathbb{N} : x_n \notin A\} \in I \\ \Rightarrow \{n \in \mathbb{N} : y_n \notin A\} \in I$$

Hence, by Definition (2.8) $\{y_n\}$ is T_{12}^* -I-convergence to same point x .

Lemma (2.5): Let (X, τ, T_1, T_2, I) be an ideal bioperator topological space, and an Ideal $I \subseteq J \subset \mathbb{N}$. If a sequence $\{x_n\}$ is T_{12}^* -I-convergence to point $x \in X$. Then, it is T_{12}^* -J-convergence to x .

Proof: Let $\{x_n\}$ be a T_{12}^* -I-convergence to point $x \in X$. By Definition (2.8) for each T_{12}^* -neighborhood A of x ,

$$\{n \in \mathbb{N} : x_n \notin A\} \in I \subseteq J$$

Implies $\{n \in \mathbb{N} : x_n \notin A\} \in J$, hence, $\{x_n\}$ is T_{12}^* -J-convergence to x .

Lemma (2.6): Let (X, τ, T_1, T_2, I) be an ideal bioperator topological space, and two Ideals $I \subseteq J \subset \mathbb{N}$. If $A \subseteq X$ is T_{12}^* -I-open. Then it is T_{12}^* -J-open.

Proof: Suppose $A \subseteq X$ is T_{12}^* -I-open, hence $X - A$ is T_{12}^* -I-closed, By Definition (2.11) (1) for every T_{12}^* -I-convergence sequence $\{x_n\}$ in $X - A$ to point x , Then $x \in X - A$, by Lemma (2.4) $\{x_n\}$ is also T_{12}^* -J-convergence in $X - A$ to x , implies $X - A$ is T_{12}^* -J-closed, hence A is T_{12}^* -J-open.

Corollary (2.6): Let (X, τ, T_1, T_2, I) be an ideal bioperator topological space, and two Ideals $I \subseteq J \subset \mathbb{N}$. Then T_{12}^* -I-sequential, then T_{12}^* -J-sequential.

Lemma (2.7): Let (X, τ, T_1, T_2, I) be an ideal bioperator topological space, an Ideals $I \subset \mathbb{N}$, and $\subseteq X$. Then the following are equivalent.

- 1) A is T_{12}^* -I-open.
- 2) For each sequence $\{x_n\}$ in X with is T_{12}^* -I-convergence to $x \in A$, where $\{n \in \mathbb{N} : x_n \in A\} \notin I$.
- 3) For each sequence $\{x_n\}$ in X with is T_{12}^* -I-convergence to $x \in A$, where $\{n \in \mathbb{N} : x_n \in A\} = \omega$.

Proof: (1) $\Rightarrow$ (2): Let A be a T_{12}^* -I-open set in X . And let $\{x_n\}$ be a T_{12}^* -I-convergence sequence in X to a point $x \in A$. Let us consider the contrary hypothesis, By Assuming $N_0 = \{n \in \mathbb{N} : x_n \in A\} \in I$, then $N_0 \neq \mathbb{N}$ and $A \neq X$. Let $x_0 \in X - A$. We define the sequence $\{y_n\}$ in X , s.t., $y_n = x_0$ for $n \in N_0$ and $y_n = x_n$ for $n \notin N_0$. By Lemma (2.4), the sequence $\{y_n\}$ is T_{12}^* -I-convergence to x . Note that $X - A$ is a T_{12}^* -I-closed and $\{y_n\}$ is T_{12}^* -I-convergence sequence in $X - A$, By definition (2.11)(1) $x \in X - A$, and this a contradiction!, Hence $N_0 \in I$.

(1) $\Rightarrow$ (2): It follows from definition (2.1) the admissible ideal.

Now to show the (3) $\Rightarrow$ (1) Assume that A is not T_{12}^* -I-open then $X - A$ is T_{12}^* -I-closed, which implies there is $\{x_n\}$ in $X - A$ with is T_{12}^* -I-convergence to $x \in A$ and

this contradicts (3) because $x_n \in A$.

3. T_{12}^* -I- SEQUENTIALLY OPEN AND T_{12}^* -I- SEQUENTIALLY SPACE

Definition (3.1): Let (X, τ, T_1, T_2, I) be an ideal bioperator topological space, and $A \subset X$. Then A is said to be T_{12}^* -I-sequentially open $\Leftrightarrow$ no sequence in $X - A$ has T_{12}^* -I-limit in A , i.e., a sequence cannot be T_{12}^* -I-converge out of T_{12}^* -I-sequentially close set.

Definition (3.2): Let (X, τ, T_1, T_2, I) be an ideal bioperator topological space and an ideal $I \subseteq \mathbb{N}$. Then X is called T_{12}^* -I-sequentially space if each T_{12}^* -I-closed set in X is T_{12}^* -I-closed.

Definition (3.3): Let (X, τ, T_1, T_2, I) be an ideal bioperator topological space. Then, the space X is T_{12}^* -I-sequentially, For any set A is T_{12}^* -open $\Leftrightarrow$ it is T_{12}^* -I-sequentially subspace of X .

Theorem (3.1): Let (X, τ, T_1, T_2, I) be an ideal bioperator topological space. Every T_{12}^* -I-sequential space is hereditary with respect to T_{12}^* -I-open (T_{12}^* -I-closed).

Proof: Assume that X be an T_{12}^* -I-sequential Space. Let A is a T_{12}^* -I-open set in X . It is clear that A is T_{12}^* -open in X . We need to show that A is T_{12}^* -I-sequential open. Let V is T_{12}^* -I-open in A . We claim that V is T_{12}^* -open in X we need to show that V is T_{12}^* -I-open in X . We can assume there is y in $X - A$. Take an arbitrary $x \in V$ and $\{x_n\}$ be a T_{12}^* -I-convergence sequence in X to a point x . Since, A is T_{12}^* -open in X and $x \in A$, the set $\{n \in \mathbb{N} : x_n \notin A\} \in I$. Define sequence $\{y_n\}$ in X , such that $y_n = x_n$ for $n \in A$ and $y_n = y$ for $n \notin A$. By lemma (2.5)(2), the sequence $\{y_n\}$ is T_{12}^* -I-convergence to x . Since $\{n \in \mathbb{N} : x_n \notin V\} = \{n \in \mathbb{N} : y_n \notin V\}$, by Lemma (2.7), V is T_{12}^* -I-open in X , by Definition (3.3) A is T_{12}^* -I-sequential open (T_{12}^* -I-sequential subspace of X).

Similarly, Assume that X be an T_{12}^* -I-sequential Space, Let A is a T_{12}^* -I-closed set in X implies A is a T_{12}^* -closed set in X . For any T_{12}^* -I-closed subset U of A set in X , we need to show U is T_{12}^* -closed set in X . Since X is a T_{12}^* -I-sequential space, by definition (3.2) it is enough to show that U is T_{12}^* -I-closed in X which implies T_{12}^* -closed set. Suppose that $\{x_n\}$ be a T_{12}^* -I-convergence sequence in to a point $x \in X$ with $x_n \in U$. By definition (2.10). Since A is T_{12}^* -closed, it has that $x \in A$ and so $x \in U$, hence U is a T_{12}^* -I-closed set of A .

Corollary (3.1): Every T_{12}^* -sequential space is hereditary with respect to T_{12}^* -I-open (T_{12}^* -I-closed).

Theorem (3.2): Let (X, τ, T_1, T_2, I) be an ideal bioperator topological space. T_{12}^* -I-sequential spaces are preserved by topological sums.

Proof: Assume $\{Y_\kappa\}$ be a family of a T_{12}^* -I-sequential spaces and $\hat{X}$ is the topological sum this family. Now, we need to show that $\hat{X}$ is a T_{12}^* -I-sequential space. Let A is a T_{12}^* -I-

closed set in $\hat{X}$, By Definition (3.2) it suffice to show that A is a T_{12}^* -closed set in $\hat{X}$. Since, $\{Y_\kappa\}$ be a family of a T_{12}^* -I-sequential implies every in $\hat{X}$ implies Y_κ is a T_{12}^* -closed set in $\hat{X}$ for each κ , so the intersection $A \cap Y_\kappa$ is a T_{12}^* -I-closed set in $\hat{X}$. It is clear that $A \cap Y_\kappa \subseteq Y_\kappa$ is a T_{12}^* -I-closed in Y_κ . By the assumption ($\{Y_\kappa\}$ be a family of a T_{12}^* -I-sequential spaces) we get that $A \cap Y_\kappa$ is a T_{12}^* -closed in Y_κ . By the definition of $\hat{X}$ implies $A \cap Y_\kappa$ is T_{12}^* -closed set in $\hat{X}$ and A is T_{12}^* -closed set in $\hat{X}$. Hence, by Definition (3.2) the sum of the family of T_{12}^* -I-sequential spaces is also, T_{12}^* -I-sequential space.

Corollary (3.2): A T_{12}^* -I-sequential spaces are preserved by topological sums.

Corollary (3.3): Let (X, τ, T_1, T_2, I) be an ideal bioperator topological space. Let $\{Y_\kappa\}$ be a family of a T_{12}^* -I-open sets

- 1) $\bigcup_\kappa Y_\kappa$ of T_{12}^* -I-open sets is also, T_{12}^* -I-open set.
- 2) The finitely many $\bigcap_\kappa Y_\kappa$ of sequentially T_{12}^* -I-open sets is also, sequentially T_{12}^* -I-open set.

Definition (3.4): [7] Let (X, τ, T_1, T_2, I) be an ideal bioperator topological space and an ideal $I \subseteq \mathbb{N}$. A sequence $\{x_n\}$ in X is I-eventually in A if $\exists B \in I$ s.t. $\{\forall n \in \mathbb{N} - B : x_n \in A\} \in I$.

Proposition (3.1): Let (X, τ, T_1, T_2, I) be an ideal bioperator topological space and I be a maximal ideal $I \subseteq \mathbb{N}$. $A \subseteq X$, A is T_{12}^* -I-open $\Leftrightarrow$ for each $\{x_n\}$ in X , with $x_n \xrightarrow{T_{12}^* I} x \in A$ is I-eventually in A .

Proof: ($\Rightarrow$) Let A is T_{12}^* -I-open and $\{x_n\}$ be a T_{12}^* -I-convergence sequence to a point $x \in A$. $\Leftrightarrow$ Since, I is maximal ideal there exist $J = \{n \in \mathbb{N} : x_n \notin A\} \in I$, $\Leftrightarrow$ by Lemma (2.10) $\forall n, \{n \in \mathbb{N} - J : x_n \in A\}$, $\Leftrightarrow$ hence by Definition (3.4), A is I-eventually in A .

Theorem (3.3): Let (X, τ, T_1, T_2, I) be an ideal bioperator topological space, and an ideal $I \subseteq \mathbb{N}$. The intersection of two T_{12}^* -I-open sets is T_{12}^* -I-open.

Proof: Let A and B are two T_{12}^* -I-open sets in X , we need to show that $A \cap B$ is T_{12}^* -I-open sets in X .

It is suffices to show that every sequence $\{x_n\}$ in X , with $x_n \xrightarrow{T_{12}^* I} x \in A \cap B$ is I-eventually in $A \cap B$, and by Proposition (3.1) $A \cap B$ is T_{12}^* -I-open sets in X .

Suppose that J and K are subset of the ideal I such that for each n $\{n \in \mathbb{N} - J : x_n \in A\}$ and $\{n \in \mathbb{N} - K : x_n \in B\}$. Since $A \cup B \in I$ implies for each $\{n \in \mathbb{N} - (J \cup K) : x_n \in A \cap B\}$. Hence, $\{x_n\}$ is I-eventually in $A \cap B$. By Proposition (3.1) $A \cap B$ is T_{12}^* -I-open sets in X .

4. REFERENCES

- [1] K. Kuratowski, Topologie, vol. 1. in Monografie matematyczne, vol. 1. Lwow, 1933. [Online]. Available: <https://books.google.iq/books?id=KFDk0AEACAAJ>.
- [2] S. Lin, "On $\mathcal{I}$ -neighborhood Spaces and $\mathcal{I}$ -quotient Spaces," Bulletin of the

- Malaysian Mathematical Sciences Society, vol. 44, no. 4, pp. 1979–2004, 2021, doi: 10.1007/s40840-020-01043-1.
- [3] X. Zhou, L. Liu, and S. Lin, “On Topological Spaces Defined by $\mathcal{I}$ -Convergence,” Bulletin of the Iranian Mathematical Society, vol. 46, no. 3, pp. 675–692, 2020, doi: 10.1007/s41980-019-00284-6.
- [4] F. Sereti, “A study of new dimensions for ideal topological spaces,” Applied General Topology, vol. 25, no. 1, pp. 183–198, Apr. 2024, doi: 10.4995/agt.2024.19760.
- [5] A. K. Banerjee and I. Debnath, “On density topology using ideals in the space of reals,” 2022. [Online]. Available: <https://arxiv.org/abs/2205.03378>.
- [6] B. Mitra and D. Chowdhury, “Ideal spaces,” Applied General Topology, vol. 22, no. 1, p. 79, Apr. 2021, doi: 10.4995/agt.2021.13608.
- [7] M. Hosny, “Topologies Generated by Two Ideals and the Corresponding M_1 - $\mathcal{I}$ -Approximations Spaces with Applications,” Journal of Mathematics, vol. 2021, pp. 1–13, Oct. 2021, doi: 10.1155/2021/6391266.
- [8] A. Njamcul and A. Pavlović, “On topology expansion using ideals,” 2023. [Online]. Available: <https://arxiv.org/abs/2312.03197>.
- [9] M. Singha and S. Roy, “Compactness with ideals,” 2021. [Online]. Available: <https://arxiv.org/abs/2107.00050>
- [10] Hadi J. Mustafa, A. Lafta. “Operator topological space”. Journal the college of Education, Al-Mustansiriya University. (2009).
- [11] Hadi J. Mustafa, and A. Abdul Hassan, T-open sets. M.Sc thesis. Mu'ta University Jordan. (2004).
- [12] Hadi J. Mustafa, and Layth M. Alabdulsada “Certain properties of contra - T_{12}^* - continuous function”. Journal of Kufa Mathematics and computer Sciences, Vol. 6, no. 3. (2019).
- [13] D. Andrijević, “On B-open Sets,” Matematički Vesnik, no. 205, pp. 59–64, 1996, [Online]. Available: <http://eudml.org/doc/252912>.
- [14] A. Caksu Guler and G. Aslim, “B I -open sets and decomposition of continuity via idealization,” Proceedings of Institute of Mathematics and Mechanics. National Academy of Sciences of Azerbaijan, vol. 22, Jan. 2005.
- [15] C. Granados, “A new notion of convergence on ideal topological spaces,” Selecciones Matemáticas, vol. 7, no. 2, pp. 250–256, Dec. 2020, doi: 10.17268/sel.mat.2020.02.07.