

# On Soc-2-Absorbing -Ideals

Khudhayer O. kadem <sup>1</sup>

<sup>1</sup> Open Educational College, Iraq, Babylon  
[Khudhayer1981@gmail.com](mailto:Khudhayer1981@gmail.com)

**Abstract:** In this paper, we aim to discover a new class of ring ideals that we have named Soc-2-Absorbing-ideal (for short S.2.A.I). We have explored many examples, proofs, and aspects that support the aforementioned definition, as well as some cases that do not fit the definition and how it links to certain previous definitions that are comparable to the work. It has been shown that the radical of a certain ideal also satisfies the definition. After that, the rings were given characteristics to see how they impacted the concept under study.

**Keywords—** 2-Absorbing ideals, multiplicatively closed, socle ideals.

## 1. INTRODUCTION

In this topic, any ring is commutative with  $1 \neq 0$ . Recall that a proper ideal  $D$  of a ring  $R$  is called 2-Absorbing ideal if whenever  $qht \in D$  with  $q, h \in R, t \in D$  implies that either  $qt \in D$  or  $ht \in D$  or  $qh \in D$ .  $F \triangleleft D$  with  $F \neq D$  named  $D$  is a 2-Absorbing if whenever  $qht \in D, \forall q, h \in R, t \in D \Rightarrow qt \in D$  or  $ht \in D$  or  $qh \in [D :_R F]$  (1, 2). Regarding the definition relevant to work, it is  $Soc(D) = \cap \{F \triangleleft D \setminus F \text{ is an essential ideal of } D\}$  (6). Research number provided an explanation of the study of the substitutional rings' structures as well as a few other characteristics and how they are transferred from the original ring to the ring that results from the combination (7, 8). In commutative rings, generalizations in ideals—including the generalization of the original ideal are crucial. From this perspective, this new model of ideals, which relies primarily on socle of  $D$ . This shows us a new type of idealism said Soc-2-Absorbing-ideal (for short S.2.A.I) its definition was as follows:  $F$  is said to be a Soc-2-Absorbing-ideal (for short S.2.A.I), whenever  $q, h, t \in D$  such that  $qht \in F$ , then  $qt \in F$  or  $ht \in F$  or  $qh \in Soc(D)$ . Some of the characteristics studied here also indicate the transfer of a trait S.2.A.I From one ring to the next through comprehensive analogy. We also used the condition  $ker(F) \subset F$  to ensure that the multiplication occurs within the ideal itself.

## 2. SOC-2-ABSORBING -IDEALS

**Definition<sup>1</sup>:** Suppose that  $F \subsetneq D$  be an ideal, where  $D$  is a ring.  $F$  is said to be a Soc-2-Absorbing-ideal (for short S.2.A.I), whenever  $q, h, t \in D$  such that  $qht \in F$ , then  $qt \in F$  or  $ht \in F$  or  $qh \in Soc(D)$ .

**Example<sup>2</sup>:**

(i) An example that fulfills the definition. If we have the ring  $D = Z_{20}$ , thus  $Soc(D) = \langle \bar{2} \rangle = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$ , let  $F = \langle \bar{0} \rangle = \{0\}$ . If  $q =$

$3 \in D, h = 2 \in D, t = 3 \in D$  and  $3 \times 2 \times 3 = 18 = 0 \in F = \langle \bar{0} \rangle = \{0\}$ , now  $\bar{3} \times 2 = 6 \in Soc(D) = \langle \bar{2} \rangle$ . Therefore  $F$  is S.2.A.I of  $D$ .

(ii) An example that does not meet the definition. Assume  $D = Z, Soc(D) = Soc(Z) = \langle \bar{0} \rangle = \{0\}$  and  $F = 12Z$ . If  $q = 2 \in D, h = 2 \in D, t = 3 \in D$  and  $2 \times 2 \times 3 = 12 \in F$ , now  $qt = 2 \times 3 = 6 \notin F$ , and  $ht = 2 \times 3 = 6 \notin F, qh = 2 \times 2 = 4 \notin Soc(Z) = \langle \bar{0} \rangle = \{0\}$ .

Therefore  $F$  is not S.2.A.I ideal of  $D$ .

**Proposition<sup>3</sup>:** Let  $qh \subsetneq D$  be an ideal, where  $D$  is a ring. Such that,  $qh \not\subseteq Soc(D)$ , if  $F_1, F_2$  are S.2.A.I of  $D$  such that  $F_1 qh = F_2 qh$  then,  $F_1 = F_2$ .

**Proof:** If  $F_1$  S.2.A.I of  $D$  and  $F_2 qh \subseteq F_1$  where  $qh \not\subseteq Soc(D)$ , Obviously  $F_2 \subseteq F_1$ . If  $F_2$  S.2.A.I of  $D$  and  $F_1 qh \subseteq F_2$  where  $qh \not\subseteq Soc(D)$ , Obviously  $F_1 \subseteq F_2$ . Thus  $F_1 = F_2$ .

**Proposition<sup>4</sup>:** Suppose that  $F \subsetneq D$  be an ideal, where  $D$  is a ring, and  $D \supset K \not\subseteq F$ , such that  $F$  is S.2.A.I of  $D$ . Then  $(F :_D K)$  is S.2.A.I of  $D$ .

**Proof:** Let  $h_1, h_2, h_3 \in D$ , such that  $h_1 h_2 h_3 \in (F :_D K)$ , then  $h_1 h_2 h_3 k \in F, \forall k \in K$ . If  $h_1 h_2 \in Soc(D)$ , then we are done. If  $h_1 h_2 \notin Soc(D)$ , Obviously,  $h_2 h_3 k \in F$ . We have  $F$  is S.2.A.I, as a result,  $h_2 h_3 \in (F :_D K)$ , thus  $(F :_D K)$  is S.2.A.I of  $D$ .

**Proposition<sup>5</sup>:** Let  $\{F_\alpha\}_{\alpha \in \Lambda}$  is a nonempty collection of S.2.A.I of  $D$ , then  $\cap_{\alpha \in \Lambda} F_\alpha$  is S.2.A.I of  $D$ .

**Proof:** Assume that  $\{F_\alpha\}_{\alpha \in \Lambda}$  is a nonempty collection of S.2.A.I of  $D$  and  $q, h, t \in D$  such that  $qht \in \cap_{\alpha \in \Lambda} F_\alpha$ . There are two cases

- If  $qh \in Soc(D)$ , this is clear.
- Assume that  $qh \notin Soc(D)$ , then,  $qht \in F_\alpha, \forall \alpha \in \Lambda$ . Now  $F_\alpha$  is S.2.A.I of  $D$ . we get  $qt \in F_\alpha, \forall \alpha \in \Lambda$  and  $ht \in F_\alpha, \forall \alpha \in \Lambda$ . Thus,  $qt \in \cap_{\alpha \in \Lambda} F_\alpha$  and  $ht \in \cap_{\alpha \in \Lambda} F_\alpha$ . Then  $\cap_{\alpha \in \Lambda} F_\alpha$  is S.2.A.I of  $D$ .

**Proposition<sup>6</sup>:** Let  $D$  be a ring. Then, the following statements hold:

- $Soc(D)$  is S.2.A.I iff  $Soc(D)$  is a 2-Absorbing ideal.

**Proof:** Let  $Soc(D)$  is S.2.A.I and  $q, h, t \in D$  such that  $qht \in Soc(D)$ . Now we have  $Soc(D)$  is S.2.A.I then  $qt \in$

$Soc(\mathcal{D})$  or  $ht \in Soc(\mathcal{D})$ , thus,  $Soc(\mathcal{D})$  is a 2-Absorbing ideal of  $\mathcal{D}$ .

- b. If  $\mathcal{D}$  be a ring and  $Soc(\mathcal{D}) = (0)$  and  $\mathcal{D}$  is not an integral domain, then  $\mathcal{D}$
- $$\{F \leq \mathcal{D}/F \text{ is S. 2. A. I.}\} \subseteq \{(0)\}.$$

**Proof:** For the sake of contradiction, if  $F$  is S. 2. A. I.  $F \subseteq Soc(\mathcal{D}) = (0)$  and  $F = Soc(\mathcal{D}) = (0)$ , now from (a)  $Soc(\mathcal{D}) = (0)$  is a 2-Absorbing ideal. This is a contradiction because  $\mathcal{D}$  is not an integral domain. Thus,  $\mathcal{D}$  admits no S. 2. A. I. Since reduced local rings are socle-free, the second statement is evident.

- c. The ring  $\mathcal{D}$  is Soc-free.  $\mathcal{D}$  is an integral domain if and only if  $(0)$  is a S. 2. A. I. of  $\mathcal{D}$ .

**Proof:** Assume that  $\mathcal{D}$  is an integral domain. Then,  $Soc(\mathcal{D}) = (0)$  is a 2-Absorbing ideal and S. 2. A. I. In contrast, let  $(0)$  is S. 2. A. I., then,  $Soc(\mathcal{D}) = (0)$  is a 2-Absorbing ideal of  $\mathcal{D}$  by (a), Which leads to  $\mathcal{D}$  is an integral domain.

**Proposition<sup>7</sup>:** Let  $F \subseteq \mathcal{D}$  be an ideal, where  $\mathcal{D}$  is a ring. The following statements are equivalent:

1.  $F$  is S. 2. A. I. of  $\mathcal{D}$ .
2.  $(F :_{\mathcal{D}} c) = F, \forall c \in \mathcal{D} \setminus Soc(\mathcal{D})$ .
3. If  $e, d$  are ideals of  $\mathcal{D}$  such that  $e.d \subseteq F$  but  $F \not\subseteq Soc(\mathcal{D})$ , then  $e \subseteq F$ .

**Proof:**

1  $\rightarrow$  2: Assume that  $F$  is S. 2. A. I. of  $\mathcal{D}$  and  $c \in \mathcal{D} \setminus Soc(\mathcal{D})$ . Let

$e \in (F :_{\mathcal{D}} c)$  then,  $c \in F$ . This demonstrates that  $(F :_{\mathcal{D}} c) \subseteq F$ , thus  $F = (F :_{\mathcal{D}} c)$ .

2  $\rightarrow$  3: If it comes true (2), and  $e, d$  are ideals of  $\mathcal{D}$  such that  $e.d \subseteq F$  but  $F \not\subseteq Soc(\mathcal{D})$ . Let  $c \in d \setminus Soc(\mathcal{D})$ .  $\forall e_1 \in e$ , we notice  $e_1 c \in e.d \subseteq F$ . Accordingly,  $e_1 \in (F :_{\mathcal{D}} c) = F$ , or it can be said  $e \subseteq F$ .

3  $\rightarrow$  1: If it comes true (3), and  $e, d \in \mathcal{D}$  such that  $e.d \in F$ . There are two cases

- (a) If  $d \in Soc(\mathcal{D})$ , this is clear.
- (b) Assume that  $qh \notin Soc(\mathcal{D})$ , then,  $(qht) \subseteq F$ , it follows that  $(ht) \subseteq F$ , from this we conclude  $(ht) \subseteq F$ . Therefore,  $F$  is S. 2. A. I. of  $\mathcal{D}$ .

**Proposition<sup>8</sup>:** Let  $\mathcal{D}$  be a ring. If  $\mathcal{D}$  admits S. 2. A. I.s. Then,  $Soc(\mathcal{D})$  contains a 2-Absorbing ideal of  $\mathcal{D}$ .

**Proof:** Assume that  $\mathcal{D}$  admits S. 2. A. I.s, and  $Q$  be the set of all S. 2. A. I.s of  $\mathcal{D}$  partially ordered by inclusion. If  $\{F_\alpha\}_{\alpha \in \Lambda}$  be a chain in  $Q$ . Let  $h_1, h_2, h_3 \in \mathcal{D}$  such that  $h_1 h_2 h_3 \in F$ . If  $h_1 h_2 \in Soc(\mathcal{D})$ , then we are done. If  $h_1 h_2 \notin Soc(\mathcal{D})$ . Since  $h_1 h_2 h_3 \in F$ , Obviously,  $h_1 h_2 h_3 \in F_\theta$ , for some  $\theta \in \Lambda$ . Since  $F_\theta$  is S. 2. A. I. and we have  $h_1 h_2 \notin Soc(\mathcal{D})$ , then  $h_1 h_3 \in F_\theta \subseteq F$ . Consequently, we have that  $F$  is S. 2. A. I. By Zorn's Lemma,  $Q$  has a maximal element S. 2. A. I.,  $n$ . We will prove  $n$  is a 2-Absorbing ideal of  $\mathcal{D}$ . Suppose that  $h_1 h_2 h_3 \in \mathcal{D}$  such that  $h_1 h_2 h_3 \in n$  but  $h_1 h_2 \notin n$ . From the above

property,  $(n :_{\mathcal{D}} h_1 h_2)$  S. 2. A. I. The maximality of  $n$  forces  $(n :_{\mathcal{D}} h_1 h_2) = n$ , and  $h_1 h_3 \in n$ , then  $n$  is a 2-Absorbing ideal of  $\mathcal{D}$ . Since  $n$  is S. 2. A. I. of  $\mathcal{D}$ , from Proposition (3),  $n \subseteq Soc(\mathcal{D})$ . Then,  $Soc(\mathcal{D})$  contains a 2-Absorbing ideal of  $\mathcal{D}$ .

**Proposition<sup>9</sup>:** Let  $\mathcal{D}$  be a ring. If,  $Soc(\mathcal{D})$  contains a 2-Absorbing ideal of  $\mathcal{D}$ . Then  $\mathcal{D}$  is a completely reducible ring or  $Soc(\mathcal{D})$  is a minimal 2-Absorbing ideal of  $\mathcal{D}$ .

**Proof:** We prove it by contradiction.  $\mathcal{D}$  is not a completely reducible ring. Let  $h$  be an arbitrary 2-Absorbing ideal of  $\mathcal{D}$ , such that  $qh \subseteq Soc(\mathcal{D})$ . Demonstrate that  $qh = Soc(\mathcal{D})$ . Let  $qh \subsetneq Soc(\mathcal{D})$ , thus,  $\exists n \subseteq \mathcal{D}$  such that  $n \not\subseteq qh$ . By Brauer's Lemma,  $n^2 = (0)$  or  $n = (g)$ , for some idempotent element  $g$  of  $\mathcal{D}$ . impossible  $n^2 = (0)$ , since  $h$  is a 2-Absorbing ideal of  $\mathcal{D}$ . Thus,  $n = (g)$  for some idempotent element  $g$  of  $\mathcal{D}$ . As a result,  $n(1 - g) \subseteq qh$  and  $h$  is a 2-Absorbing ideal with  $n \not\subseteq h$  then,  $(1 - g) \subseteq h$ . But,  $1 = g + 1 - g \in Soc(\mathcal{D})$ , so  $Soc(\mathcal{D}) = \mathcal{D}$  which cannot hold since  $\mathcal{D}$  is not a completely reducible ring, this contradiction. Then,  $qh = Soc(\mathcal{D})$  hence  $Soc(\mathcal{D})$  is a minimal 2-Absorbing ideal of  $\mathcal{D}$ .

**Corollary<sup>10</sup>:** Let  $\mathcal{D}$  be a completely reducible ring. Then  $\mathcal{D}$  admits S. 2. A. I.s.

**Proposition<sup>11</sup>:** Assume that  $\mathcal{D}$  be a ring. Then, every proper ideal of  $\mathcal{D}$  is a S. 2. A. I. if and only if  $\mathcal{D}$  is a completely reducible ring.

**Proof:** We prove it by contradiction. If  $\mathcal{D}$  is not a completely reducible ring. Let  $\mathcal{D}$  contains two distinct maximal ideals  $h$  and  $k$ . Since  $h$  and  $k$  are S. 2. A. I.s, we see that  $h, k \subseteq Soc(\mathcal{D})$  and so  $\mathcal{D} = h + k \subseteq Soc(\mathcal{D})$ , since  $\mathcal{D}$  is not a completely reducible ring.

**Proposition<sup>12</sup>:** Assume that  $\mathcal{D}_1, \mathcal{D}_2$  be rings. The assertion that follows is true:

- a. The  $Rad(\mathcal{D}_1)$  and  $Rad(\mathcal{D}_2)$  are non-zero, then  $(\mathcal{D}_1 \oplus \mathcal{D}_2)$  admits no S. 2. A. I.
- b. The  $Rad(\mathcal{D}_1)$  and  $Rad(\mathcal{D}_2)$  are non-zero, then  $(\mathcal{D}_1 \times \mathcal{D}_2)$  admits no S. 2. A. I.

**Proof:**(a)  $Soc(\mathcal{D}_1 \oplus \mathcal{D}_2) = Soc(\mathcal{D}_1) \oplus Soc(\mathcal{D}_2)$ , as the  $Rad(\mathcal{D}_1)$  and  $Rad(\mathcal{D}_2)$  are non-zero, thus  $Soc(\mathcal{D}_1) = \mathcal{D}_1$  and  $Soc(\mathcal{D}_2) = \mathcal{D}_2$ . Therefore  $Soc(\mathcal{D}_1) \oplus Soc(\mathcal{D}_2) = \mathcal{D}_1 \oplus \mathcal{D}_2$ . Includes either  $\mathcal{D}_1 \oplus (0)$  or  $\mathcal{D}_2 \oplus (0)$  ( $\mathcal{D}_1 \oplus \mathcal{D}_2$ ), thus admits no S. 2. A. I.

**Proof:** (b) In the same way as the solution (a).

**Corollary<sup>13</sup>:** Assume that  $F : \mathcal{D} \rightarrow Q$  be a homomorphism ring, then  $F(Soc(\mathcal{D})) \subseteq Soc(Q)$ .

**Proof:** If  $F$  be a large ideal of  $\mathcal{D}$ . Obviously,  $F^{-1}(F)$  be an a large ideal of  $\mathcal{D}$ . Thus,  $F^{-1}((Soc(\mathcal{D})) = F^{-1}(\cap_{F \subseteq e Q} F) = \cap_{F \subseteq e Q} F^{-1}(F) \supseteq \cap_{h \subseteq e Q} h = Soc(\mathcal{D})$ . This explains  $F(Soc(\mathcal{D})) \subseteq Soc(Q)$ .

**Proposition<sup>14</sup>:** Assume that  $g : \mathcal{D} \rightarrow Q$  be an epimorphism ring and  $ker(F) \subsetneq F$ .

If  $F$  is S. 2. A. I. of  $\mathcal{D}$ , then  $g(F)$  is S. 2. A. I. of  $Q$ .

**Proof:** Let  $q, h, t \in \mathcal{D}$  such that  $qht \in F(F)$ . As  $g$  is epimorphism,  $\exists a, b, c \in \mathcal{D}$ , such that  $qt = g(a)$ ,  $ht = g(b)$  and  $qh = gF(c)$  Which leads to,  $qhtqh = g(abc)$ . We have  $ker(F) \subsetneq F$ , thus  $abc \in F$ . From above  $F$  is

$S. 2. A. I$ , then neither  $ac \in \mathcal{F}$ , Which leads to,  $g(ac) \in g(\mathcal{F})$  nor  $bc \in \mathcal{F}$ , Which leads to,  $g(bc) \in g(\mathcal{F})$  nor  $ab \in Soc(\mathcal{D})$ , Which leads to,  $g(ab) \in g(Soc(\mathcal{D}))$ . Thus  $g(Soc(\mathcal{D})) \subseteq Soc(Q)$ . Then  $g(ab) \in Soc(Q)$ . Thus,  $g(\mathcal{F})$  is  $S. 2. A. I$  of  $Q$ .

**Proposition<sup>15</sup>:** Assume that  $g: \mathcal{D} \rightarrow Q$  be an epimorphism ring and  $g^{-1}((Soc(Q)) \subseteq Soc(\mathcal{D}))$ . If  $g$  is  $S. 2. A. I$  of  $Q$ , then  $g^{-1}(\mathcal{F})$  is  $S. 2. A. I$  of  $\mathcal{D}$ .

**Proof:** Let  $q, h, t \in \mathcal{D}$  such that  $qht \in g^{-1}(\mathcal{F})$ . From the above, we can say,  $g(qht) \in g(g^{-1}(\mathcal{F}))$ , Which can be said,  $g(qt)g(ht)g(qh) \in \mathcal{F}$ , but  $\mathcal{F}$  is  $S. 2. A. I$  of  $Q$ , subsequently, neither  $g(qt) \in \mathcal{F}$ , from this we can say,  $qt \in g^{-1}(\mathcal{F})$  nor  $g(ht) \in \mathcal{F}$ , from this we can say,  $ht \in g^{-1}(\mathcal{F})$  nor  $g(qh) \in Soc(Q)$ , from this we can say,  $qh \in g^{-1}(Soc(Q))$ , but we have  $g^{-1}((Soc(Q)) \subseteq Soc(\mathcal{D}))$ , then  $qh \in Soc(\mathcal{D})$ .

Therefore,  $g^{-1}(\mathcal{F})$  is  $S. 2. A. I$  of  $\mathcal{D}$ .

**Theorem<sup>16</sup>:** Assume  $\mathcal{D}$  is a ring and  $g \subseteq \mathcal{D}$  be multiplicatively closed. If  $g$  is  $\mathcal{F}$ -saturated ideals of  $\mathcal{D}$ , and  $g^{-1}\mathcal{F}$  is  $S. 2. A. I$  of  $g^{-1}\mathcal{S}$ , thus  $g$  is  $S. 2. A. I$  of  $\mathcal{D}$ .

**Proof:** We know  $Soc(\mathcal{D})$ ,  $Soc(g^{-1}\mathcal{D})$  are minimal 2-Absorbing ideals of  $\mathcal{D}$ . It is known that  $g^{-1}((Soc(\mathcal{D})) \subseteq Soc(g^{-1}\mathcal{D})$  take,  $g^{-1}((Soc(\mathcal{D})) = Soc(g^{-1}\mathcal{D})$ . Assume  $q, h, t \in \mathcal{D}$  such that  $qht \in \mathcal{F}$ . It is obvious that,  $qt = \frac{qt}{1}$ ,  $ht = \frac{ht}{1}$  and  $qh = \frac{qh}{1}$  thus,  $\frac{qt}{1} \frac{ht}{1} \frac{qh}{1} = \frac{qhtqh}{1} \in \mathcal{F}^{-1}(\mathcal{F})$ . Now we have  $g^{-1}\mathcal{F}$  is  $S. 2. A. I$ , then  $\frac{qt}{1} \in g^{-1}\mathcal{F}$  or  $\frac{ht}{1} \in g^{-1}\mathcal{F}$  or  $\frac{qh}{1} \in Soc(g^{-1}\mathcal{D}) = g^{-1}((Soc(\mathcal{D}))$ . Therefore  $\mathcal{F}$  is  $S. 2. A. I$  of  $\mathcal{D}$ .

### 3. CONCLUSION

The goal of this work has been accomplished since we were able to identify a new kind of idealism that is mostly based on the sum of minimal idealisms and the degree to which elements belong to them. This has made it possible to identify further novel forms of this kind of idealism. Furthermore, this new definition enables future researchers to identify and discover numerous ideals and establish connections between them, as it affords them significant scope for exploration.

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